Schrodinger's Cat

The superposition principle states that if $|\phi_a\rangle$ and $|\phi_b\rangle$ are two possible states of a quantum system, the quantum superposition $\frac{1}{\sqrt{2}}\left(|\phi_a\rangle + |\phi_b\rangle\right)$ is also an allowed state for this system. This principle is essential in explaining quantum interference phenomena. However, when it is applied to "large" or "macroscopic" objects, it leads to paradoxical situations where a system can be in a superposition of states which is classical self-contradictory.

The most famous example is Schrodinger's "cat paradox" where the cat is in a superposition of the "dead" and "alive" states. The purpose of this discussion is to show that such superposition of macroscopic states are not detectable in practice. They are extremely fragile, and very weak coupling to the environment suffices to destroy the quantum superposition of the two states $|\phi_a\rangle$ and $|\phi_b\rangle$.

1. The Quasi-Classical States of a Harmonic Oscillator

We consider the high energy excitations of a one-dimensional harmonic oscillator or mass m and frequency ω. The Hamiltonian is written

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

We denote the eigenstates of $\hat{H}$ by $\{|n\rangle\}$ where the energy eigenvalues are given by

$$\hat{H}|n\rangle = E_n|n\rangle = \hbar\omega(n+1/2)|n\rangle$$

Preliminaries

We introduce the operators

$$\hat{X} = \sqrt{m\omega/\hbar}\,\hat{x} \quad , \quad \hat{P} = \hat{p}/\sqrt{m\hbar\omega}$$

and the annihilation and creation operators

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\hat{X} + i\hat{P}\right) \quad , \quad \hat{a}^+ = \frac{1}{\sqrt{2}}\left(\hat{X} - i\hat{P}\right) \quad , \quad \hat{N} = \hat{a}^+\hat{a}$$

The commutator $[\hat{x},\hat{p}] = i\hbar$ leads to the commuators $[\hat{X},\hat{P}] = i$, $[\hat{a},\hat{a}^+] = 1$ and the relations

$$\hat{H} = \hbar\omega(\hat{N}+1/2) \quad , \quad \hat{N}|n\rangle = n|n\rangle$$

We also have the relations

$$\hat{P} = -i\frac{\partial}{\partial X} \quad , \quad \hat{X} = i\frac{\partial}{\partial P}$$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \quad , \quad \hat{a}^{+}|n\rangle = \sqrt{n+1}|n+1\rangle \tag{1.1}$$

We can use these relations to derive the ground state wave function in the position representation as follows:

$$0 = \langle X|\hat{a}|0\rangle = \frac{1}{\sqrt{2}}\langle X|\left(\hat{X} + i\hat{P}\right)|0\rangle = \frac{1}{\sqrt{2}}X\langle X|0\rangle + \frac{i}{\sqrt{2}}\left(-i\frac{\partial}{\partial X}\right)\langle X|0\rangle$$

$$\left(X + \frac{\partial}{\partial X}\right)\langle X|0\rangle = 0 \rightarrow \langle X|0\rangle = Ae^{-X^2/2} = \psi_0(X)$$

$$\psi_0(x) = Ae^{-m\omega x^2/2\hbar}$$

Similarly, we can derive its the ground state wave function in the momentum representation as follows:

$$0 = \langle P|\hat{a}|0\rangle = \frac{1}{\sqrt{2}}\langle P|\left(\hat{X} + i\hat{P}\right)|0\rangle = \frac{1}{\sqrt{2}}i\frac{\partial}{\partial P}\langle P|0\rangle + \frac{i}{\sqrt{2}}P\langle P|0\rangle$$

$$\left(P + \frac{\partial}{\partial P}\right)\langle P|0\rangle = 0 \rightarrow \langle P|0\rangle = Ae^{-P^2/2} = \varphi_0(P)$$

$$\varphi_0(p) = Ae^{-p^2/2m\omega\hbar}$$

These two wave functions are related by the Fourier transform, that is,

$$\varphi_0(p) = e^{-p^2/2m\omega\hbar} \propto \int_{-\infty}^{\infty} e^{-m\omega x^2/2\hbar}e^{-ipx/\hbar}dx \propto \int_{-\infty}^{\infty} \psi_0(x)e^{-ipx/\hbar}dx$$

The Quasi-Classical States

The eigenstates of the operator $\hat{a}$ are called "quasi-classical" states, for reasons we will now discuss.

Since we are considering the question: what are the eigenstates of the lowering operator $\hat{a}$?. We can write

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \quad \text{where} \quad \alpha = |\alpha|e^{i\varphi}$$

where $|\alpha\rangle$ is the eigenvector of $\hat{a}$ and α is the eigenvalue, which is not necessarily real since $\hat{a}$ is not Hermitian.

