Quantum-Bayesian coherence

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Abstract

In the quantum-Bayesian interpretation of quantum theory (or QBism), the Born rule cannot be interpreted as a rule for setting measurement-outcome probabilities from an objective quantum state. But if not, what is the role of the rule? In this paper, the argument is given that it should be seen as an empirical addition to Bayesian reasoning itself. Particularly, it is shown how to view the Born rule as a normative rule in addition to usual Dutch-book coherence. It is a rule that takes into account how one should assign probabilities to the consequences of various intended measurements on a physical system, but explicitly in terms of prior probabilities for and conditional probabilities consequent upon the imagined outcomes of a special counterfactual reference measurement. This interpretation is exemplified by representing quantum states in terms of probabilities for the outcomes of a fixed, fiducial symmetric informationally complete measurement. The extent to which the general form of the new normative rule implies the full state-space structure of quantum mechanics is explored.

1 Introduction: Unperformed Measurements Have No Outcomes

In the opening chapter on quantum mechanics in the famous Feynman Lectures on Physics (Feynman, Leighton, and Sands, 1965), Richard Feynman wrote

> We choose to examine a phenomenon which is impossible, *absolutely* impossible, to explain in any classical way, and which has in it the heart of quantum mechanics. In reality, it contains the *only* mystery. We cannot make the mystery go away by "explaining" how it works. We will just tell you how it works. In telling you how it works we will have told you about the basic peculiarities of all quantum mechanics.

With these words, Feynman plunged into a discussion of the double-slit experiment using *individual* electrons. Imagine if you will, however, someone well versed in the quantum foundations debates of the last 30 years - since the Aspect experiment say (Aspect, Dalibard, and Roger, 1982) - yet naively unaware of when Feynman wrote this. What might he conclude that Feynman was talking about? Would it be the double-slit experiment? Probably not. To the modern mind- set, a good guess would be that Feynman was talking about something to do with quantum entanglement or Bell- inequality violations. In the history of foundational thinking, the double-slit experiment has fallen by the wayside.

So, what is it that quantum entanglement teaches us (via EPR-type considerations and Bell-inequality violations) that the double-slit experiment does not? A common answer is that quantum mechanics does not admit a "local hidden-variable" formulation.[1] By this one usually means the conjunction of two statements (Bell, 1964, 1981): (1) that experiments in one region of spacetime cannot instantaneously affect matters of fact at far away regions of spacetime, and (2) that there exist "hidden variables" that in some way "explain" measured values or their probabilities. Bell-inequality violations imply that one or the other or some combination of both these statements fails. This, many would say, is the deepest "mystery" of quantum mechanics.

This mystery has two sides. A number of physicists who care about these things think it is locality (condition 1) that has to be abandoned through the force of the experimentally observed Bell-inequality violations, i.e., they think there really are "spooky actions at a distance."[2] Yet, there are others who think the abandonment of condition 2 is the more warranted conclusion (Peres, 1978; Wheeler, 1982; Zeilinger, 1996; Mermin, 1999; Zukowski, 2005; Plotnitsky, 2006; D'Ariano, 2009; Demopoulos, 2010). Among these are the *quantum Bayesians* (Schack, Brun, and Caves, 2001; Caves, Fuchs, and Schack, 2002a, 2007; Fuchs, 2002a, 2003, 2010a, 2010b, 2012; Schack,

[1] Too quick, some would say (Norsen, 2006). However, the conclusion drawn there (that a Bell-inequality violation implies the failure of locality, full stop) is based in part on taking the EPR criterion of reality or variants of it as sacrosanct. As will become clear in this review, we do not take it so.

[2] Indeed, it flavors almost everything they think of quantum mechanics, including the *interpretation* of the toy models they use to better understand the theory. For instance, Popescu-Rohrlich boxes (Popescu and Rohrlich, 1994) are imaginary devices that give rise to greater-than-quantum violations of various Bell inequalities. Importantly, another common name for these devices is the term "nonlocal boxes" (Barrett and Pironio, 2005). Their exact definition comes via the magnitude of a Bell-inequality violation, which entails the non-pre-existence of values or a violation of locality or both, but the commonly used name opts only to recognize nonlocality. They are not called no-hidden-variable boxes, for instance.

2003; Fuchs and Schack, 2004; Appleby, 2005b, 2005c; Timpson, 2008; Mermin, 2012).[3] Giving up on hidden variables implies, in particular, that measured values do not *preexist* the act of measurement. A measurement does not merely "read off" the values, but enacts or creates them by the process itself. In a phrase inspired by Asher Peres (Peres, 1978), "unperformed measurements have no outcomes."

Among the various arguments the quantum Bayesians use to come to this conclusion, not least in importance is a thorough-going personalist account of all probabilities (Ramsey, 1931; de Finetti, 1931, 1990; Savage, 1954; Bernardo and Smith, 1994; Jeffrey, 2004), including probabilities for quantum measurement outcomes and even the probability-1 assignments among these (Caves, Fuchs, and Schack, 2007). From the quantum-Bayesian point of view, this is the only sound interpretation of probability. Moreover, this move for quantum probabilities frees up the quantum state from any objectivist obligations. In so doing it wipes out the mystery of quantum-state change at a distance (Einstein, 1951; Fuchs and Peres, 2000; Timpson, 2008) and much of the mystery of wave function collapse as well (Fuchs, 2002a, 2010b, 2013).

Apparently Feynman too saw something of a truth in the idea that "unperformed measurements have no outcomes." Yet, he did so because of the double-slit experiment. Later in the lecture he wrote,

> Is it true, or is it not true that the electron either goes through hole 1 or it goes through hole 2? The only answer that can be given is that we have found from experiment that there is a certain special way that we have to think in order that we do not get into inconsistencies. What we must say (to avoid making

[3] For alternative developments of several Bayesian-inspired ideas in quantum mechanics, see Youssef (2001), Baez (2003), Leifer (2006, 2007), Pitowsky (2003, 2005), Srednicki (2005), Caticha (2007), Leifer and Spekkens (2011), Porta Mana (2007), Rau (2009), Goyal (2008), Warmuth and Kuzmin (2009), and Bub (2011). We keep these citations separately because of various distinctions within each from what we are calling quantum Bayesianism - these distinctions range from (1) the particular strains of Bayesianism each adopts to (2) whether quantum mechanics is a generalized probability theory or rather simply an application within Bayesian probability *per se* to (3) the level of the agent's involvement in bringing about the outcomes of quantum measurements. There are nonetheless sometimes striking kinships between the ideas of these papers and the effort here, and the references are well worth studying.

wrong predictions) is the following. If one looks at the holes or, more accurately, if one has a piece of apparatus which is capable of determining whether the electrons go through hole 1 or hole 2, then one can say that it goes either through hole 1 or hole 2. But, when one does not try to tell which way the electron goes, when there is nothing in the experiment to disturb the electrons, then one may not say that an electron goes either through hole 1 or hole 2. If one does say that, and starts to make any deductions from the statement, he will make errors in the analysis. This is the logical tightrope on which we must walk if we wish to describe nature successfully.

Returning to our quote from Feynman's opening chapter on quantum mechanics, we are left with the feeling that this is the very thing Feynman saw to be the "basic peculiarity of all quantum mechanics."

One should ask though, is his conclusion really compelled by so simple a phenomenon as the double slit? How could simple "interference" be so far reaching in its metaphysical implications? Water waves interfere and there is no great mystery there. Most importantly, the double-slit experiment is a story of measurement on a single quantum system, whereas the story of EPR and Bell is that of measurement on two seemingly disconnected systems.

Two systems are introduced for good reason. Without the guarantee of arbitrarily distant parts within the experiment (so that one can conceive of measurements on one and draw inferences about the other) what justification would one have to think that changing the conditions of the experiment (from one slit closed to both slits open) should not make a deep conceptual difference to its analysis? Without such a guarantee for underwriting a belief that some matter of fact stays constant in the consideration of two experiments, one, it might seem, would be quite justified in responding, "Of course, you change an experiment, and you get a different probability distribution arising from it. So what?"[4] It seems that Feynman's logical path

[4]This is a point Koopman (1957) and Ballentine (1986) seem to stop the discussion with. For instance, Ballentine writes, "One is well advised to beware of probability statements of the form, $P(X)$, instead of $P(X|C)$. The second argument may be safely omitted only if the conditional event or information is clear from the context, and only if it is constant throughout the problem. This is not the case in the double-slit experiment ..... We observe from experiment that $P(X|C_3) \neq P(X|C_1) + P(X|C_2)$. This fact, however, has no bearing

from *example* to *conclusion*, a conclusion that we indeed agree with, is simply unwarranted. The argument just does not seem to hold to the same stringent standards as Bell-inequality analyses.

This dismissal of Feynman's argument is premature. The argument is just not so easily appreciated without the proper mind-set. The key point is that the so-called interference in the example is not in a material field (of course it was never purported to be) but in something so ethereal as probability itself (a logical, not a physical, construct). Most particularly, Feynman makes use of a beautiful and novel move: He analyzes the probabilities in an experiment that *will be* done in terms of the probabilities from experiments that *will not be* done. He does not simply conditionalize the probabilities to the two situations and let it go at that.[5] Rather he tries to see the probabilities in the two situations not as functions of a condition, but functions (or at least relations) of each other. This is an important point. There is no necessity that the world gives a relation between these probabilities, yet it does: Quantum mechanics is what makes the link precise. Feynman seems to hint at the idea that the essence of the quantum mechanical formalism is to provide a tool for analyzing the factual in terms of a counterfactual.

Here is the way Feynman put it in a paper titled, "The Concept of Probability in Quantum Mechanics," (Feynman, 1951):

> I should say, that in spite of the implication of the title of this talk the concept of probability is not altered in quantum mechanics. When I say the probability of a certain outcome of an experiment is p, I mean the conventional thing, that is, if the experiment is repeated many times one expects that the fraction of those which give the outcome in question is roughly p.[6] I will not be at all concerned with analyzing or defining this concept in more detail,

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on the validity of ... probability theory."

[5]See footnote 4.

[6]Note that Feynman was not a frequentist in his thinking about probability. For instance, in the Lectures on Physics, chapter 1-6, he says (Feynman, Leighton, and Sands, 1965)

> An experimental physicist usually says that an "experimentally determined" probability has an "error," and writes
>
> $$P(H) = \frac{N_H}{N} \pm \frac{1}{2\sqrt{N}}$$

> for no departure from the concept used in classical statistics is required.
>
> What is changed, and changed radically, is the method of calculating probabilities.

We believe that Feynman is thinking here explicitly about the relation between outcomes of factual and counterfactual measurements. The "radical change" is that these outcomes are related by the complex-amplitudes formalism developed by Feynman, rather than by the ordinary rules of probability theory. As shown in Sec. 2, Bayesian probability theory by itself does not provide any relation between the probabilities for the outcomes of factual and counterfactual measurements. Rather than changing the way probabilities are calculated, the complex-amplitudes formalism provides a connection between probabilities which, *a priori*, need not be connected at all. The following phrase, adapted from Feynman?s words, captures therefore more accurately the starting point of this paper.

> The concept of probability is not altered in quantum mechanics (it is personalistic Bayesian probability). What is radical is the recipe it gives for calculating new probabilities from old.

We plan to show in this review that quantum mechanics gives a resource that raw Bayesian probability theory does not: It gives a rule for forming probabilities for the outcomes of *factualizable*[7] experiments (experiments that may actually be performed) from the probabilities one assigns for the outcomes of a designated *counterfactual* experiment (an experiment only imagined, and though possible to do, never actually performed). So, yes, unperformed

> There is an implication in such an expression that there is a "true" or "correct" probability which could be computed if we knew enough, and that the observation may be in "error" due to a fluctuation. There is, however, no way to make such thinking logically consistent. It is probably better to realize that the probability concept is in a sense subjective, that it is always based on uncertain knowledge, and that its quantitative evaluation is subject to change as we obtain more information.

[7] We coin this term because it stands as a better counterpoint to the term "counterfactual" than the term "actualizable" seems to. We also want to capture the following idea a little more plainly: Both measurements being spoken of here are only potential measurements - it is just that one will always be considered in the imaginary realm, whereas the other may one day become a fact of the matter if it is actually performed.

measurements have no outcomes as Peres expressed; nonetheless, imagining their performance can aid in analyzing the probabilities one ought to assign for an experiment that may factually be performed. Quantum mechanics can thus be seen as providing an empirical addition to the laws of Bayesian probability.

In this review, we offer a development along quantum-Bayesian lines of Feynman's ideas by making intimate use of a potential[8] representation of quantum states unknown in his time: It is one based on symmetric informationally complete observables (SICs) (Caves, 1999; Zauner, 1999; Fuchs, 2004; Renes et al., 2004; Appleby, 2005a; Appleby, Dang, and Fuchs, 2007). The goal is to make it more transparent than ever that the content of the Born rule is not that it gives a procedure for setting probabilities (from some independent entity called "the quantum state"), but that it represents a "method of calculating probabilities," new ones from old.

That this must be the meaning of the Born rule more generally for quantum Bayesianism has been argued from several angles by Caves, Fuchs, and Schack (2007) and Fuchs (2013). What is new in this review is the emphasis on a single designated observable for the counterfactual thinking, as well as a detailed exploration of the rule for combining probabilities in this picture. Particularly, we will see that a significant part of the structure of quantum-state space arises from the consistency of that rule, a single formula we designate the *urgleichung* (German for "primal equation"). We are thus putting a simple Feynman-style scenario (if not the double-slit experiment *per se*, nonetheless one wherein probabilities for the outcomes of factualizable experiments are obtained from probabilities in a family of designated counterfactual ones) at the heart of quantum mechanics. If our considerations turn out to give rise to the full formalism of quantum theory, we will be able to say with Feynman that this scenario contains the "basic peculiarities of all quantum mechanics."

The plan of the paper is as follows. In Sec. 2, we review the personalist Bayesian account of probability, showing how some Dutch-book arguments work and emphasizing a point we have not seen emphasized before: Bayes's rule and the law of total probability, Eqs. (1) and (3), are not necessities in

[8]We say "potential" because so far the representation has been seen to exist only for finite dimensional quantum systems with dimension 100; see Sec. 3.

a Bayesian account of probability. These rules are enforceable when there is a *conditional lottery* in the picture that can be gambled upon. But when there is no such lottery, the rules hold no force; without a conditional lottery there is nothing in Dutch-book coherence itself that can be used to compel the rules.

In Sec. 3, we review the notion of a SIC and show a sense in which it is a special measurement. Most importantly we delineate the full structure of quantum-state space in SIC terms. It turns out that, by making use of a SIC instead of any other informationally complete measurement, the formalism be- comes uniquely simple and compact. We also show that unitary time evolution, when written in SIC terms, looks (formally at least) almost identical to classical stochastic evolution.

In Sec. 4, we introduce the idea of thinking of an imaginary (counterfactual) SIC behind all quantum measurements, so as to give an imaginary conditional lottery with which to define conditional probabilities. We then show how to write the Born rule in these terms and find it strikingly similar to the law of total probability, Eq. (3). We then note how this move in interpretation is radically different from the one offered by the followers of "objective chance" in the sense of Lewis (1986a, 1986b).

In Sec. 5, we show that one can derive some of the features of quantum-state space by taking this modified or quantum law of total probability as a postulate. Particularly, we show that with a small number of further assumptions, it gives rise to a generalized Bloch sphere and seems to define an underlying "dimensionality" for a system that matches the one given by its quantum mechanical state space. We also demonstrate other features of the geometry these considerations give rise to.