Since the vectors $|n\rangle$ are eigenvectors of a Hermitian operator, they form a orthonormal complete set and can be used as an orthonormal basis for the vector space. We can then write

$$|\alpha\rangle = \sum_{m=0}^{\infty} b_m|m\rangle$$

where

$$\langle k|\alpha\rangle = \sum_{m=0}^{\infty} b_m\langle k|m\rangle = \sum_{m=0}^{\infty} b_m\delta_{km} = b_k \quad .$$

Now

$$\langle n-1|\hat{a}|\alpha\rangle = \alpha\langle n-1|\alpha\rangle = \alpha b_{n-1}$$

and using

$$\hat{a}^{+}|n-1\rangle = \sqrt{n}|n\rangle \rightarrow \langle n-1|\hat{a} = \sqrt{n}\langle n|$$

we have

$$\langle n-1|\hat{a}|\alpha\rangle = \sqrt{n}\langle n|\alpha\rangle = \sqrt{n} b_n$$

or

$$b_n = \frac{\alpha}{\sqrt{n}} b_{n-1}$$

This says that

$$b_1 = \frac{\alpha}{\sqrt{1}} b_0 \quad , \quad b_2 = \frac{\alpha}{\sqrt{2}} b_1 = \frac{\alpha^2}{\sqrt{2!}} b_0$$

or

$$b_n = \frac{\alpha^n}{\sqrt{n!}} b_0$$

We thus get the final result

$$|\alpha\rangle = b_0 \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} |m\rangle$$

Let us now normalize this state (choose b_0). We have

$$\langle\alpha|\alpha\rangle = 1 = |b_0|^2 \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{\alpha^{*m}\alpha^k}{\sqrt{m!}\sqrt{k!}} \langle k|m\rangle = |b_0|^2 \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{\alpha^{*m}\alpha^k}{\sqrt{m!}\sqrt{k!}} \delta_{km}$$

$$= |b_0|^2 \sum_{m=0}^{\infty} \frac{|\alpha|^2}{m!} = |b_0|^2 e^{|\alpha|^2}$$

which says that

$$b_0 = e^{-\frac{1}{2}|\alpha|^2}$$

and thus

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} |m\rangle \tag{1.2}$$

Now

$\langle n|\alpha\rangle$ = probability amplitude that the system in the state $|\alpha\rangle$ will be found in the state $|n\rangle$

We have

$$\langle n|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}\langle n|m\rangle = e^{-\frac{1}{2}|\alpha|^2}\ \frac{\alpha^n}{\sqrt{n!}}$$

which then says that

$$P_n = \left|\langle n|\alpha\rangle\right|^2 = \frac{e^{-|\alpha|^2}|\alpha|^{2n}}{n!} = \frac{e^{-N}N^n}{n!} = \text{probability amplitude that the system in the state } |\alpha\rangle \text{ will be found in the state } |n\rangle$$

where we have defined $N = |\alpha|^2$. We note that

$$\langle\alpha|\hat{a}^+\hat{a}|\alpha\rangle = \left|\alpha^2\right|\langle\alpha|\alpha\rangle = \left|\alpha^2\right| = N = \langle\alpha|\hat{N}_{op}|\alpha\rangle$$

or N = the average value or expectation value of the $\hat{N}_{op}$ operator in the state $|\alpha\rangle$. This type of probability distribution is called a Poisson distribution, i.e., the state $|\alpha\rangle$ has the number states or energy eigenstates distributed in a **"Poisson"** manner.

Since the states $|n\rangle$ are energy eigenstates, we know their time dependence, i.e.,

$$|n,t\rangle = e^{-i\frac{E_n}{\hbar}t}|n\rangle$$

Therefore, we have for the time dependence of the state $|\alpha\rangle$

$$|\alpha,t\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}|m,t\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}e^{-i\frac{E_m}{\hbar}t}|m\rangle$$

This simple operation clearly indicates the fundamental importance of the energy eigenstates when used as a basis set.

> If we are able to expand an arbitrary vector representing some physical system in the energy basis, then we immediately know the time dependence of that state vector and hence we know the time dependence of all the probabilities associated with the state vector and the system.