In Sec. 6, we give a brief discussion of where we stand at this stage of research. Finally in Sec. 7, we conclude by mapping out a tentative path toward a formal expression of the ontology underlying a quantum-Bayesian vision of quantum mechanics: It has to do with the Peres phrase "unperformed measurements have no outcomes," but tempered with a kind of "realism" that he probably would not have accepted forthrightly.[9] It is not a

[9]We say this because of Peres's openly acknowledged positivist tendencies. See Chapters 22 and 23 in Fuchs (2010a) where Peres would sometimes call himself a "recalcitrant

realism that we expect to be immediately accepted by most modern philosophers of science either.[10] This is because it is already clear that whatever it will ultimately turn out to be, it is based on (a) a rejection of the ontology of the block universe (James, 1940, 1956a, 1956b), and (b) a rejection of the ontology of the detached observer (Laurikainen, 1988; Pauli, 1994; Gieser, 2005). The realism in vogue in philosophy-of-science circles, which makes heavy use of both these ele- ments, is, as Wolfgang Pauli once said, "too narrow a concept" for our purposes (Pauli, 1994). Reality, the stuff of which the world is made, the stuff that was here before agents and observers, is more interesting than that.

2 Personalist Bayesian Probability

From the Bayesian point of view, probability is a degree of belief as measured by action. More precisely, we say one has (explicitly or implicitly) assigned a probability $p(A)$ to an event A if, before knowing the value of A, one is willing to either buy or sell a lottery ticket of the form

$$\text{worth \$1 if } A$$

for an amount $\$p(A)$.[11] The personalist Bayesian position adds only that this is the full meaning of probability; it is nothing more and nothing less than this definition. Particularly, nothing intrinsic to the event or proposition A can help declare $p(A)$ right or wrong, or more or less rational. The value $p(A)$ is solely a statement about the agent who assigns it.

Nonetheless, even for a personalist Bayesian, probabilities do not wave in the wind. Probabilities are held together by a normative principle: That whenever an agent declares probabilities for various events [say A, $\neg B$ ("not B"), $A \vee B$, ("A or B"), $A \wedge B$ ("A and C"), etc] he should strive to never gamble (i.e., buy and sell lottery tickets) so as to incur what he believes will be a sure loss. This normative principle is known as Dutch-book coherence.

positivist." Also see the opening remarks of Peres (2005).

[10]See Nagel (1989), Price (1997), and Dennett (2004) for introductions to the "view from nowhere" and "view from nowhen" *weltanschauungen*.

[11]In other words, the personalist Bayesian agent regards $\$p(A)$ as the fair price of the lottery ticket. He would regard it as advantageous to buy it for any price less than $\$p(A)$, or to sell it for any price greater than $\$p(A)$.

And from it, one can derive the usual calculus of probability theory.

This package of views about probability (that in value it is personal, but that in function it is akin to the laws of logic) had its origin in the mid-1920s and early 1930s with the work of Ramsey (1931) and de Finetti (1931). Keynes (1951) characterizes Ramsey's position succinctly:

> (Ramsey) succeeds in showing that the calculus of probabilities simply amounts to a set of rules for ensuring that the system of degrees of belief which we hold shall be a *consistent* system. Thus the calculus of probabilities belongs to formal logic. But the basis of our degrees of belief - or the *a priori*, as they used to be called - is part of our human outfit, perhaps given us merely by natural selection, analogous to our perceptions and our memories rather than to formal logic.

We now go through some of the derivation of the probability calculus from Dutch-book coherence so that we may better make a point concerning quantum mechanics afterward.[12] We first establish that our normative principle requires $0 \leq P(A) \leq 1$. Suppose $P(A) < 0$. This means an agent will sell a ticket for negative money, i.e., he will pay someone $\$p(A)$ to take the ticket off his hands. Regardless of whether A occurs or not, the agent will then be sure he will lose money. This violates the normative principle. Now take the case $P(A) > 1$. This means the agent will buy a ticket for more than it is worth even in the best case, again a sure loss for him and a violation of the normative principle. So, probability in the sense of ticket pricing should obey the usual range of values.

Now we establish the probability sum rule. Suppose our agent believes two events A and B to be mutually exclusive; i.e., he is sure that if A occurs, B will not, or if B occurs, A will not. We can contemplate three distinct lottery tickets:

worth \$1 if $A \vee B$
worth \$1 if A
worth \$1 if B

[12] Here we basically follow the development in Richard Jeffrey's posthumously published book *Subjective Probability, The Real Thing* (Skyrms, 1987a; Jeffrey, 2004), but with our own emphasis.

Clearly the value of the first ticket should be the same as the total value of the other two. For instance, suppose an agent had set $P(A \vee B)$, $P(A)$, and $P(B)$ such that $P(A \vee B) > P(A) + P(B)$. Then, by definition, when confronted with a seller of the first ticket, he must be willing to buy it, and when confronted with a buyer of the other two tickets, he must be willing to sell them. But then the agent's initial balance sheet would be negative: $= \$P(A \vee B) + \$P(A) + \$P(B) < \$0$. And whether A or B or neither event occurs, it would not improve his finances: If a dollar flows in (because of the bought ticket), it will also flow out (because of the agent's responsibilities for the sold tickets), and still the balance sheet is negative. The agent is sure of a loss. A similar argument goes through if the agent had set his ticket prices so that $P(A \vee B) < P(A) + P(B)$. Thus whatever values are set, the normative principle prescribes that it had better be the case that $P(A \vee B) = P(A) + P(B)$.

Consider now the following lottery ticket of a slightly different structure:

$$\text{worth } \$\frac{m}{n} \text{ if } A$$

where $m \leq n$ are integers. Does Dutch-book coherence say anything about the value of this ticket in comparison to the value of the standard ticket, i.e., one worth \$1 if A? It does. An argument quite like the one above dictates that it should be valued $\$(m/n)P(A)$. If a real number α were in place of the m/n, a similar result follows from a limiting argument.

Now we come to the most interesting case, which is Bayes's rule relating joint to conditional probabilities:

$$p(A \wedge B) = p(A)p(B|A) \tag{1}$$

Like the rest of the structure of probability theory within the Bayesian conception, this rule must arise from an application of Dutch-book coherence.

That application is a conditional lottery (Kyburg and Smokler, 1980). In such a lottery, it is revealed to the agent first whether or not the event A happens. If A obtains, the lottery proceeds to the revelation of event B, and finally all monies are settled up. If, on the other hand, $\neg A$ obtains, the remainder of the lottery is called off, and the monies put down for any "conditional tickets" are returned. That is to say, the meaning of $p(B|A)$ is taken to be the price $\$p(B|A)$ at which one is willing to buy or sell a lottery ticket of the following form:

$$\text{worth \$1 if } A \wedge B, \quad \text{but return price if } \neg A$$

Explicitly inserting the definition of $p(B|A)$, this becomes

$$\text{worth \$1 if } A \wedge B, \quad \text{but return } \$p(B|A) if \neg A$$

Now comes the coherence argument. If you think about it, the price for this ticket had better be the same as the total price for these two tickets:

$$\begin{aligned} &\text{worth \$1 if } A \wedge B \\ &\text{worth } \$p(B|A) \text{ if } \neg A \end{aligned}$$

That is to say, to guard against a sure loss, we must have

$$\begin{aligned} p(B|A) &= p(A \wedge B) + P(B|A)p(\neg A) \\ &= p(A \wedge B) + P(B|A) - p(B|A)p(A) \end{aligned} \tag{2}$$

Consequently, Eq. (1) should hold whenever there is a conditional lottery under consideration.

When a conditional lottery is not without consequence. - But what if, in the above scenario introduced to derive Bayes's rule, the conditional lottery is called off because the draw that was to give rise to the event A does not take place? In this case the probabilities $p(A)$ and $p(B|A)$ refer to a counterfactual and there is no reason to assume the validity of Eq. (1).

It is worth investigating the idea of counterfactuals in more detail. Suppose an agent makes a measurement of a variable X that takes on mutually exclusive values x, followed by a measurement of a variable Y with mutually exclusive values y. A Dutch bookie asks him to commit on various unconditional and conditional lottery tickets. What can we say of the probabilities he ought to ascribe? A minor variation of the Dutch-book arguments above tells us that whatever values of $p(x)$, $p(y)$, and $p(y|x)$ he commits to, they ought, if he is coherent, satisfy the law of total probability:

$$p(y) = \sum_x p(x)p(y|x) \tag{3}$$

Imagine now that the X measurement is called off, so there will only be the Y measurement. Is the agent still normatively committed to buying and selling Y-lottery tickets for the price $\$p(y)$ in Eq. (3) that he originally expressed? Not at all. That would clearly be irrational in some situations. The action bringing about the result of the X measurement might change the situation for bringing about Y so that he simply would not gamble on it in the same way. To hold fast to the $\$p(y)$ valuation of a Y-lottery ticket then is not a necessity enforced by coherence, but a judgment that might or might not be the right thing to do.

In fact, one might regard holding fast to the initial value $\$p(y)$ in spite of the nullification of the conditional lottery as the formal definition of precisely what it means to judge an unperformed measurement to have an outcome. It means one judges that looking at the value of X is incidental to the whole affair, and this is reflected in the way one gambles on Y (Fuchs and Schack, 2012a). So, if $q(y)$ represents the probabilities with which the agent gambles supposing the X lottery nullified, then a formal statement of the Peresian phrase that the unperformed X measurement had no outcome (i.e., measuring X matters, and it matters even if one makes no note of the outcome) is that

$$q(y) \neq p(y) \tag{4}$$

Still one might imagine situations in which even if an agent judges that equality does not hold for them, he nonetheless judges that $q(y)$ and $p(y)$ should bear a precise relation to each other. In Sec. 4, we show that in fact the positive content of the Born rule as an addition to Bayesianism is to connect the probabilities for two measurements, one factual and one counterfactual, for which Dutch-book coherence alone does not provide a precise relationship.

3 Expressing Quantum-State Space in terms of SICs

Let $\mathcal{H}_d$ be a finite-dimensional Hilbert space associated with some physical system. A quantum state for the system is usually expressed as a unit-trace positive semidefinite linear operator $\rho \in \mathcal{L}(\mathcal{H}_d)$. However, through the use of a SIC as a reference observable, we can find an alternative representation of quantum states directly in terms of an associated set of probability distributions.

A SIC is an example of a generalized measurement or positive operator-valued measure (POVM) (Peres, 1993). A POVM is a collection $\{E_i\}, i = 1, ..., n$, of positive semi-definite operators E_i on $\mathcal{H}_d$ such that

$$\sum_i E_i = I \tag{5}$$

where n is in general unrelated to d and may be larger or smaller than d. Supposing a quantum state ρ, the probability of the measurement outcome labeled i is then given by

$$p(i) = \mathrm{Tr}\rho E_i \tag{6}$$

The POVMs represent the most general kinds of quantum measurement that can be made on a system. A von Neumann measurement is a special POVM where the E_i are mutually orthogonal projection operators. Mathematically, any POVM can be written as a unitary interaction with an ancillary quantum system, followed by a von Neumann measurement on the ancillary system (Nielsen and Chuang, 2000).

We can provide an injective mapping between the convex set of density operators and the set of probability distributions[13]

$$\|p\rangle\!\rangle = (p(1), p(2), ..., p(d^2))^T \tag{7}$$

over d^2 outcomes (the probability simplex Δ_{d^2}) by first fixing any so-called minimal informationally complete fiducial measurement $\{E_i\}, i = 1, ..., d^2$. This is a POVM with all the E_i linearly independent. With respect to such a measurement, the probabilities $p(i)$ for its outcomes completely specify ρ. This follows because the E_i form a basis for $\mathcal{L}(\mathcal{H}_d)$, and the probabilities $\mathrm{Tr}\rho E_i$ can be viewed as instances of the Hilbert-Schmidt inner product

$$(A, B) = \mathrm{Tr}A^\dagger B \tag{8}$$

[13]Please note our pseudo-Dirac notation $\|v\rangle\!\rangle$ for vectors in a real vector space of d^2 dimensions. The relevant probability simplex for us, the one we are mapping quantum states ρ to, denoted Δ_{d^2}, is a convex body within this linear vector space. Its points may be expressed with the notation $\|p\rangle\!\rangle$ as well. The choice of a pseudo-Dirac notation for probability distributions also emphasizes that one should think of the valid $\|p\rangle\!\rangle$ as a direct expression of the set of quantum states.

The quantities $p(i)$ thus merely express the projections of the vector ρ onto the basis vectors E_i. These projections completely fix the vector ρ.

One can see how to calculate ρ in terms of the vector $\|p\rangle\rangle$ in the following way. Since the E_i form a basis, there must be some expansion

$$\rho = \sum_j \alpha_j E_j \tag{9}$$

where the α_j are real numbers making up a vector $\|\alpha\rangle\rangle$. Thus,

$$p(i) = \sum_j \alpha_j \mathrm{Tr} E_i E_j \tag{10}$$

If we let a matrix M be defined by entries

$$M_{ij} = \mathrm{Tr} E_i E_j \tag{11}$$

this just becomes

$$\|p\rangle\rangle = M\|\alpha\rangle\rangle \tag{12}$$

Using the fact that M is invertible because the E_i are linearly independent, we have finally

$$\|\alpha\rangle\rangle = M^{-1}\|p\rangle\rangle \tag{13}$$

The most important point of this exercise is that with such a mapping established, one has every right to think of a quantum state as a probability distribution. Fuchs (2002a) argued that conceptually it is nothing more. However, the mapping $\rho \mapsto \|p\rangle\rangle$, although injective, cannot be surjective; only some probability distributions in the simplex are valid for representing quantum states (see Fig. 1). Understanding the range of shapes available under these mappings (Bengtsson and Życzkowski, 2006) is an important problem. If quantum states are nothing more than probability distributions, a significant part of understanding quantum mechanics is understanding what restrictions there are on the set of valid distributions.

Informationally complete measurements abound - they come in all forms and sizes. What is the best measurement one can use for a mapping $\rho \mapsto \|p\rangle\rangle$? One would not want to unduly burden the representation with extra terms and calculations if one does not have to. For instance, it would be advantageous if one could take the informationally complete measurement $\{E_i\}$ so that M is simply a diagonal matrix or even the identity matrix itself. But these maximal simplifications cannot be achieved.

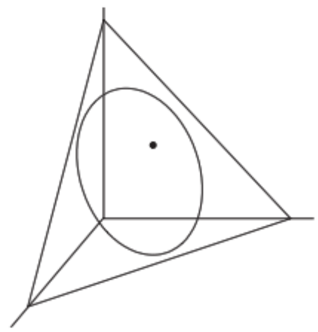

Figure 1: The planar surface represents the convex set of all probability distributions over d^2 outcomes - the probability simplex Δ_{d^2}. With respect to any fiducial POVM, the probability distributions valid for representing the set of quantum states, however, form a smaller convex set within the simplex, here depicted as an ellipsoid. In actual fact, however, the convex shape is quite complex. The choice of a SIC for defining the mapping makes the shape as simple as it can be with respect to the natural geometry of the simplex.

If one cannot make M diagonal, one might still want to make M as close to the identity as possible. A convenient measure for how far M is from the identity is the squared Frobenius distance:

$$F = \sum_{ij}(\delta_{ij} - M_{ij})^2 = \sum_i (1 - \mathrm{Tr} E_i^2)^2 + \sum_{i \neq j} (\mathrm{Tr} E_i E_j)^2 \tag{14}$$

We can place a lower bound on this quantity with the help of a special instance of the Schwarz inequality: If λ_r is any set of n non-negative numbers, then

$$\sum_r \lambda_r^2 \geq \frac{1}{n} \left(\sum_r \lambda_r \right)^2 \tag{15}$$

with equality if and only if $\lambda_1 = \cdots = \lambda_n$. Applying the inequality (15) to each term in Eq. (14) gives

$$F \geq \frac{1}{d^2} \left(1 - \mathrm{Tr} E_i^2\right)^2 + \frac{1}{d^4 - d^2} \left(\sum_{i \neq j} \mathrm{Tr} E_i E_j \right)^2 \tag{16}$$

Equality holds in this if and only if there are constants m and n such that $\mathrm{Tr} E_i^2 = m$ for all i and $\mathrm{Tr} E_i E_j = n$ for all $i \neq j$. We have

$$d = \mathrm{Tr} I^2 = \sum_{ij} \mathrm{Tr} E_i E_j = \sum_i \mathrm{Tr} E_i^2 + \sum_{i \neq j} \mathrm{Tr} E_i E_j \tag{17}$$

Dividing both sides of Eq. (17) by d^2 gives

$$\frac{1}{d} = m + (d^2 - 1)n \tag{18}$$

On the other hand, with the conditions for F achieving its lower bound fulfilled, the E_i must all have the same trace. For

$$\mathrm{Tr}E_k = \sum_i \mathrm{Tr}E_k E_i = m + (d^2 - 1)n \tag{19}$$

Consequently,

$$\mathrm{Tr}E_k = \frac{1}{d} \tag{20}$$

Now how large can m be? Take a positive semidefinite matrix A with $\mathrm{Tr}A = 1$ and eigenvalues λ_i. Then $\lambda_i \leq 1$, and clearly $\mathrm{Tr}A^2 \leq \mathrm{Tr}A$ with equality if and only if the largest λ_i is equal to 1. Hence, dE_k will give the largest allowed value m if $E_i = (1/d)\Pi_i$, where

$$\Pi_i = |\psi_i\rangle\langle\psi_i| \tag{21}$$

for some rank-1 projection operator Π_i. If this holds, n takes the form

$$n = \frac{1}{d^2(d+1)} \tag{22}$$

In total we have shown that a measurement $\{E_i\}, i = 1, ..., d^2$, will only if achieve the best lower bound for F if and only if

$$E_i = \frac{1}{d}\Pi_i \tag{23}$$

with

$$\mathrm{Tr}\Pi_i\Pi_j = |\langle\psi_i|\psi_j\rangle|^2 = \frac{d\delta_{ik} + 1}{d+1} \tag{24}$$

It turns out that measurements of this variety also have the property of being necessarily informationally complete (Caves, 1999). This follows from the fact that the E_i are linearly independent. Suppose there are some numbers α_i such that

$$\sum_i \alpha_i E_i = 0 \tag{25}$$

Taking the trace of this equation, we infer that

$$\sum_i \alpha_i = 0 \tag{26}$$

Now multiply Eq. (25) by an arbitrary E_k and take the trace of the result. We get

$$\frac{1}{d^2}\sum_i \alpha_i \frac{d\delta_{ik}+1}{d+1} = 0 \tag{27}$$

which, in view of Eq. (26), gives $\alpha_k = 0$. So the E_i are linearly dependent.

These kinds of measurements are presently a hot topic of study in quantum information theory and have come to be known as "symmetric informationally complete" quantum measurements (Caves, 1999). As such, the measurement $\{E_i\}$, the associated set of projection operators $\{\Pi_i\}$, and even the set of $\{|\psi_i\rangle\}$ are often simply called SIC. We adopt that terminology here.

Here is an example of a SIC in dimension 2, expressed in terms of the Pauli operators:

$$\begin{aligned}
|\psi_1\rangle\langle\psi_1| &= \frac{1}{2}\left(I + \frac{1}{\sqrt{3}}(\sigma_x + \sigma_y + \sigma_z)\right)\\
|\psi_2\rangle\langle\psi_2| &= \frac{1}{2}\left(I + \frac{1}{\sqrt{3}}(\sigma_x - \sigma_y - \sigma_z)\right)\\
|\psi_3\rangle\langle\psi_3| &= \frac{1}{2}\left(I + \frac{1}{\sqrt{3}}(-\sigma_x - \sigma_y + \sigma_z)\right)\\
|\psi_4\rangle\langle\psi_4| &= \frac{1}{2}\left(I + \frac{1}{\sqrt{3}}(-\sigma_x + \sigma_y - \sigma_z)\right)
\end{aligned} \tag{28}$$

i.e., the vectors $|\psi_i\rangle$ are spin-1/2 eigenstates of the spin components along the four diagonals of a cube.

And here is an example of a SIC in dimension 3 (Tabia, 2012). Taking

$\omega = e^{2\pi i/3}$ to be a third root of unity and $\bar{\omega}$ to be its complex conjugate, let

$$
\begin{aligned}
|\psi_1\rangle &= \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \ |\psi_2\rangle = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \ |\psi_3\rangle = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \\
|\psi_4\rangle &= \begin{pmatrix} 0 \\ \omega \\ -\bar{\omega} \end{pmatrix}, \ |\psi_5\rangle = \begin{pmatrix} -1 \\ 0 \\ \bar{\omega} \end{pmatrix}, \ |\psi_6\rangle = \begin{pmatrix} 1 \\ -\omega \\ 0 \end{pmatrix} \\
|\psi_7\rangle &= \begin{pmatrix} 0 \\ \bar{\omega} \\ -\omega \end{pmatrix}, \ |\psi_8\rangle = \begin{pmatrix} -1 \\ 0 \\ \omega \end{pmatrix}, \ |\psi_9\rangle = \begin{pmatrix} 1 \\ -\bar{\omega} \\ 0 \end{pmatrix}
\end{aligned}
\tag{29}
$$

be defined up to normalization. One can check by quick inspection that (after normalization) these vectors do indeed satisfy Eq. (24).

Do SICs exist for every finite dimension d? Despite many efforts in the last 14 years [see Caves (1999), Zauner (1999), Fuchs (2004), Renes et al. (2004), Appleby (2005a), and Appleby, Dang, and Fuchs (2007) and particularly the extensive reference lists in Scott and Grassl (2010) and Appleby et al. (2012)] no one presently knows. However, there is a strong feeling in the community that they do, as analytical proofs have been obtained for all dimensions $d =$ 2-16, 19, 24, 28, 31, 35, 37, 43,and 48,[14] and within a numerical precision of 10^{-38}, they have been observed by computational means (Scott and Grassl, 2010) in all dimensions d = 2-67. To lesser numerical precision Schnetter (2013) also found SICs in d = 68-76, 78-81, 83-85, 87, 89, 93, and 100.

We now spell out in some detail what the set of quantum states written as SIC probability vectors $\|p\rangle\!\rangle$ looks like. Perhaps the most remarkable thing about a SIC is the level of simplicity it lends to Eq. (13). On top of the theoretical justification that SICs are as near as possible to an orthonormal basis, from Eqs. (6), (23), and (24) one gets the simple expression (Caves, 2002; Fuchs, 2004)

$$\rho = \sum_i \left((d+1)p(i) - \frac{1}{d} \right) \Pi_i \tag{30}$$

[14]Dimensions 2-5 were published in Zauner (1999). Dimensions 6-15, 24, 28, 35, and 48 are due to M. Grassl in various publications; see Scott and Grassl (2010) and Appleby et al. (2012). Dimensions 7, 19, 31, 37, and 43 are due to D. M. Appleby, with the latter three as yet unpublished (Appleby et al., 2012).

The components $(d+1)p(i)-1/d$ are obtained by a universal scalar readjustment from the probabilities $p(i)$. This will have important implications.

Still, one cannot place just any probability distribution $p(i)$ into Eq. (30) and expect to obtain a positive semidefinite ρ. Only some probability vectors $\|p\rangle\!\rangle$ are *valid* ones. Which ones are they? For instance, $p(i) \leq 1/d$ follows from Eqs. (6) and (20) alone, and this already restricts the class of valid probability assignments. But there are more requirements than that.

In preparation for characterizing the set of valid probability vectors $\|p\rangle\!\rangle$, we note that since the Π_k form a basis on the space of operators, we can define operator multiplication in terms of them. This is done by introducing the so-called structure coefficients α_{ijk} for the algebra:

$$\Pi_i \Pi_j = \sum_k \alpha_{ijk} \Pi_k \tag{31}$$

A couple of properties immediately follow. Taking the trace of both sides of Eq. (31), one has

$$\sum_k \alpha_{ijk} = \frac{d\delta_{ik} + 1}{d+1} \tag{32}$$

Using this, one gets straightforwardly that

$$\mathrm{Tr}(\Pi_i \Pi_j \Pi_k) = \frac{1}{d+1} \left(d\alpha_{ijk} + \frac{d\delta_{ik} + 1}{d+1} \right) \tag{33}$$

In other words,

$$\alpha_{ijk} = \frac{1}{d} \left((d+1)\mathrm{Tr}(\Pi_i \Pi_j \Pi_k) - \frac{d\delta_{ik} + 1}{d+1} \right) \tag{34}$$

For the analog of Eq. (32) but with summation over the first or second index, one gets

$$\sum_i \alpha_{ijk} = d\delta jk \quad \text{and} \quad \sum_j \alpha_{ijk} = d\delta ik \tag{35}$$

With Eqs. (31)-(35) in hand, one sees a direct connection between the structure of the algebra of quantum states when written in operator language and the structure of quantum states when written in probability-vector language. The complete convex set of quantum states is fixed by the set of its extreme points, i.e., the pure quantum states or rank-1 projection operators.

To characterize this set algebraically, one method is to note that these are the only Hermitian operators satisfying $\rho^2 = \rho$. Using Eq. (30), we find that a quantum state $\|p\rangle\!\rangle$ is pure if and only if its components satisfy these d^2 simultaneous quadratic equations:

$$p(k) = \frac{1}{3}(d+1)\sum_{ij}\alpha_{ijk}p(i)p(j) + \frac{2}{3d(d+1)} \tag{36}$$

Another way to characterize this algebraic variety (an algebraic variety is defined as the set of solutions of a system of polynomial equations) is to make use of a theorem of Flammia (2004) and Jones and Linden (2005): A Hermitian operator A is a rank-1 projection operator if and only if $\mathrm{Tr}A^2 = \mathrm{Tr}A^3 = 1$.[15] So, in fact our d^2 simultaneous quadratic equations reduce to just two equations instead, one a quadratic and one a cubic:

$$\sum_i p(i)^2 = \frac{2}{d(d+1)} \tag{37}$$

and

$$\sum_{ijk}\alpha_{ijk}p(i)p(j)p(k) = \frac{4}{d(d+1)^2} \tag{38}$$

Note that Eqs. (36) and (38) are complex equations, but one could symmetrize them and make them purely real if one wanted to.

There are also some advantages to working out these equations more explicitly in terms of the completely sym- metric 3-index tensor

$$c_{ijk} = \mathrm{Re}\ \mathrm{Tr}(\Pi_i\Pi_j\Pi_k) \tag{39}$$

In terms of these quantities, the analogs of Eqs. (36) and (38) become

$$p(k) = \frac{(d+1)^2}{3d}\sum_{ij}c_{ijk}p(i)p(j) - \frac{1}{3d} \tag{40}$$

[15] The theorem is nearly trivial to prove once one's attention is ??drawn to it: Since A is Hermitian, it has a real eigenvalue spectrum λ_i. From the first condition, one has that $\sum_i \lambda_i^2 = 1$; from the second, $\sum_i \lambda_i^3 = 1$. The first condition, however, implies that $|\lambda_i| \le 1$ for all i. Consequently $1 - \lambda_i \ge 0$ for all i. Now taking the difference of the two conditions, one sees that $\sum_i(\lambda_i^2(1-\lambda_i) = 0$. In order for this to hold, it must be the case that λ_i is always 0 or 1 exclusively. That there is only one nonzero eigenvalue then follows from using the first condition again. Thus the theorem is proved. However, it seems to not have been widely recognized previous to 2004 and 2005.

and

$$\sum_{ijk} c_{ijk} p(i)p(j)p(k) = \frac{d+7}{(d+1)^3} \tag{41}$$

respectively. The reason for noting this comes from the simplicity of the d^2 matrices C_k with matrix entries $(C_k)_{ij} = c_{ijk}$ from Eq. (39), which was explored by Appleby, Flammia, and Fuchs (2011). To give an idea of the results, we note for instance that, for each value of k, C_k turns out to have the form (Appleby, Flammia, and Fuchs, 2011)

$$C_k = \|m_k\rangle\!\rangle\langle\!\langle m_k\| + \frac{d}{2(d+1)} Q_k \tag{42}$$

where the kth vector $\|m_k\rangle\!\rangle$ is defined by

$$\|m_k\rangle\!\rangle = \left(\frac{1}{d+1}, ..., 1, ..., \frac{1}{d+1}\right)^T \tag{43}$$

and Q_k is a $(2d-2)$-dimensional projection operator on the real vector space embedding the probability simplex Δ_{d^2}. Furthermore, using this, one obtains a useful expression for the pure states; they are probabilities satisfying a simple class of quadratic equations

$$p(k) = dp(k)^2 + \tfrac{1}{2}(d+1)\langle\!\langle p\|Q_k\|p\rangle\!\rangle \tag{44}$$

With Eqs. (37), (40), and (41) we have now discussed the extreme points of the convex set of quantum states - the pure states. The remainder of the set of quantum states is then constructed by taking convex combinations of the pure states. This is an implicit expression of quantum-state space. But SICs can also help give an explicit parametrization of the convex set.

We can see this by starting not with density operators, but with "square roots" of density operators. This is useful because a matrix ρ is positive semidefinite if and only if it can be written as $\rho = B^2$ for some Hermitian B. Thus, let

$$B + \sum_i b_i \Pi_i \tag{45}$$

with b_i a set of real numbers. Then

$$\rho = \sum_k \left(\sum_{ij} b_i b_j \alpha_{ijk} \right) \Pi_k \tag{46}$$

will represent a density operator so long as $\mathrm{Tr}\rho = 1$. This condition requires simply that

$$\left(\sum_i b_i\right)^2 + d\sum_i b_i^2 = d + 1 \tag{47}$$

so that the vectors $(b_1, ..., b_{d^2})$ lie on the surface of an ellipsoid.

Putting these ingredients together with Eq. (6), we have the following parametrization of valid probability vectors $\|p\rangle\!\rangle$:

$$p(k) = \frac{1}{d}\sum_{ij} c_{ijk} b_i b_j \tag{48}$$

Here the c_{ijk} are the triple-product constants defined in Eq. (39) and the b_i satisfy the constraint (47).

Finally, we note what the Hilbert-Schmidt inner product of two quantum states looks like in SIC terms. If a quantum state ρ is mapped to $\|p\rangle\!\rangle$ via a SIC, and a quantum state σ is mapped to $\|q\rangle\!\rangle$, then

$$\mathrm{Tr}\rho\sigma = d(d+1)\sum_i p(i)q(i) - 1 = d(d+1)\langle\!\langle p\|q\rangle\!\rangle - 1 \tag{49}$$

Notice a particular consequence of this: Since $\mathrm{Tr}\rho\sigma \geq 0$, the distributions associated with distinct quantum states can never be too nonoverlapping:

$$\langle\!\langle p\|q\rangle\!\rangle \geq \frac{1}{d(d+1)} \tag{50}$$

With this development we gave a broad outline of the shape of quantum-state space in SIC terms. We do this because that shape is our target. Particularly, we are obliged to answer the following question: If one takes the view that quantum states are nothing more than probability distributions with the restrictions (47) and (48), what could motivate that restriction? That is, what could motivate it other than knowing the usual formalism for quantum mechanics? The answer has to do with rewriting the Born rule in terms of SICs, which we do in Sec. 4.

Unitary time evolution. - We now take a moment to move beyond statics and rewrite quantum dynamics in SIC terms: We do this because the result will have a striking resemblance to the Born rule itself, once developed in the next section.

Suppose we start with a density operator ρ and let it evolve under unitary time evolution to a new density operator $\sigma = U\rho U^\dagger$. If ρ has a representation $p(i)$ with respect to a certain given SIC, σ will have a SIC representation as well - we call it $q(j)$. We use the different index j (contrasting with i) to help indicate that we are talking about the quantum system at a later time than the original.