Now let us try to understand the physics contained in the $|\alpha\rangle$ state vector. In a given energy eigenstate the expectation value of the position operator is given by

$$\langle n,t|\hat{x}|n,t\rangle = \sqrt{\frac{\hbar}{2m\omega_0}}\langle n,t|(\hat{a}+\hat{a}^+)|n,t\rangle = \sqrt{\frac{\hbar}{2m\omega_0}}\langle n|e^{i\frac{E_n}{\hbar}t}(\hat{a}+\hat{a}^+)e^{-i\frac{E_n}{\hbar}t}|n\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega_0}}\langle n|(\hat{a}+\hat{a}^+)|n\rangle = \sqrt{\frac{\hbar}{2m\omega_0}}\langle n|\left(\sqrt{n}|n-1\rangle + \sqrt{n+1}|n+1\rangle\right) = 0$$

i.e., it is equal to zero and is a constant.

On the other hand, in the state $|\alpha\rangle$ we find

$$\langle\alpha,t|\hat{x}|\alpha,t\rangle = \sqrt{\frac{\hbar}{2m\omega_0}}\sum_m\sum_k b_m^* b_k e^{i\frac{(E_m-E_k)}{\hbar}t}\langle m|(\hat{a}+\hat{a}^+)|k\rangle$$

Now

$$\langle m|(\hat{a}+\hat{a}^+)|k\rangle = \langle m|\left(\sqrt{k}|k-1\rangle + \sqrt{k+1}|k+1\rangle\right) = \sqrt{k}\delta_{m,k-1} + \sqrt{k+1}\delta_{m,k+1}$$

Using this result we have

$$\begin{aligned}
\langle\alpha,t|\hat{x}|\alpha,t\rangle &= \sqrt{\frac{\hbar}{2m\omega_0}}\left(\sum_{k=1}^{\infty} b_{k-1}^* b_k\sqrt{k}e^{i\frac{(E_{k-1}-E_k)}{\hbar}t} + \sum_{k=0}^{\infty} b_{k+1}^* b_k\sqrt{k+1}e^{i\frac{(E_{k+1}-E_k)}{\hbar}t}\right)\\
&= \sqrt{\frac{\hbar}{2m\omega_0}}\left(\sum_{k=1}^{\infty} b_{k-1}^* b_k\sqrt{k}e^{-i\omega_0 t} + \sum_{k=0}^{\infty} b_{k+1}^* b_k\sqrt{k+1}e^{i\omega_0 t}\right)\\
&= \sqrt{\frac{\hbar}{2m\omega_0}}\left(\sum_{k=0}^{\infty} b_k^* b_{k+1}\sqrt{k}e^{-i\omega_0 t} + \sum_{k=0}^{\infty} b_{k+1}^* b_k\sqrt{k+1}e^{i\omega_0 t}\right)\\
&= \sqrt{\frac{\hbar}{2m\omega_0}}b_0^2\left(\sum_{k=0}^{\infty}\frac{\alpha^{*k}\alpha^{k+1}}{\sqrt{(k+1)!k!}}\sqrt{k}e^{-i\omega_0 t} + \sum_{k=0}^{\infty}\frac{\alpha^{*k+1}\alpha^{k}}{\sqrt{(k+1)!k!}}\sqrt{k+1}e^{i\omega_0 t}\right)\\
&= \sqrt{\frac{\hbar}{2m\omega_0}}b_0^2\sum_k\frac{1}{k!}|\alpha|^{2k}\left(\alpha e^{-i\omega_0 t} + \alpha^* e^{i\omega_0 t}\right)
\end{aligned}$$

Now using $\alpha = |\alpha|e^{i\varphi}$ we get

$$\begin{aligned}
\langle\alpha,t|\hat{x}|\alpha,t\rangle &= \sqrt{\frac{\hbar}{2m\omega_0}}b_0^2 2|\alpha|\sum_k\frac{|\alpha|^{2k}}{k!}\mathrm{Re}\,al(e^{i\varphi}e^{-i\omega_0 t})\\
&= 2x_0|\alpha|\cos(\omega_0 t-\varphi)(b_0^2\sum_k\frac{|\alpha|^{2k}}{k!}) \quad , \quad x_0 = \sqrt{\frac{\hbar}{2m\omega_0}}\\
&= 2x_0|\alpha|\cos(\omega_0 t-\varphi)
\end{aligned}$$

The expectation value in the state $|\alpha\rangle$ behaves like that of a **classical oscillator.**

Before proceeding with the discussion, we will repeat the derivation using an alternate but very powerful technique.