What is the form of the mapping that takes $\|p\rangle\!\rangle$ to $\|q\rangle\!\rangle$? It is simple enough to find with the help of Eqs. (23) and (30):

$$q(j) = \frac{1}{d}\mathrm{Tr}\sigma\Pi_j = \frac{1}{d}\sum_i \left((d+1)p(i) - \frac{1}{d}\right)\mathrm{Tr}(U\Pi_i U^\dagger \Pi_j) \tag{51}$$

If we now define

$$r_U(j|i) = \frac{1}{d}\mathrm{Tr}(U\Pi_i U^\dagger \Pi_j) \tag{52}$$

and remember, e.g., Eq. (21), we have that

$$0 \le r_U(j|i) \le 1 \tag{53}$$

and

$$\sum_j r_U(j|i) = 1 \;\; \forall\, i \quad \text{and} \quad \sum_i r_U(j|i) = 1 \;\; \forall\, j \tag{54}$$

In other words, the $d^2 \times d^2$ matrix $[r_U(j|i)]$ is a doubly stochastic matrix (Horn and Johnson, 1985).

Most importantly, one has

$$q(j) = (d+1)\sum_{i=1}^{d^2} p(i) r_U(j|i) - \frac{1}{d} \tag{55}$$

Without the $d+1$ factor and the $1/d$ term, Eq. (55) would represent classical stochastic evolution. Unitary time evolution in a SIC representation is thus formally close to classical stochastic evolution. As we shall shortly see, this teaches us something about unitarity and its connection to the Born rule itself.

4 Expressing the Born Rule in terms of SICs

In this section we come to the main focus of the review: We rewrite the Born rule in terms of SICs, using the expansion in Eq. (30). We first do it for an arbitrary von Neumann measurement - that is, any measurement specified by a set of rank-1 projection operators $P_j = |j\rangle\langle j|$, $j = 1, ..., d$. Expressing the Born rule the usual way, we obtain these probabilities for the measurement outcomes:

$$q(j) = \mathrm{Tr}\rho P_j \tag{56}$$

Then, by defining

$$r(j|i) = \mathrm{Tr}\Pi_i P_j \tag{57}$$

one sees that the Born rule becomes

$$q(j) = (d+1)\sum_{i=1}^{d^2} p(i)r(j|i) - 1 \tag{58}$$

We now take a moment to seek out a good interpretation of this equation. It should be viewed in terms of the considerations developed in Sec. 2. Imagine that before performing the P_j measurement, which we call the "measurement on the ground," we were to perform a SIC measurement Π_i. We call the latter the "measurement in the sky."

Starting with an initial quantum state ρ, we assign a probability distribution $p(i)$ to the outcomes of the SIC measurement. In order to be able to say something about probabilities conditional on a particular outcome of the SIC measurement, we need to specify the post-measurement quantum state for that outcome, i.e., the quantum operation associated with the measurement. Since the quantum operation associated with a POVM is not determined by the POVM, we are free to make the most convenient choice. Here we adopt the standard Lüders rule (Busch, Grabowski, and Lahti, 1995; Busch and Lahti, 2009) that ρ transforms to Π_i when outcome i occurs. The conditional probability for getting j in the subsequent von Neumann measurement on the ground, consequent upon i, is then precisely $r(j|i)$ as defined in Eq. (57). With these assignments, Dutch-book coherence demands an assignment $s(j)$ or the outcomes on the ground that satisfies

$$s(j) = \sum_{i=1}^{d^2} p(i)r(j|i) \tag{59}$$

i.e., a probability that comes about via the law of total probability, Eq. (3).

But now imagine the measurement in the sky nullified, i.e., imagine it does not occur after all, and that the quantum system goes directly to the measurement device on the ground. Quantum mechanics tells us to make the probability assignment $q(j)$ given in Eq. (58) instead. So

$$q(j) = (d+1)s(j) - 1 \tag{60}$$

That $q(j) \neq s(j)$ holds, for any assignment $s(j) \neq 1/d$, is a formal expression of the idea that the "unperformed SIC had no outcomes," as explained in Sec. 2. But Eq. (60) tells us more detailed information than this. It expresses a kind of "empirically extended coherence," not implied by Dutch-book coherence alone, but formally similar to the kind of relation one gets from Dutch-book coherence. It contains a surprising amount of information about the structure of quantum mechanics.

To support this, we try to glean some insight from Eq. (60). The most obvious thing one can note is that $\|s\rangle\!\rangle$ cannot be too sharp a probability distribution. Otherwise, $q(j)$ will violate the bounds $0 \leq q(j) \leq 1$ set by Dutch-book coherence. In particular,

$$\frac{1}{d+1} \leq s(j) \leq \frac{2}{d+1} \tag{61}$$

This in turn has implications for the range of values possible for $p(i)$ and $r(j|i)$. Indeed if either of these distributions become too sharp (in the latter case, for too many values of i), again the bounds will be violated. This suggests that an essential part of quantum-state space structure, as expressed by its extreme points satisfying Eqs. (37) and (38), arises from the very requirement that $q(j)$ be a proper probability distribution. In the next section, we explore this question in greater depth.

First though, we must note the most general form of the Born rule, when the measurement on the ground is not restricted to being of the simple von Neumann variety. So, let

$$q(j) = \mathrm{Tr}\rho F_j \tag{62}$$

and

$$r(J|i) = \mathrm{Tr}\Pi_i F_j \tag{63}$$

for some general POVM $\{F_j\}$ on the ground, with any number of outcomes, $j = 1, ..., m$. Then the Born rule becomes

$$q(j) = (d+1)\sum_{i=1}^{d^2} p(i)r(j|i) - \frac{1}{d}\sum_{i=1}^{d^2} r(j|i) \tag{64}$$

As stated, this is the most general form of the quantum law of total probability. It has two terms, a term composed of the classical law of total probability, and a term dependent only upon the sum of the conditional probabilities.

When the measurement on the ground is itself another SIC (any SIC) it reduces to

$$q(j) = (d+1)\sum_{i=1}^{d^2} p(i)r(j|i) - \frac{1}{d} \tag{65}$$

Notice the formal resemblance between this and Eq. (55) expressing unitary time evolution.

Why empirically extended coherence instead of objective quantum states? - What we are suggesting is that perhaps Eq. (64) should be taken as one of the basic axioms of quantum theory, since it provides a particularly clear way of thinking of the Born rule as an addition to Dutch-book coherence. This addition is empirically based and gives extra normative rules, beyond the standard rules of probability theory, to guide the agent's behavior when he interacts with the physical world.

But, one may well ask, what is wrong with the standard way of expressing the Born rule? How is introducing an addition to Dutch-book coherence conceptually any more palatable than introducing objective quantum states or objective probability distributions? If the program is successful, then the demand that $q(j)$ be a proper probability distribution will place necessary restrictions on $p(i)$ and $r(j|i)$. This, a skeptic would say, is the very sign that one is dealing with objective (or agent-independent) probabilities in the first place. Why would a personalist Bayesian accept any *a priori* restrictions on his probability assignments? And particularly, restrictions supposedly of empirical origin? It is true that through an axiom like Eq. (64) one gets a restriction on the ranges of the various probabilities one can contemplate holding. But that restriction in no way diminishes the functional role of prior beliefs in the makings of an agent's particular assignments $p(i)$ and

$r(j|i)$. That is, this addition to Dutch-book coherence preserves the points ex- pressed in the quote by Keynes in Sec. 2 in a way that objective chance cannot.

Take the usual notion of objective chance, as given operational meaning through David Lewis's "principal principle" (Lewis, 1986a, 1986b). If an event A has objective chance $ch(A) = x$, hen the subjective, personalist probability an agent (any agent) should ascribe to A on the condition of knowing the chance proposition is

$$\mathrm{Prob}(A|``ch(A) = x'' \wedge E) = x \tag{66}$$

where E is any "admissible" proposition. There is some debate about what precisely constitutes an admissible proposition, but an example of a proposition universally accepted to be admissible in spite of these interpretive details is

E = "All my experience causes me to believe A with probability 75%"

That is, upon knowing an objective chance, all prior beliefs should be overridden. Regardless of the agent's firmly held belief about A, that belief becomes irrelevant once he is apprised of the objective chance.

When it comes to quantum mechanics, philosophers of science who find something digestible in Lewis's idea, often view the Born rule itself as a healthy serving of principal principle. Only, it has the quantum state ρ filling the role of chance. That is, for any agent contemplating performing a measurement $\{P_j\}$, his subjective, personal probabilities for the outcomes j should condition on knowledge of the quantum state just as one conditions with the principal principle:

$$\mathrm{Prob}(j|\rho \wedge E) = \mathrm{Tr}\rho P_j \tag{67}$$

where E is any admissible proposition. Beliefs are beliefs, but quantum states are something else: They are the facts of nature that power a quantum version of the principal principle. In other words, in this context one has conceptually

$$\rho \to ``ch(j) = \mathrm{Tr}\rho P_j" \tag{68}$$

But the quantum-Bayesian view cannot sanction this. The essential point for a quantum Bayesian is that there is no such thing as the true quantum state. There are potentially as many states for a given quantum system as there are agents. And that point is not diminished by accepting the addition to Dutch-book coherence described in this review. Indeed, it is just as with standard (nonquantum) probabilities, where their subjectivity is not diminished by normatively satisfying standard Dutch-book coherence.

The most telling reason for this arises directly from quantum statistical practice. The way one comes to a quantum-state assignment is ineliminably dependent on one's priors (Fuchs, 2002a; Caves, Fuchs, and Schack, 2007; Fuchs et al., 2009). Quantum states are not God given, but have to be fought for via measurement, updating, calibration, computation, and any number of related activities. The only place quantum states are "given" outright (that is to say, the model on which much of the notion of an objective quantum state arises from in the first place) is in a textbook homework problem. For instance, a textbook exercise might read, "Assume a hydrogen atom in its ground state. Calculate . . .." But outside the textbook it is not difficult to come up with examples where two agents looking at the same data, differing only in their prior beliefs, will asymptotically update to distinct (even orthogonal) pure quantum- state assignments for the same system (Fuchs et al., 2009).[16] Thus the basis for one's particular quantum-state assignment is always outside the formal apparatus of quantum mechanics.[17]

This is the key difference between the set of ideas being developed here and the position of the objectivists: added relations for probabilities, yes, but not one of those probabilities can be objective in the sense of being any less a function of the agent. A way to put it more prosaically is that these norma-

[16]Here is a simple if contrived example. Consider a two-qubit system for which two agents have distinct quantum-state assignments ρ_+ and ρ_-, defined by $\rho_\pm = \frac{1}{2}(|0\rangle\langle 0|^{\otimes 2} \pm |\pm\rangle\langle\pm|^{\otimes 2})$ where $|\pm\rangle = 2^{-1/2}(|0\rangle \pm |1\rangle)$. These state assignments are "compatible" (Brun, Finkelstein, and Mermin, 2002; Caves, Fuchs, and Schack, 2002b), yet suppose the first qubit is measured in the basis $\{|0\rangle, |1\rangle\}$ and outcome 1 is found. The two agents' postmeasurement states for the second qubit are $|+\rangle$ and $|-\rangle$, respectively. See Fuchs et al. (2009) for a more thorough discussion.

[17]Nor does it help to repeat over and over as one commonly hears coming from the philosophy-of-physics community, "quantum probabilities are specified by physical law." The simple reply is, "No, they are not." The phrase has no meaning once one has accepted that quantum states are born in probabilistic considerations, rather than being the parents of them, as laboratory practice clearly shows (Paris and Řeháček, 2004; Kaznady and James, 2009).

tive considerations may narrow the agent from the full probability simplex to the set of quantum states, but beyond that, the formal apparatus of quantum theory gives him no guidance on which quantum state he should choose. Instead, the role of a normative reading of the Born rule is as it is with the usual Dutch book. Here is the way L. J. Savage put it (Savage, 1954, p. 57).

> According to the personalistic view, the role of the mathematical theory of probability is to enable the person using it to detect inconsistencies in his own real or envisaged behavior. It is also understood that, having detected an inconsistency, he will remove it. An inconsistency is typically removable in many different ways, among which the theory gives no guidance for choosing.

If an agent does not satisfy Eq. (64) with his personal probability assignments, then he is not properly recognizing the change of conditions (or perhaps we could say "context"[18]) that a potential SIC measurement would bring about. The theory gives no guidance for which of his probabilities should be adjusted or how, but it does say that they must be adjusted or "undesirable consequences" will become unavoidable.

Expanding on this point, Bernardo and Smith (1994) put it this way.

> Bayesian Statistics offers a rationalist theory of personalistic beliefs in contexts of uncertainty, with the central aim of characterizing how an individual should act in order to avoid certain kinds of undesirable behavioral inconsistencies .... The goal, in effect, is to establish rules and procedures for individuals concerned with disciplined uncertainty accounting. The theory is not descriptive, in the sense of claiming to model actual behavior. Rather, it is prescriptive, in the sense of saying "if you wish to avoid the possibility of these undesirable consequences you must act in the following way."

So much, indeed, we imagine for the *full* formal structure of quantum mechanics (including dynamics, tensor-product structure, etc.) - that it is all or nearly all an addition to Dutch-book coherence. And specifying those

[18]We add this alternative formulation so as to place the discussion within the context of various other analyses of the idea of "contextuality" (Mermin, 1993; Grangier, 2002, 2005; Appleby, 2005d; Spekkens, 2005, 2008; Ferrie and Emerson, 2008, 2009).

undesirable consequences is a significant part of the project of specifying the ontology underlying the quantum-Bayesian position. But that is a goal we leave for future work. Let us now explore the consequences of adopting Eq. (64) as a basic statement, acting as if we do not yet know the underlying Hilbert-space structure that gave rise to it.

5 Deriving Quantum-State Space from Empirically Extended Coherence

We now examine how far we can go toward deriving the structure of quantum-state space from the conceptual apparatus portrayed in Fig. 2. In this development we do not make use of amplitudes, Hilbert space, or any other part of the usual apparatus of quantum mechanics. Remember that we are representing quantum states by probability vectors $\|p\rangle\rangle$ lying in the probability simplex Δ_{d^2}. The set of pure states is given by the solutions of either Eq. (40) or Eqs. (37) and (38), and can thus be thought of as an algebraic variety within Δ_{d^2}. The set of all quantum states, pure or mixed, is the convex hull of the set of pure states, i.e., the set of all convex combinations of vectors $\|p\rangle\rangle$ representing pure states. We want to explore how much of this structure can be recovered from the considerations summarized in Fig. 2. We have to add at least three other assumptions on the nature of quantum measurement, but at first, we try to forget as much about quantum mechanics as we can.

Namely, start with Fig. 2 but forget about quantum mechanics and forget about SICs. Simply visualize an imaginary experiment in the sky S, supplemented with various real experiments we might perform on the ground G. We postulate that the probabilities we ascribe for the outcomes of G are determined by the probabilities we ascribe to the imaginary outcomes in the sky, and the conditional probabilities for the outcomes of G consequent upon them, were the measurement in the sky factualized. In particular, we take the quantum law of total probability, Eq. (64), as a postulate.

Assumption 0: The urgleichung (see Fig. 2). Degrees of belief for outcomes in the sky and degrees of belief for outcomes on the ground are related by the following normative rule:

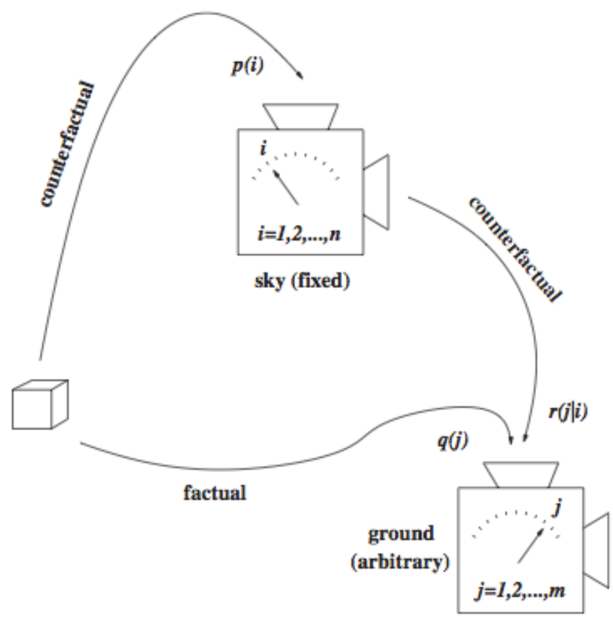

Figure 2: The basic conceptual apparatus of this review. The measurement on the ground, with outcomes $j = 1, ..., m$, is some potential measurement that could be performed in the laboratory, i.e., one that could be factualized. The measurement in the sky, on the other hand, with outcomes $i = 1, ..., n$, is a fixed measurement one can contemplate independently. The probability distributions $p(i)$ and $r(j|i)$ represent how an agent would gamble if a conditional lottery based on the measurement in the sky were operative. The probability distribution $q(j)$ represents instead how the agent would gamble on outcomes of the ground measurement if the measurement in the sky and the associated conditional lottery were nullified, i.e., they were to never take place at all. In the quantum case, the measurement in the sky is a SIC with $n = d^2$ outcomes; the measurement on the ground is any POVM. In pure Bayesian reasoning, there is no necessity that $q(j)$ be related to $p(i)$ and $r(j|i)$ at all. In quantum mechanics, however, there is the specific relation $q(j) = (d+1)\sum_{i=1}^{d^2} p(i)r(j|i) - (1/d)\sum_{i=1}^{d^2} r(j|i)$. This equation contains the sum content of the Born rule to which it is equivalent.

$$q(j) = (d+1)\sum_{i=1}^{d^2} p(i)r(j|i) - \frac{1}{d}\sum_{i=1}^{d^2} r(j|i) \tag{69}$$

We call this postulate the "urgleichung" (German for primal equation) to emphasize its primary status to all our thinking. As before, $p(i)$ represents the probabilities in the sky and $q(j)$ represents the probabilities on the ground. The index i is assumed to range from 1 to d^2, for some fixed natural number d. The range of j will not be fixed, but in any case considered will be denoted as running from 1 to m. (For example, for some cases m might be d^2, for some cases it might be d, but it need be neither and may be something else entirely - it will depend upon which experiment we are talking about for G.) We write $r(j|i)$ to represent the conditional probability for obtaining j on the ground, given that the experiment in the sky was actually performed and resulted in outcome i. When we want to suppress components, we write vectors $\|p\rangle\!\rangle$ and $\|q\rangle\!\rangle$, and write R for the matrix with entries $r(j|i)$. By definition, R is a stochastic matrix, i.e., $\sum_j r(j|i) = 1$, but not necessarily a doubly stochastic matrix, i.e., $\sum_i r(j|i) = 1$ does not necessarily hold (Horn and Johnson, 1985).

One of the main features we require, of course, is that ?calculated by Eq. (69), $\|q\rangle\!\rangle$ must satisfy $0 \le q(j) \le 1$ for all j. We call

$$0 \le (d+1)\sum_{i=1}^{d^2} p(i)r(j|i) - \frac{1}{d}\sum_{i=1}^{d^2} r(j|i) \le 1 \tag{70}$$

the *fundamental inequality.*

We define two sets $\mathcal{P}$ and $\mathcal{R}$, the first consisting of priors for the sky $\|p\rangle\!\rangle$ and the second consisting of stochastic matrices R.[19] We sometimes call $\mathcal{P}$ our state space, and its elements states. What we want to pin down are the properties of $\mathcal{P}$ and $\mathcal{R}$ under the assumption that they are consistent and maximal, by which we mean the following. We say that

(1) $\mathcal{P}$ and $\mathcal{R}$ are consistent if all pairs $(\|p\rangle\!\rangle, R) \in \mathcal{P}\times\mathcal{R}$ obey the fundamental inequality.

(2) $\mathcal{P}$ and $\mathcal{R}$ are maximal whenever $\mathcal{P}' \subseteq \mathcal{P}$ and $\mathcal{R}' \subseteq \mathcal{R}$ imply $\mathcal{P}' = \mathcal{P}$ and $\mathcal{R}' = \mathcal{R}$ for any consistent $\mathcal{P}'$ and $\mathcal{R}'$

With respect to consistent sets $\mathcal{P}$ and $\mathcal{R}$, for convenient terminology, we call a general $\|p\rangle\!\rangle \in \Delta_{d^2}$ valid if it is within the state space $\mathcal{P}$; if it is not within $\mathcal{P}$, we call it invalid.

The key idea behind the demand for maximality is that we want the urgleichung Eq. (69) to be as least exclusionary as possible in limiting an agent's

[19]The matrices R could also be regarded as part of the agent's prior, but since we keep R fixed once the measurement on the ground is fixed, we reserve the term "prior" for members of the set $\mathcal{P}$.

probability assignments. There is, of course, no guarantee without further assumptions there will be a unique maximal $\mathcal{P}$ and $\mathcal{R}$ consistent with the fundamental inequality, or even whether there will be a unique set of them up to rotations or other kinds of transformations, but we can say some things.

One important result follows immediately: If $\mathcal{P}$ and $\mathcal{R}$ are consistent and maximal, both sets must be convex. For instance, if $\|p\rangle\!\rangle$ and $\|p'\rangle\!\rangle$ satisfy Eq. (70) for all $R \in \mathcal{R}$ it is clear that, for any $x \in [0,1]$, $\|p''\rangle\!\rangle = x\|p\rangle\!\rangle + (1-x)\|p'\rangle\!\rangle$ will as well. Thus, if $\|p''\rangle\!\rangle$ were not in $\mathcal{P}$, the set would not have been maximal to begin with.[20] Furthermore, maximality and the boundedness of Eq. (70) ensure that $\mathcal{P}$ and $\mathcal{R}$ are closed sets, thus convex sets with extreme points (Appleby, Ericsson, and Fuchs, 2011).

Now is there any obvious connection between $\mathcal{P}$ and $\mathcal{R}$? We define the state of maximal ignorance

$$\|c\rangle\!\rangle = \left(\frac{1}{d^2}, \frac{1}{d^2}, \ldots\ldots, \frac{1}{d^2}\right)^T \tag{71}$$

and make the assumption that one can be completely ignorant of the outcomes in the sky.

Assumption 1: The state of maximal ignorance $\|c\rangle\!\rangle$ is included in $\mathcal{P}$.

Suppose now that the experiment in the sky really is performed as well as the experiment on the ground. Before either experiment, the agent is ignorant of both the outcome i in the sky and the outcome j on the ground. Using Bayes's rule, he can find the conditional probability for i given j, which has

[20] It is important to recognize that the considerations leading to the convexity of the state space here are distinct from the arguments one finds in the "convex sets" and "operational" approaches to quantum theory. See, for instance, Holevo (1982) and Busch, Grabowski, and Lahti (1995) and more recently the Barnum, Barrett, Leifer, and Wilce (BBLW) school starting in Barnum et al. (2006), as well as the work of Hardy (2001). There the emphasis is on the idea that a state of ignorance about a finer preparation is a preparation itself. The present argument even differs from some of our own earlier Bayesian considerations (where care was taken not to view "preparation" as an objective matter of fact, independent of prior beliefs, as talk of preparation would seem to imply) (Fuchs, 2002a; Schack, 2003). Here, instead, the emphasis is on the closure of the fundamental inequality, i.e., maximal $\mathcal{P}$ and $\mathcal{R}$

the form of a posterior probability,

$$\text{Prob}(i|j) = \frac{r((j|i)}{\sum_k r(j|k)} \tag{72}$$

We now make the next assumption.

Assumption 2: Principle of reciprocity. Posteriors from maximal ignorance are priors. For any $R \in \mathcal{R}$, a posterior probability $\text{Prob}(i|j)$ as in Eq. (72) is a valid prior $p(i)$ for the outcomes of the measurement in the sky. Moreover, for each valid $p(i)$, there exists some $R \in \mathcal{R}$ and some j such that $p(i) = \text{Prob}(i|j)$ as in Eq. (72).

Quantum mechanics certainly has this property. Suppose a completely mixed state for our quantum system and a POVM $\mathcal{G} = \{G_j\}$ measured on the ground. Upon noting an outcome j on the ground, the agent uses Eqs. (63) and (72) to infer

$$\text{Prob}(i|j) = \frac{\text{Tr}\Pi_i G_j}{d\text{Tr}G_j} \tag{73}$$

Defining

$$\rho_j = \frac{G_j}{\text{Tr}G_j} \tag{74}$$

this says that

$$\text{Prob}(i|j) = \frac{1}{d}\text{Tr}\rho_j \Pi_i \tag{75}$$

In other words, $\text{Prob}(i|j)$ is itself a SIC representation of a quantum state. Moreover, ρ_j can be any quantum state whatsoever, simply by adjusting which POVM $\mathcal{G}$ is under consideration.

5.1 Basis distributions

Since we are free to contemplate any measurement on the ground, we consider the case where the ground measurement is set to be the same as the reference measurement in the sky. We denote $r(j|i)$ by $r_S(j|i)$ in this special case. Remembering that the probabilities on the ground $q(j)$ refer to the case in which the measurement in the sky remains counterfactual, we must then have that $p(j) = q(j)$ for any valid $\|p\rangle\!\rangle$, or, using the urgleichung Eq. (69),

$$p(j) = (d+1)\sum_i p(i) r_S(j|i) - \frac{1}{d}\sum_i r_S(j|i) \tag{76}$$

where we used that the $r_S(j|i)$ depend only on the measurements and not on the prior $\|p\rangle\!\rangle$. Take the case where $p(i) = 1/d^2$ specifically. Substituting for $p(i)$ into Eq. (76), we find that $r_S(j|i)$ must satisfy

$$\sum_i r_S(j|i) = 1 \tag{77}$$

Therefore, when going back to more general priors $\|p\rangle\!\rangle$, one has in fact the simpler relation

$$p(j) = (d+1)\sum_i p(i) r_S(j|i) - \frac{1}{d} \tag{78}$$

Introducing an appropriately sized matrix M of the form

$$M = \begin{pmatrix} (d+1)-\frac{1}{d} & -\frac{1}{d} & \cdots & -\frac{1}{d} \\ -\frac{1}{d} & (d+1)-\frac{1}{d} & \cdots & -\frac{1}{d} \\ \vdots & & \ddots & \vdots \\ -\frac{1}{d} & -\frac{1}{d} & \cdots & (d+1)-\frac{1}{d} \end{pmatrix} \tag{79}$$

we can rewrite Eq. (78) in vector form,

$$MR_S\|p\rangle\!\rangle = \|p\rangle\!\rangle \tag{80}$$

where R_S is a matrix with matrix elements $r_S(j|i)$.

Now, we make a minor assumption on our state space.

Assumption 3: The elements $\|p\rangle\!\rangle \in \mathcal{P}$ span the full simplex Δ_{d^2}.

This is a very natural assumption: If $\mathcal{P}$ did not span the simplex, one would be justified in simply using a smaller simplex for all considerations.

With assumption 3, the only way Eq. (80) can be satisfied is if

$$MR_S = I \tag{81}$$

Since M is a circulant matrix, its inverse is a circulant matrix as well, and one can easily work out that

$$r_S(j|i) = \frac{1}{d+1}\left(\delta_{ij} + \frac{1}{d}\right) \tag{82}$$

It follows by the principle of reciprocity (our assumption 2) then that among the distributions in $\mathcal{P}$, along with the uniform distribution, there are at least d^2 other ones, namely,

$$\|e_k\rangle\!\rangle = \left(\frac{1}{d(d+1)}, \ldots, \frac{1}{d}, \ldots, \frac{1}{d(d+1)}\right)^T \tag{83}$$

with a $1/d$ in the kth slot and $1/d(d+1)$ in all other slots. We call these d^2 special distributions, appropriately enough, the *basis distributions*.

Notice that, in the special case of quantum mechanics, the basis distributions are just the SIC states themselves, now justified in a more general setting. Also, like the SIC states, we have

$$\sum_i e_k(i)^2 = \frac{2}{d(d+1)} \quad \forall\, k \tag{84}$$

in accordance with Eq. (37).

In Sec. 5.2 we consider arbitrary extreme points of the set $\mathcal{P}$ of valid states.

5.2 A Bloch sphere

Consider a class of measurements for the ground that have a property we call *in-step unpredictability* (ISU). The property is as follows: Whenever one assigns a uniform distribution for the measurement in the sky, one also assigns a uniform distribution for the measurement on the ground. This is meant to express the idea that the measurement on the ground has no in-built bias with respect to one's expectations of the sky. (In the full-blown quantum mechanical setting, this corresponds to a POVM $\{G_j\}$ such that $\mathrm{Tr}G_j$ is a constant value - von Neumann measurements with d outcomes being one special case of this.)

Denote the $r(j|i)$ and corresponding matrix R in this special case by $r_{ISU}(j|i)$ and R_{ISU}, respectively, and suppose the measurement being spoken of has m outcomes. Our requirement is that

$$\frac{1}{m} = \frac{(d+1)}{d^2} \sum_i r_{ISU}(j|i) - \frac{1}{d} \sum_i r_{ISU}(j|i) \tag{85}$$

To meet this, we must have

$$\sum_i r_{ISU}(j|i) = \frac{d^2}{m} \tag{86}$$

and the urgleichung Eq. (69) becomes

$$q(j) = (d+1)\sum_i p(i) r_{ISU}(j|i) - \frac{d}{m} \tag{87}$$

Suppose now that a prior $\|s\rangle\!\rangle$ for the sky happens to arise in accordance with Eq. (72) for one of these ISU measurements. That is,

$$s(i) = \frac{r_{ISU}(j|i)}{\sum_k r_{ISU}(j|k)} \tag{88}$$

for some R_{ISU} and some j. hen Eq. (87) tells us that for any $\|p\rangle\!\rangle \in \mathcal{P}$ we must have

$$0 \le \frac{d^2}{m}(d+1)\sum_i p(i)s(i) - \frac{d}{m} \le 1 \tag{89}$$

In other words, for any $\|s\rangle\!\rangle$ of our specified variety and any $\|p\rangle\!\rangle \in \mathcal{P}$, the following constraint must be satisfied:

$$\frac{1}{d(d+1)} \le \sum_i p(i)s(i) \le \frac{d+m}{d^2(d+1)} \tag{90}$$

Think particularly of the case where $\|s\rangle\!\rangle = \|p\rangle\!\rangle$. Then we must have

$$\sum_i p(i)^2 \le \frac{d+m}{d^2(d+1)} \tag{91}$$

Now suppose there are ISU measurements (distinct from simply bringing the sky measurement down to the ground) that have the basis distributions $\|e_k\rangle\!\rangle$ as their posteriors as defined in assumption 2, the principle of reciprocity. If this is so, then according to Eq. (84) the bound in Eq. (91) will be violated unless $m \ge d$. Moreover, it will not be tight for the basis states unless $m = d$ precisely.

Thinking of a basis distribution as the prototype of an extreme-point state (after all, they give the most predictability possible for the measurement in the sky), this motivates the next assumption.

Assumption 4: Every extreme point $\|p\rangle\!\rangle \in \mathcal{P}$ arises as in Eq. (88) as the posterior of an ISU measurement with $m = d$ and achieves equality in Eq. (91).

Thus, for any two extreme points $\|p\rangle\!\rangle$ and $\|s\rangle\!\rangle$, we assume

$$\frac{1}{d(d+1)} \leq \sum_i p(i)s(i) \leq \frac{2}{d(d+1)} \tag{92}$$

with equality on the right-hand side when $\|s\rangle\!\rangle = \|p\rangle\!\rangle$.[21]

Thus, the extreme points of $\mathcal{P}$ live on a sphere

$$\sum_i p(i)^2 \leq \frac{2}{d(d+1)} \tag{93}$$

Further trivial aspects of quantum-state space follow immediately from the requirement of Eq. (92) for any two extreme points. For instance, since the basis distributions are among the set of valid states, for any other valid state $\|p\rangle\!\rangle$ no component in it can be too large. This follows because

$$\langle\!\langle p\|e_k\rangle\!\rangle = \frac{1}{d(d+1)} + \frac{1}{d+1}p(k) \tag{94}$$

The right-hand side of Eq. (92) then requires

$$p(k) \leq \frac{1}{d} \tag{95}$$

But, do we have enough to get us all the way to Eq. (41) in addition to Eq. (37)? We analyze aspects of this in Sec. 5.3. First, however, we focus on the significance of the sphere.

[21] It should be noted that this inequality establishes the fact that if $\mathcal{P}$ at least contains the actual quantum-state space, it can contain no more than that. That is, the full set of quantum states is, in fact, a maximal set. Suppose a SIC exists, yet $\|s\rangle\!\rangle$ corresponds to some non-positive-semidefinite operator via the mapping in Eq. (30). Then there will be some $\|p\rangle\!\rangle \in \mathcal{P}$ corresponding to a pure quantum state such that the left-hand side of Eq. (92) is violated. This follows immediately from the definition of positive semidefiniteness and the expression for Hilbert-Schmidt inner products in Eq. (49).

What we have postulated in a natural way is that the extreme points of $\mathcal{P}$ must live on a d^2-1 sphere centered at the zero vector. But then it comes for free that these extreme points must also live on a smaller-radius d^2-2 sphere centered at the state of maximal ignorance $\|c\rangle\!\rangle$ defined in Eq. (71). This is because the $\|p\rangle\!\rangle$ live on the probability simplex Δ_{d^2}. Let $\|w\rangle\!\rangle = \|p\rangle\!\rangle - \|c\rangle\!\rangle$, where $\|p\rangle\!\rangle$ is any point satisfying Eq. (93). Then

$$r^2 = \langle\!\langle w\|w\rangle\!\rangle = \frac{d-1}{d^2(d+1)} \tag{96}$$

gives the radius of the lower-dimensional sphere.

The sphere in Eq. (96) is actually the more natural sphere for us to consider as most of the sphere in Eq. (93) (all but a set of measure zero) is thrown away anyway. In fact, it may legitimately be considered the higher-dimensional analog of the Bloch sphere from the quantum-Bayesian point of view. Indeed, when $d=2$, we have a 2-sphere, and it is isomorphic to the usual Bloch sphere.

It is natural to think of the following statement:

$$\sum_i p(i)^2 \leq \frac{2}{d(d+1)} \quad \text{for all } \|p\rangle\!\rangle \in \mathcal{P} \tag{97}$$

in information theoretic terms. This is because two well- known measures of the uncertainty associated with a probability assignment [the Renyi and Daróczy entropies (Aczél and Daróczy, 1975) of order 2] are simple functions of the left-hand side of it. To put it in a short phrase (Caves et al., 1996; Fuchs, 2010a), "In quantum mechanics, maximal in- formation is not complete and cannot be completed." The sharpest predictability one can have for the outcome of a SIC measurement is specified by Eq. (37). This is an old idea, of course, but quantified here in yet another way. It is related to the basic idea underlying the toy model of Spekkens (2007), with its "knowledge balance principle." In that model, which combines local hidden variables with an "epistemic constraint" on an agent's knowledge of the variables' values, more than 20 well-known quantum information theoretic phenomena [like no cloning (Dieks, 1982; Wootters and Zurek, 1982), no broadcasting (Barnum et al., 1996), teleportation (Bennett et al., 1993), correlation monogamy (Coffman, Kundu, and Wootters, 2000), and "nonlocality without entanglement" (Bennett et al., 1999), etc.] are readily reproduced, at least in a qualitative way.

Despite the toy model's successes, however, we suspect that an information constraint alone cannot support the more sweeping part of the quantum-Bayesian program, that "the possible outcomes cannot correspond to actualities, existing objectively prior to asking the question," i.e., that unperformed measurements have no outcomes. There are crucial differences between the present considerations having to do with an addition to Dutch-book coherence and "epistemic restriction" approaches. First, it is hard to see how that line of thought can get beyond the possibility of an underlying hidden-variable model (as the toy model illustrates). Second, and more importantly, in the present approach the Bloch sphere may well express an epistemic constraint - a constraint on an agent?s advised certainty. But the epistemic constraint is itself a result of a deeper consideration to do with the coherence between factual and counterfactual gambles, not a starting point. Furthermore, the constraint is not expressible in terms of a single information function; instead it involves pairs of distributions. Below we explain this point further.

5.3 An underlying dimensionality?

The state space implied by Eq. (92) does not lead to the full sphere in Eq. (96). According to the left-hand side of Eq. (92), when two points are too far away from each other, at least one of them cannot be in $\mathcal{P}$. Before we explore the implications of this more carefully, we note that Eq. (96) by itself tells us that we cannot have the full sphere. An argument due to Plunk (2002) established the fact that the radius of the sphere is such that the sphere extends beyond the boundary of the probability simplex Δ_{d^2}. Hence, $\mathcal{P}$ is contained within a nontrivial intersection of sphere and simplex.

Returning to the left-hand side of Eq. (92), it signifies that the "most orthogonal" two valid distributions $\|p\rangle\!\rangle$ and $\|q\rangle\!\rangle$ can ever be is

$$\langle\!\langle p\|q\rangle\!\rangle = \sum_i p(i)q(i) = \frac{1}{d(d+1)} \tag{98}$$

Their overlap can never approach zero; they can never be truly orthogonal. Now suppose we have a collection of distributions $\|p_k\rangle\!\rangle, k = 1, ..., n$, all of which live on the sphere - that is, they individually satisfy Eq. (93). We can ask, how large can the number n be while maintaining that each of the $\|p_k\rangle\!\rangle$ be maximally orthogonal to each other? Another way to put it is what is the maximum number of "mutually maximally distant" states?

In other words, we require

$$\langle\!\langle p_k \| p_l \rangle\!\rangle = \sum_i p(i)q(i) = \frac{\delta_{kl} + 1}{d(d+1)} \tag{99}$$

for as many values as possible. It turns out that there is a nontrivial constraint on how large n can be, and it is nothing other than $n = d$, the same thing one sees in quantum mechanics.

To see this, we reference again the center of the probability simplex with all our vectors. Define

$$\|w_k\rangle\!\rangle = \|p_k\rangle\!\rangle - \|c\rangle\!\rangle \tag{100}$$

In these terms, our constraint becomes

$$\langle\!\langle w_k \| w_l \rangle\!\rangle = \sum_i p(i)q(i) = \frac{d\delta_{kl} - 1}{d^2(d+1)} \tag{101}$$

We are thus asking for a set of vectors whose Gram matrix $G = [\langle\!\langle w_k \| w_l \rangle\!\rangle]$ is an $n \times n$ matrix of the form

$$G = \begin{pmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ \vdots & & & \ddots & \vdots \\ b & b & b & \cdots & a \end{pmatrix} \tag{102}$$

with

$$a = \frac{d-1}{d^2(d+1)} \quad \text{and} \quad b = \frac{-1}{d^2(d+1)} \tag{103}$$

Using an elementary theorem in linear algebra, a matrix G is the Gram matrix of a set of vectors if and only if G is positive semidefinite (Horn and Johnson, 1985). Moreover, the rank of G represents the number of linearly independent such vectors.

Since G in Eq. (102) is a circulant matrix, its eigenvalues can be readily calculated:

$$\lambda_0 = a + (n-1)b = \frac{d-n}{d^2(d+1)} \tag{104}$$

while the $n-1$ other eigenvalues are

$$\lambda_k = a - b = \frac{1}{d(d+1)} \tag{105}$$

To make G positive semidefinite then, we must have $n \le d$, with $n = d$ being the maximal value. At that point G is a rank-$(d-1)$ matrix, so that only $d-1$ of the $\|w_l\rangle\!\rangle$ are linearly independent.

On the other hand, all d vectors $\|p_k\rangle\!\rangle = \|w_k\rangle\!\rangle + \|c\rangle\!\rangle$ are actually linearly independent. To see this, suppose there are numbers α_i, such that $\sum_i \alpha_i \|p_i\rangle\!\rangle = 0$. Acting from the left on this equation with $\langle\!\langle c\|$, one obtains

$$\sum_i \alpha_i = 0 \tag{106}$$

and acting on it with $\langle\!\langle p_k\|$, we obtain

$$\begin{aligned} 0 &= \frac{2}{d(d+1)}\alpha_k + \frac{1}{d(d+1)} \sum_{i \neq k} \alpha_i \\ &= \frac{1}{d(d+1)}\alpha_k + \frac{1}{d(d+1)} \sum_i \alpha_i = \frac{1}{d(d+1)}\alpha_k \end{aligned} \tag{107}$$

So $\alpha_k = 0$ for all k as required.

This result is suggestive of a "dimension" for the valid states on the surface of the sphere: it admits no more than d maximally equidistant points. At this stage, however, "dimension" must remain in quotes. Ultimately one must see that the Hausdorff dimension of the manifold of valid extreme states is $2d-2$ (i.e., what it is in quantum theory), and the present result does not get that far.

5.4 Further results and ongoing work

In summary, from the urgleichung Eq. (69) and four further assumptions, we derived that the basis distributions $\|e_k\rangle\!\rangle$ should be among the valid states $\mathcal{P}$, that for any $\|p\rangle\!\rangle \in \mathcal{P}$ the probabilities are bounded above by $p(k) \le 1/d$, and that the extreme points of the valid $\|p\rangle\!\rangle$ should live on the surface of a sphere that at times pokes outside the probability simplex. We derived that for any two valid distributions $\|p\rangle\rangle$ and $\|s\rangle\rangle$ (including the case where $\|p\rangle\rangle = \|s\rangle\rangle$), it must hold that

$$\frac{1}{d(d+1)} \leq \sum_i p(i)s(i) \leq \frac{2}{d(d+1)} \tag{108}$$

From the latter, it follows that no more than d extreme points $\|p\rangle\rangle$ can ever be mutually maximally distant from each other.

What really needs to be derived is that the extreme points of such a convex set correspond to an algebraic variety of the form given in Eq. (40), with a set of c_{ijk} that can be written in the form of Eq. (39). The key idea is to supplement assumption 0 with as little extra structure as possible for getting all the way to full-blown quantum mechanics. Many questions remain, at both the technical and conceptual levels.

One such question is that of the origin of the urgleichung Eq. (69). A small step toward a deeper understanding of the particular form Eq. (69) takes was made by Fuchs and Schack (2011), where the starting point is again the setup in Fig. 2, but without the assumption that the number of outcomes for the measurement in the sky is a perfect square d^2. The fundamental postulate in Fuchs and Schack (2011) is the "generalized urgleichung"

$$q(j) = \sum_{i=1}^{n} [\alpha p(i) - \beta] r(j|i) \tag{109}$$

where α and β are fixed non-negative real numbers. Then it was shown that the numerical relations between the constants α, β, and n, and, in particular, the fact that n is a perfect square, follow from a set of simple assumptions given purely in terms of the personalist probabilities a Bayesian agent may assign to the outcomes of certain experiments. In this section we have only scratched the surface of the problem of characterizing maximal consistent sets, i.e., maximal subsets of the probability simplex consistent with the urgleichung as defined at the beginning of the section. Maximal consistent sets are an active area of ongoing re- search. For example, Appleby, Huangjun Zhu, and one of us have shown recently that quantum-state space is a maximally consistent set if and only if a SIC exists, and that hidden within every maximally consistent set is a subgroup of the orthogonal group (Appleby, Fuchs, and Zhu, 2013). Much work, however, remains to be done.

6 Summary: From Quantum Interference to Quantum-Bayesian Coherence

In this review, we have given a new way to think of quantum interference: Particularly, we have shown how to view it as an empirical addition to Dutch-book coherence, operative when one calculates probabilities for the outcomes of a factualizable quantum experiment in terms of one explicitly assumed counterfactual. We did this and not once did we use the idea of a probability *amplitude*. Thus we have brought the idea of quantum interference formally much closer to its roots in probabilistic considerations. For this, we made use of the mathematical machinery of SIC measurements.

In doing so we showed that the Born rule can be viewed as a relation between probabilities, rather than a *setter* of probabilities from a quantum state regarded as more firm or secure than probability itself, i.e., rather than facilitating a probability assignment from the true quantum state. From the quantum-Bayesian point of view there is no such thing as the true quantum state, there being as many quantum states for a system as there are agents interested in considering it. This last point makes it particularly clear why we needed a way of viewing the Born rule as an extension of Dutch-book coherence: One can easily invent situations where two agents will update to divergent quantum states (even pure states, and even orthogonal pure states, see footnote 17) by looking at the same empirical data (Fuchs, 2002a; Fuchs and Schack, 2004; Fuchs et al., 2009) - a quantum state is always ultimately dependent on the agent's priors.

But, as we have seen, there is much more to do. We gave an indication that the urgleichung Eq. (69) and considerations to do with it already specify a significant fraction of the structure of quantum states - and for that reason one might want to take it as one of the fundamental axioms of quantum mechanics. We did not, however, get all the way back to a set based on the manifold of pure quantum states, Eq. (40). A further open question concerns the origin of the urgleichung. An intriguing idea would be to justify it Dutch-book style in terms of bought and returned lottery tickets consequent upon the nullification step in our standard scenario. Then the positive content of the Born rule might be viewed as a kind of cost excised whenever one factualizes a SIC. But this is just speculation.

What is firm is that we have a new setting for quantifying the old idea that, in quantum mechanics, unperformed measurements have no outcomes.

7 Outlook: The Paulian Idea and the Jamesian Pluriverse

This review has focused on adding a new girder to the developing structure of quantum Bayesianism ("QBism" hereafter). As such, we have taken much of the previously developed program as a background for the present efforts. For instance, the core arguments for why we choose a more "personalist Bayesianism" rather than a so-called "objective Bayesianism" can be found in Fuchs (2002a, 2013) and Fuchs and Schack (2004), and the argument for why a subjective, personalist account of certainty is crucial for breaking the impasse set by the EPR criterion of reality is explained in Caves, Fuchs, and Schack (2007) and Fuchs (2013).

William James wrote

> Of every would be describer of the universe one has a right to ask immediately two general questions. The first is: "What are the materials of your universe's composition?" And the second: "In what manner or manners do you represent them to be connected?" - William James, notebook entry, 1903 or 1904.

Fearing James?s injunction, in this section we discuss anew the term "measurement," which we have been using uncritically in this review. Providing a deeper understanding of the proclamation "unperformed measurements have no outcomes" is the first step toward characterizing "the materials of our Universe's composition."

We take our cue from Bell (1990). Despite our liberal use of the term so far, we think the word measurement should indeed be banished from fundamental discussions of quantum theory.[22] However, it is not because the word is "unprofessionally vague and ambiguous," as Bell stated (Bell, 1987). Rather, it is because, from the QBist perspective, the word suggests a misleading notion of the very subject matter of quantum mechanics.

[22] For a related argument, see N. D. Mermin's "In Praise of Measurement" (Mermin, 2006).

To make the point more clear, we put quantum theory aside for a moment and consider instead basic Bayesian probability theory. There the subject matter is an agent's expectations for various outcomes. For instance, an agent might write down a joint probability distribution $P(h_i, d_j)$ for various mutually exclusive hypotheses j_i and data values d_j appropriate to some phenomenon. As discussed above, a major role of the theory is that it provides a scheme (Dutch-book coherence) for how these probabilities should be related to other probabilities, $P(h_i)$ and $P(d_j)$ say, as well as to any other degrees of belief the agent has for other phenomena. The theory also prescribes that if the agent is given a specific data value d_j, he should update his expectations for everything else within his interest. For instance, under the right conditions (Diaconis and Zabell, 1982; Skyrms, 1987b; Fuchs and Schack, 2012a), he should reassess his probabilities for the h_i by conditionalizing:

$$P_{new}(h_i) = \frac{P(h_i, d_j)}{P(d_j)} \tag{110}$$

But what is this phrase "given a specific data value"? What does it really mean in detail? Should not one specify a mechanism or at least a chain of logical and/or physical connectives for how the raw fact signified by d_j comes into the field of the agent's consciousness? And who is this "agent" reassessing his probabilities anyway? Indeed, what is the precise definition of an agent? How would one know one when one sees one? Can a dog be an agent? Or must it be a person? Maybe it should be a person with a Ph.D.?[23]

Probability theory cannot answer these questions because they are not questions within the subject matter of the theory. Within probability theory, the notions of agent and given a data value are primitive and irreducible. The whole theory is constructed to guide agents' decisions based on data. Agents and data are at the bottom of the structure of probability theory - they are not to be constructed from it, but rather agents are there to receive the theory's guidance, and data are there to designate the world external to the agent.

QBism says that, if all of this is true of Bayesian probability theory in general, it is true of quantum theory as well. As the foundations of probability theory dismiss the questions of where data come from and what constitutes

[23]See Bell (1990).

an agent (these questions never even come to its attention) so can the foundations of quantum theory dis- miss them too.

A likely reaction at this point will be along these lines: "It is one thing to say all this of probability theory, but quantum theory is a wholly different story. Quantum mechanics is no simple branch of mathematics, like probability or statistics. Nor can it plausibly be a theory about the insignificant specks of life in our vast universe making gambles and decisions. Quantum mechanics is one of our best theories of the world. It is one of the best maps we have drawn yet of what is actually out there." But this is where QBism differs. Just as probability theory is not a theory of the world, so quantum theory is not a theory of the world: It is a theory for the use of agents immersed in and interacting with a world of a particular nature, the "quantum world."

This last statement is crucial. Regarding the idea of a world external to the agent, as Gardner (1983) says,

> The hypothesis that there is an external world, not dependent on human minds, made of *something*, is so obviously useful and so strongly confirmed by experience down through the ages that we can say without exaggerating that it is better confirmed than any other empirical hypothesis. So useful is the posit that it is almost impossible for anyone except a madman or a professional metaphysician to comprehend a reason for doubting it.

Yet there is no implication in these words that quantum theory, for all its success in chemistry, physical astronomy, laser making, etc., must be read as a theory of the world. There is room for a significantly more interesting form of dependence: Quantum theory is conditioned by the character of the world, but yet is not a theory directly of it. Confusion on this very point is what has caused most of the discomfort in quantum foundations in the 85 years since the theory's coming to a relatively stable form in 1927.

Returning to our discussion of Bell and the word measurement, we want the word banished because it subliminally whispers the philosophy of its origin: That quantum mechanics should be conceived in a way that makes no ultimate reference to agency, and that agents are constructed out of the theory, rather than taken as the primitive entities the theory is meant to aid. In a nutshell, the word deviously carries forward the impression that quantum mechanics should be viewed as a theory directly of the world, rather than as a theory guiding agents in their interactions with the world.

Quantum mechanics, making no direct reference to the world but only to our actions and gambles within it, has no explicit ontological content. Nevertheless, the structure of quantum mechanics has fundamental implications for the character of the world. The interpretation of the term measurement is particularly relevant for questions of ontology. In the following, we look more closely at what measurement means from a QBist perspective.

The Paulian idea and the Jamesian pluriverse. - The best way to begin a more thoroughly QBist delineation of quantum mechanics is to start with two quotes on personalist Bayesianism itself. The first is from Hampton, Moore, and Thomas (1973):

> Bruno de Finetti believes there is no need to assume that the probability of some event has a uniquely determinable value. His philosophical view of probability is that it expresses the feeling of an individual and cannot have meaning except in relation to him.

and the second from Lindley (1982):

> The Bayesian, subjectivist, or coherent, paradigm is egocentric. It is a tale of one person contemplating the world and not wishing to be stupid (technically, incoherent). He realizes that to do this his statements of uncertainty must be probabilistic.

These two quotes make it clear that personalist Bayesianism is a "single-user theory." Thus, QBism must inherit at least this much egocentrism in its view of quantum states ρ. The "Paulian idea" (see Fig. 3) (Fuchs, 2010a), which is also essential to the QBist view, goes further still. It says that the outcomes to quantum measurements are single user as well. That is to say, when an agent writes down her degrees of belief for the outcomes of a quantum measurement, what she is writing down are her degrees of belief about her potential *personal* experiences arising in consequence of her actions upon the external world (Fuchs, 2007, 2010b, 2012; Mermin, 2012).

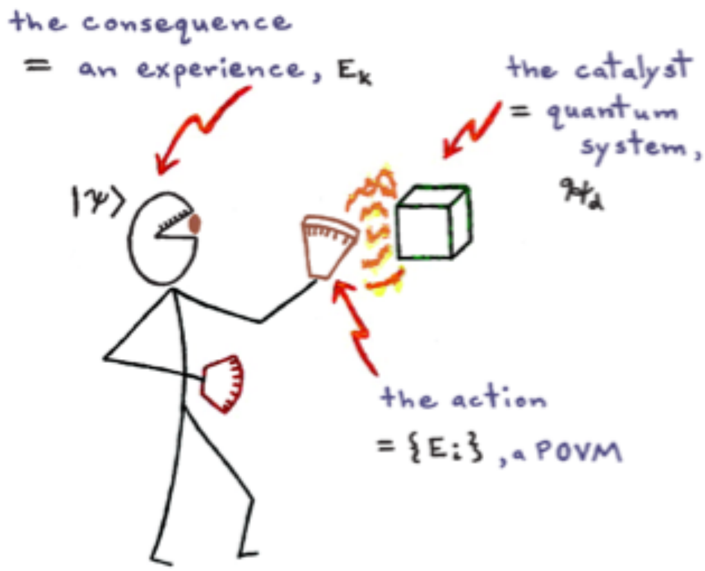

Figure 3: The Paulian idea (Fuchs, 2010a) - in the form of a figure inspired by John Archibald Wheeler (Patton and Wheeler, 1975). In contemplating a quantum measurement, one makes a conceptual split in the world: one part is treated as an agent, and the other as a kind of reagent or catalyst (one that brings about change in the agent). In older terms, the former is an observer and the latter a quantum system of some finite dimension d. [For arguments in support of finite fundamental theories see, e.g., Parikh and Verlinde (2005), Zinn-Justin (2007), and Braunstein, Pirandola, and Życzkowski (2013).] A quantum measurement consists first of the agent taking an *action* on the quantum system. The action is formally captured by some POVM $\{E_i\}$. The action leads generally to an incompletely predictable *consequence*, a particular personal experience E_i for the agent (Fuchs, 2007). The quantum state $|\psi\rangle$ makes no appearance but in the agent's head; for it only captures his degrees of belief concerning the consequences of his actions, and (in contrast to the quantum system itself) has no existence in the external world. Measurement devices are depicted as prosthetic hands to make it clear that they should be considered an integral part of the agent. (This contrasts with Bohr's view where the measurement device is always treated as a classically describable system external to the observer.) The sparks between the measurement-device hand and the quantum system represent the idea that the consequence of each quantum measurement is a unique creation within the previously existing universe (Fuchs, 2007). Wolfgang Pauli characterized this picture as a "wider form of the reality concept?" than that of Einstein?s, which he labeled "the ideal of the detached observer" (Laurikainen, 1988; Pauli, 1994; Gieser, 2005). The particular character of the catalysts (James's "materials of your universe's composition") leaves its trace in the formal rules that allow us to conceptualize factualizable measurements in terms of a standard counterfactual one.

Before exploring this further, we give a quick outline of some elements of quantum theory, retaining the usual mathematical formulation of the theory, but starting the process of changing the English descriptions of what the term "quantum measurement" means.

The primitive notions of the theory are (a) the agent, (b) things external to the agent, or, more commonly, "systems," (c) the agent's actions on the systems, and (d) the consequences of those actions for her experience. The formal structure of quantum mechanics is a theory of how the agent ought to organize her Bayesian probabilities for the consequences of all her potential actions on the things around her. Implicit in this is a theory of the structure of actions. The theory works as follows: When the agent posits a system, she posits a Hilbert space $\mathcal{H}_d$ as the arena for all her considerations. Actions upon the system are captured by positive-operator valued measures $\{E_i\}$ on $\mathcal{H}_d$. Potential consequences of the action are labeled by the individual elements E_i within the set,[24] i.e.,

$$\text{action} = \{E_i\} \text{ and consequence} = E_k$$

Quantum mechanics organizes the agent's beliefs by saying that she should strive to find a single density operator ρ such that her degrees of belief will always satisfy

$$\text{Prob(consequence|action)} = \text{Prob}(E_k|\{E_i\}) = \text{Tr}\rho E_k$$

no matter what action $\{E_i\}$ is under consideration.

Regarding our usage of the word measurement, this means that one should think of it simply as an action upon the system of interest. Actions lead to consequences within the experience of the agent, and that is what a quantum measurement is. A quantum measurement finds nothing, but very much makes something.

Thus, in a QBist painting of quantum mechanics, quantum measurements are "generative" in a real sense. Measurement outcomes come about for the agent herself. Quantum mechanics is a single-user theory through and through - first in the usual Bayesian sense with regard to personal beliefs, and

[24] There is a formal similarity between this and the development in Cox (1961), where "questions" are treated as sets, and "answers" are treated as elements within the sets.

second in that quantum measurement outcomes are personal experiences.

Of course, as a single-user theory, quantum mechanics is available to any agent to guide and better prepare her for her own encounters with the world. Furthermore, although quantum mechanics has nothing to say about another agent's wholly personal experiences, it is important to distinguish between an agent Bob's experience, which is internal to him and accessible only to himself, and his report of that experience, which he gives to Alice and so becomes accessible to her as part of her own external world. Alice can then use the report about Bob's experience to update her own probability assignments.

In the spirit of the Paulian idea, however, eliciting a report from another agent means taking an action on him. Whenever "I" encounter a quantum system, and take an action upon it, it catalyzes a consequence in my experience that my experience could not have foreseen. Similarly, by a Copernican-style principle, I should assume the same for "you": Whenever you encounter a quantum system, taking an action upon it, it catalyzes a consequence in your experience. To ourselves we are each agents. To another agent, we are all physical systems. When we take actions upon each other the distinctions between which of us are agents and which of us are physical systems are symmetrical. Like with the Rubin vase, the best the eye can do is flit back and forth between the two formulations.

Viewing quantum mechanics as a single-user theory does not mean there is only one user. QBism does not lead to solipsism. Any charge of solipsism is further refuted by two points central to the Paulian idea. (Fuchs, 2002b). One is the conceptual split of the world into two parts (one an agent and the other an external quantum system) that gets the discussion of quantum measurement off the ground. If such a split were not needed for making sense of the question of actions (actions upon what? in what? with respect to what?), it would not have been made. Imagining a quantum measurement without an autonomous quantum system participating in the process would be as paradoxical as the Zen koan of the sound of a single hand clapping. The second point is that once the agent chooses an action $\{E_i\}$, the particular consequence E_k of it is beyond her control. That is to say, the particular outcome of a quantum measurement is not a product of her desires, whims, or fancies - this is the very reason she uses the calculus of probabilities: they quantify her uncertainty (Lindley, 2006), an uncertainty that, try as she might, she cannot get around. So, implicit in this whole picture (this whole Paulian idea) is an "external world ... made of something." It is only that quantum theory is a rather small theory: Its boundaries are set by being a handbook for agents immersed within that "world made of something."

But a small theory can still have grand importance, and quantum mechanics most certainly does. This is because it tells us how a user of the theory sees his role in the world. Even if quantum mechanics (viewed as an addition to probability theory) is not a theory of the world itself, it is conditioned by the particular character of this world. Its empirical content is exemplified by the urgleichung Eq. (69), which takes one specific form rather than an infinity of other possibilities. Even though quantum theory is now understood as a theory of acts, decisions, and consequences (Savage, 1954), it tells us about the character of our particular world. Apparently, the world is made of a stuff that does not have preestablished "consequences" waiting around to respond to our "actions" - it is a world in which the consequences are generated on the fly. One starts to get a sense of a world picture that is part personal (truly personal) and part the joint product of all that interacts. This is a world of indeterminism, but one with no place for objective chance. From within any part, the future is undetermined. If one of those parts is an agent, then it is an agent in a situation of uncertainty. And where there is uncertainty, agents should use the calculus of Bayesian probability in order to make the best go at things.

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References

Aczél, J., and Z. Daróczy, 1975, On Measures of Information and Their Characterizations, Mathematics in Science and Engineering Vol. 115 (Academic Press, New York).

Appleby, D.M., 2005a, SIC-POVMs and the Extended Clifford Group, J. Math. Phys. (N.Y.) 46, 052107.

Appleby, D. M., 2005b, Facts, Values and Quanta, Found. Phys. 35, 627.

Appleby, D.M., 2005c, Probabilities Are Single-Case, or Nothing, Opt. Spectrosc. 99, 447.

Appleby, D. M., 2005d, The Bell-Kochen-Specker Theorem, Stud. Hist. Phil. Mod. Phys. 36, 1.

Appleby, D. M., I. Bengtsson, S. Brierley, M. Grassl, D. Gross, and J.-A. Larsson, 2012, The monomial representations of the Clifford group, Quantum Inf. Comput. 12, 404.

Appleby, D. M., H. B. Dang, and C. A. Fuchs, 2007, Symmetric Informationally-Complete Quantum States as Analogues to Orthonormal Bases and Minimum Uncertainty States, arXiv:0707.2071v2.

Appleby, D. M., Å. Ericsson, and C. A. Fuchs, 2011, Properties of QBist State Spaces, Found. Phys. 41, 564.

Appleby, D. M., S. T. Flammia, and C. A. Fuchs, 2011, The Lie Algebraic Significance of Symmetric Informationally Complete Measurements, J. Math. Phys. (N.Y.) 52, 022202.

Appleby, D. M., C. A. Fuchs, and Huangjun Zhu, 2013, unpublished.

Aspect, A., J. Dalibard, and G. Roger, 1982, Experimental Test of Bell's Inequalities Using Time-Varying Analyzers, Phys. Rev. Lett. 49, 1804.

Baez, J., 2003, Bayesian Probability Theory and Quantum Mechanics, http://math.ucr.edu/home/baez/bayes.html.

Ballentine, L. E., 1986, in New Techniques and Ideas in Quantum Measurement Theory, Annals of the New York Academy of Sciences Vol. 480, edited by D.M. Greenberger (New York Academy of Sciences, New York), pp. 382?392.

Barnum, H., J. Barrett, M. Leifer, and A. Wilce, 2006, Cloning and Broadcasting in Generic Probabilistic Models, arXiv:quant-ph/0611295v1.

Barnum, H., C.M. Caves, C.A. Fuchs, R. Jozsa, and B. Schumacher, 1996, Noncommuting Mixed States Cannot Be Broadcast, Phys. Rev. Lett. 76, 2818.

Barrett, J., and S. Pironio, 2005, Popescu-Rohrlich Correlations as a Unit of Nonlocality, Phys. Rev. Lett. 95, 140401.

Bell, J., 1990, Against 'Measurement', Phys. World, August 1990, 33.

Bell, J.S., 1964, On the Einstein Podolsky Rosen Paradox, Physics (Long Island City, N.Y.) 1, 195.

Bell, J. S., 1981, Bertlmann's Socks and the Nature of Reality, J. Phys. (Paris), Colloq. 42, C2-41.

Bell, J. S., 1987, Speakable and Unspeakable in Quantum Mechanics (Cambridge University Press, Cambridge, England).

Bengtsson, I., and K. Życzkowski, 2006, Geometry of Quantum States: An Introduction to Quantum Entanglement (Cambridge University Press, Cambridge, England).

Bennett, C.H., G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, 1993, Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels, Phys. Rev. Lett. 70, 1895.

Bennett, C. H., D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters, 1999, Quantum Nonlocality without Entanglement, Phys. Rev. A 59, 1070.

Bernardo, J. M., and A. F. M. Smith, 1994, Bayesian Theory (Wiley, Chichester).

Braunstein, S. L., S. Pirandola, and K. Życzkowski, 2013, Better Late Than Never: Information Retrieval from Black Holes, Phys. Rev. Lett. 110, 101301.

Brun, T. A., J. Finkelstein, and N. D. Mermin, 2002, How Much State Assignments Can Differ, Phys. Rev. A 65, 032315.

Bub, J., 2011, in Probabilities in Physics, edited by C. Beisbart, and S. Hartmann (Oxford University Press, Oxford).

Busch, P., M. Grabowski, and P.J. Lahti, 1995, Operational Quantum Mechanics (Springer, Berlin).

Busch, P., and P. Lahti, 2009, in Compendium of Quantum Physics, edited by F. Weinert, K. Hentschel, and D. Greenberger (Springer-Verlag, Berlin), pp. 356?358.

Caticha, A., 2007, in Foundations of Probability and Physics-4, edited by G. Adenier, C. A. Fuchs, and A. Yu. Khrennikov, AIP Conf. Proc. No. 889 (AIP, Melville, NY).

Caves, C. M., 1999, Symmetric Informationally Complete POVMs, `http://info.phys.unm.edu/~caves/reports/infopovm.pdf`.

Caves, C. M., 2002 (private communication).

Caves, C. M., and C. A. Fuchs, Quantum Information: How Much Information in a State Vector?, 1996, Ann. Isr. Phys. Soc. 12, 226.

Caves, C. M., C. A. Fuchs, and R. Schack, 2002a, Unknown Quantum States: The Quantum de Finetti Representation, J. Math. Phys. (N.Y.) 43, 4537.

Caves, C. M., C. A. Fuchs, and R. Schack, 2002b, Conditions for Compatibility of Quantum-State Assignments,? Phys. Rev. A 66, 062111.

Caves, C. M., C. A. Fuchs, and R. Schack, 2007, Subjective Probability and Quantum Certainty, Stud. Hist. Phil. Mod. Phys. 38, 255.

Coffman, V., J. Kundu, and W.K. Wootters, 2000, Distributed Entanglement, Phys. Rev. A 61, 052306.

Cox, R. T., 1961, The Algebra of Probable Inference (Johns Hopkins Press, Baltimore).

D'Ariano, G.M., 2009, Probabilistic Theories: What Is Special about Quantum Mechanics?, in Philosophy of Quantum Information and Entanglement, edited by A. Bokulich and G. Jaeger (Cambridge University Press, Cambridge, England).

de Finetti, B., 1931, Probabilismo, Logos 14, 163 [Probabilism, Erkenntnis 31, 169 (1989)].

de Finetti, B., 1990, Theory of Probability (Wiley, New York), two volumes.

Demopoulos, W., 2010, Effects and Propositions, Found. Phys. 40, 368.

Dennett, D. C., 2004, Freedom Evolves (Penguin Group, New York).

Diaconis, P., and S. L. Zabell, 1982, Updating Personal Probability, J. Am. Stat. Assoc. 77, 822.

Dieks, D., 1982, Communication by EPR devices, Phys. Lett. 92A, 271.

Einstein, A., 1951, in Albert Einstein: Philosopher-Scientist, edited by P. A. Schilpp (Tudor Co. Publishing, New York), pp. 81-87.

Ferrie, C., and J. Emerson, 2008, Frame Representations of Quantum Mechanics and the Necessity of Negativity in Quasi-probability Representations, J. Phys. A 41, 352001.

Ferrie, C., and J. Emerson, 2009, Framed Hilbert Space: Hanging the Quasi-probability Pictures of Quantum Theory, New J. Phys. 11, 063040.

Feynman, R.P., 1951, in Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, edited by J. Neyman (University of California Press, Berkeley), pp. 533-541.

Feynman, R. P., R. B. Leighton, and M. Sands, 1965, The Feynman Lectures on Physics, Vol. 3 (Addison-Wesley, Reading, MA).

Flammia, S. T., 2004, unpublished.

Fuchs, C. A., 2002a, Quantum Mechanics as Quantum Information (and only a little more), arXiv:quant-ph/0205039v1.

Fuchs, C.A., 2002b, in Quantum Theory: Reconsideration of Foundations, edited by A. Khrennikov (Văxjŏ University Press, Văxjŏ, Sweden), pp. 99-116.

Fuchs, C. A., 2003, Quantum Mechanics as Quantum Information, Mostly, J. Mod. Opt. 50, 987.

Fuchs, C.A., 2004, On the Quantumness of a Hilbert Space, Quantum Inf. Comput. 4, 467.

Fuchs, C. A., 2007, in Foundations of Probability and Physics-4, edited by G. Adenier, C. A. Fuchs, and A. Yu. Khrennikov, AIP Conf. Proc. Vol. 889 (American Institute of Physics, Melville, NY), pp. 438-462.

Fuchs, C.A., 2010a, Coming of Age with Quantum Information (Cambridge University Press, Cambridge, England).

Fuchs, C.A., 2010b, QBism, the Perimeter of Quantum Bayesianism, arXiv:1003.5209v1.

Fuchs, C. A., 2012, Interview with a Quantum Bayesian, arXiv:1207.2141v1.

Fuchs, C. A., 2013, My Struggles with the Block Universe, `http://www.perimeterinstitute.ca/personal/cfuchs/`.

Fuchs, C.A., and A. Peres, 2000, Quantum Theory Needs No 'Interpretation', Phys. Today 53, No. 3, 70.

Fuchs, C. A., and R. Schack, 2004, in Quantum Estimation Theory, edited by M.G.A. Paris and J.Řeháček (Springer-Verlag,Berlin), pp. 151-190.

Fuchs, C. A., and R. Schack, 2011, A Quantum-Bayesian Route to Quantum-State Space, Found. Phys. 41, 345.

Fuchs, C.A., and R. Schack, 2012a, in Probability in Physics, edited by Y. Ben-Menahem and M. Hemmo (Springer, Berlin), pp. 233-247.

Fuchs, C. A., and R. Schack, 2009, in Foundations of Probability and Physics-5, edited by L. Accardi et al., AIP Conf. Proc. Vol. 1101 (American Institute of Physics, Melville, NY), pp. 255-259.

Gardner, M., 1983, The Whys of a Philosophical Scrivener (W. Morrow, New York).

Gieser, S., 2005, The Innermost Kernel: Depth Psychology and Quantum Physics. Wolfgang Pauli's Dialogue with C. G. Jung (Springer, Berlin).

Goyal, P., 2008, An Information-Geometric Reconstruction of Quantum Theory, I: The Abstract Quantum Formalism, arXiv:0805.2761v1.

Grangier, P., 2002, Contextual objectivity: a realistic interpretation of quantum mechanics, Eur. J. Phys. 23, 331.

Grangier, P., 2005, Contextual objectivity and the quantum formalism, Int. J. Quantum. Inform. 03, 17.

Hampton, J. M., P. G. Moore, and H. Thomas, 1973, Probability and Its Measurement, J. R. Sta. Soc. A 136, 21.

Hardy, L., 2001, Quantum Theory From Five Reasonable Axioms, arXiv:quant-ph/0101012v4.

Holevo, A.S., 1982, Probabilistic and Statistical Aspects of Quantum Theory, North-Holland Series in Statistics and Probability Vol. 1 (North-Holland, Amsterdam).

Horn, R. A., and C. R. Johnson, 1985, Matrix Analysis (Cambridge University Press, Cambridge, England).

James, W., 1940, Some Problems of Philosophy: A Beginning of an Introduction to Philosophy (Longmans, Green, New York).

James, W., 1956a, The Will to Believe, and Other Essays in Popular Philosophy (Dover, New York), pp. 263-298.

James, W., 1956b, The Will to Believe, and Other Essays in Popular Philosophy (Dover, New York), pp. 145-183.

Jeffrey, R., 2004, Subjective Probability. The Real Thing (Cambridge University Press, Cambridge, England).

Jones, N. S., and N. Linden, 2005, Parts of Quantum States, Phys. Rev. A 71, 012324.

Kaznady, M. S., and D. F. V. James, 2009, Numerical Strategies for Quantum Tomography: Alternatives to Full Optimization, Phys. Rev. A 79, 022109.

Keynes, J.M., 1951, Essays in Biography (W. W. Norton, New York), pp. 243-244.

Koopman, B. O., 1957, in Proceedings of the Symposia in Applied Mathematics, Volume VII: Applied Probability, edited by L.A. MacColl (McGraw-Hill, New York), pp. 97-102.

Kyburg, Jr., H. E., and H. E. Smokler, 1980, Eds., Studies in Subjective Probability (Robert E. Krieger Publishing, Huntington, NY), 2nd ed.

Laurikainen, K.V., 1988, Beyond the Atom: The Philosophical Thought of Wolfgang Pauli (Springer-Verlag, Berlin).

Leifer, M.S., 2006, Quantum Dynamics As an Analog of Conditional Probability, Phys. Rev. A 74, 042310.

Leifer, M. S., 2007, in Foundations of Probability and Physics-4, AIP Conf. Proc. Vol. 889, edited by G. Adenier, C. A. Fuchs, and A. Yu. Khrennikov (AIP, Melville, NY), pp. 172-186.

Leifer, M. S., and R. W. Spekkens, 2011, Formulating Quantum Theory as a Causally Neutral Theory of Bayesian Inference, arXiv:1107.5849v3.

Lewis, D., 1986a, Studies in Inductive Logic and Probability, Vol. II (Oxford University Press, Oxford), pp. 83-112.

Lewis, D., 1986b, Philosophical Papers, Vol. II (Oxford University Press, Oxford), pp. 114-132.

Lindley, D.V., 1982, Comment on A. P. Dawid's, 'The Well- Calibrated Bayesian', J. Am. Stat. Assoc. 77, 611.

Lindley, D. V., 2006, Understanding Uncertainty (Wiley Interscience, Hoboken, NJ).

Mermin, N.D., 1993, Hidden Variables and Two Theorems of John Bell, Rev. Mod. Phys. 65, 803.

Mermin, N. D., 1999, Nonlocality and the absurd, Found. Phys. 29, 571.

Mermin, N. D., 2006, In Praise of Measurement, Quantum Inf. Process. 5, 239.

Mermin, N.D., 2012, Quantum Mechanics: Fixing the Shifty Split, Phys. Today 65, No. 7, 8.

Nagel, T., 1989, The View from Nowhere (Oxford University Press, Oxford, England).

Nielsen, M. A., and I. L. Chuang, 2000, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, England).

Norsen, T., 2006, Bell Locality and the Nonlocal Character of Nature, Found. Phys. Lett. 19, 633.

Parikh, M.K., and E. Verlinde, 2005, Sitter Holography with a Finite Number of States, J. High Energy Phys. 01 054.

Paris, M. G. A., and J. Řeháček, 2004, Eds., Quantum Estimation Theory (Springer-Verlag, Berlin).

Patton, C. M., and J. A. Wheeler, 1975, in Quantum Gravity: An Oxford Symposium, edited by C. J. Isham, R. Penrose, and D. W. Sciama (Clarendon Press, Oxford), pp. 538-605.

Pauli, W., 1994, Writings on Physics and Philosophy, edited by C. P. Enz and K. von Meyenn (Springer-Verlag, Berlin).

Peres, A., 1978, Unperformed Experiments Have No Results, Am. J. Phys. 46, 745.

Peres, A., 1993, Quantum Theory: Concepts and Methods (Kluwer, Dordrecht).

Peres, A., 2005, Einstein, Podolsky, Rosen, and Shannon, Found. Phys. 35, 511.

Pitowsky, I., 2003, Betting on the Outcomes of Measurements: A Bayesian Theory of Quantum Probability, Stud. Hist. Phil. Mod. Phys. 34, 395.

Pitowsky, I., 2005, ??Quantum Mechanics as a Theory of Probability,?? arXiv:quant-ph/0510095v1.

Plotnitsky, A., 2006, Reading Bohr: Physics and Philosophy (Springer, Dordrecht).

Plunk, G.G., 2002, Investigations on Informationally Complete Measurements, Bell Labs Summer Research Program, Summer (unpublished); see also Sec. VC of arXiv:1301.3274v1.

Popescu, S., and D. Rohrlich, 1994, Quantum Nonlocality as an Axiom, Found. Phys. 24, 379.

Porta Mana, P. G. L., 2007, Ph.D. thesis (KTH, Stockholm, Sweden).

Price, H., 1997, Time's Arrow & Archimedes? Point: New Directions for the Physics of Time (Oxford University Press, Oxford, England).

Ramsey, F. P., 1931, in The Foundations of Mathematics and other Logical Essays, edited by R. B. Braithwaite (Harcourt, Brace and Company, New York), pp. 156-198.

Rau, J., 2009, "On Quantum vs Classical Probability," Ann. Phys. (Amsterdam) 324, 2622.

Renes, J. M., R. Blume-Kohout, A. J. Scott, and C. M. Caves, 2004, Symmetric Informationally Complete Quantum Measurements, J. Math. Phys. (N.Y.) 45, 2171.

Savage, L. J., 1954, The Foundations of Statistics (Wiley, New York).

Schack, R., 2003, Quantum Theory from Four of Hardy's Axioms, Found. Phys. 33, 1461.

Schack, R., T. A. Brun, and C. M. Caves, 2001, Quantum Bayes Rule, Phys. Rev. A 64, 014305.

Schnetter, E., 2013 (private communication).

Scott, A. J., and M. Grassl, 2010, SIC-POVMs: A New Computer Study, J. Math. Phys. (N.Y.) 51, 042203.

Skyrms, B., 1987a, in Scientific Inquiry in Philosophical Perspective, edited by N. Rescher (University of Pittsburgh Press, Pittsburgh), pp. 225?242.

Skyrms, B., 1987b, Dynamic Coherence and Probability Kinematics, Philos. Sci. 54, 1.

Spekkens, R. W., 2005, Contextuality for Preparations, Transformations, and Unsharp Measurements, Phys. Rev. A 71, 052108.

Spekkens, R.W., 2007, Evidence for the Epistemic View of Quantum States: A Toy Theory, Phys. Rev. A 75, 032110.

Spekkens, R. W., 2008, Negativity and Contextuality are Equivalent Notions of Nonclassicality, Phys. Rev. Lett. 101, 020401.

Srednicki, M., 2005, Subjective and Objective Probabilities in Quantum Mechanics, Phys. Rev. A 71, 052107.

Tabia, G. N. M., 2012, Experimental Scheme for Qubit and Qutrit Symmetric Informationally Complete Positive Operator-Valued Measurements Using Multiport Devices, Phys. Rev. A 86, 062107.

Timpson, C. J., 2008, Quantum Bayesianism: A Study, Stud. Hist. Phil. Mod. Phys. 39, 579.

Warmuth, M.K., and D. Kuzmin, 2009, Bayesian Generalized Probability Calculus for Density Matrices, arXiv:0901.1273v1.

Wheeler, J.A., 1982, in Mind in Nature: Nobel Conference XVII, Gustavus Adolphus College, St. Peter, Minnesota, edited by R.Q. Elvee (Harper & Row, San Francisco, CA), pp. 1-23.

Wootters, W. K., and W. H. Zurek, 1982, A Single Quantum Cannot Be Cloned, Nature (London) 299, 802.

Youssef, S., 2001, Physics with Exotic Probability Theory, arXiv: hep-th/0110253v2.

Zauner, G., 1999, Quantum Designs - Foundations of a Non- Commutative Theory of Designs, Ph.D. thesis (University of Vienna) (in German).

Zeilinger, A., 1996, On the Interpretation and Philosophical Foundation of Quantum Mechanics, in Vastakohtien todellisuus, Juhlakirja Professori K. V. Laurikaisen 80-vuotispäivänä , edited by U. Ketvel et al. (Helsinki University Press, Helsinki), pp. 167-178.

Zinn-Justin, J., 2007, Phase Transitions and Renormalization Group (Oxford University Press, Oxford, England).

Zukowski, M., 2005, On the Paradoxical Book of Bell, Stud. Hist. Phil. Mod. Phys. 36, 566.