Using the Translation Operator

In general, a **displaced state** $|\lambda\rangle$ is given in terms of the displacement operator (in one dimension) by

$$|\lambda\rangle = e^{-\frac{i}{\hbar}\hat{p}\lambda}|0\rangle$$

For the harmonic oscillator system

$$\hat{p} = \frac{1}{i}\sqrt{\frac{m\hbar\omega}{2}}(\hat{a} - \hat{a}^+)$$

If we choose $|0\rangle$ to be the ground state of the oscillator, then we have for the corresponding displaced ground-state

$$|\lambda\rangle = e^{\sqrt{\frac{m\omega}{2\hbar}}(\hat{a}^+ - \hat{a})\lambda}|0\rangle$$

By Glauber's theorem $e^{(\hat{A}+\hat{B})} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]}$, we have

$$e^{\sqrt{\frac{m\omega}{2\hbar}}(\hat{a}^+ - \hat{a})\lambda} = e^{\sqrt{\frac{m\omega}{2\hbar}}\hat{a}^+\lambda}e^{-\sqrt{\frac{m\omega}{2\hbar}}\hat{a}\lambda}e^{\frac{1}{2}\frac{m\omega}{2\hbar}[\hat{a}^+,\hat{a}]\lambda^2} = e^{\sqrt{\frac{m\omega}{2\hbar}}\hat{a}^+\lambda}e^{-\sqrt{\frac{m\omega}{2\hbar}}\hat{a}\lambda}e^{-\frac{1}{4}\frac{m\omega}{\hbar}\lambda^2}$$

and thus

$$|\lambda\rangle = e^{\sqrt{\frac{m\omega}{2\hbar}}\hat{a}^+\lambda}e^{-\sqrt{\frac{m\omega}{2\hbar}}\hat{a}\lambda}e^{-\frac{1}{4}\frac{m\omega}{\hbar}\lambda^2}|0\rangle$$

Now

$$e^{-\sqrt{\frac{m\omega}{2\hbar}}\hat{a}\lambda}|0\rangle = \left(\hat{I} + \left(-\sqrt{\frac{m\omega}{2\hbar}}\lambda\hat{a}\right) + \frac{1}{2}\left(-\sqrt{\frac{m\omega}{2\hbar}}\lambda\hat{a}\right)^2 + ...\right)|0\rangle = |0\rangle$$

using $\hat{a}|0\rangle = 0$.

Similarly, using $\left(\hat{a}^+\right)^n|0\rangle = \sqrt{n!}|n\rangle$ we have

$$e^{\sqrt{\frac{m\omega}{2\hbar}}\hat{a}^+\lambda}|0\rangle = \left(\hat{I} + \left(\sqrt{\frac{m\omega}{2\hbar}}\lambda\hat{a}^+\right) + \frac{1}{2}\left(\sqrt{\frac{m\omega}{2\hbar}}\lambda\hat{a}^+\right)^2 + ...\right)|0\rangle$$

$$= |0\rangle + \sqrt{\frac{m\omega}{2\hbar}}\lambda|1\rangle + \frac{1}{2}\left(\sqrt{\frac{m\omega}{2\hbar}}\lambda\right)^2|2\rangle + ... = \sum_{n=0}^{\infty}\frac{\left(\sqrt{\frac{m\omega}{2\hbar}}\lambda\right)^n}{\sqrt{n!}}|n\rangle$$

or

$$|\lambda\rangle = e^{-\frac{1}{4}\frac{m\omega}{\hbar}\lambda^2}\sum_{n=0}^{\infty}\frac{\left(\sqrt{\frac{m\omega}{2\hbar}}\lambda\right)^n}{\sqrt{n!}}|n\rangle$$

Thus,

$$|\lambda\rangle = \sum_{n=0}^{\infty}b_n|n\rangle$$

where

$$b_n = \frac{e^{-\frac{N}{2}}N^{\frac{n}{2}}}{\sqrt{n!}} \quad , \quad \frac{N}{2} = \frac{m\omega}{4\hbar}\lambda^2$$

or

$$P_n = \text{probability of find the system in the state } |n\rangle$$
$$= |b_n|^2 = \frac{e^{-N}N^n}{n!}$$

which is a Poisson distribution. Thus, we obtain the coherent states once again.

Let us now return to the original discussion. In the state $|\alpha\rangle$ we have

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \to \langle\alpha|\hat{a}^+ = \alpha^*\langle\alpha|$$

so that

$$\langle E\rangle = \langle\alpha|\hat{H}|\alpha\rangle = \hbar\omega\langle\alpha|\left(\hat{N}+1/2)\right)|\alpha\rangle = \hbar\omega\left(|\alpha|^2+1/2)\right)$$

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle\alpha|(\hat{a}+\hat{a}^+)|\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\alpha+\alpha^*)$$

$$\langle p\rangle = -i\sqrt{\frac{m\hbar\omega}{2}}\langle\alpha|(\hat{a}-\hat{a}^+)|\alpha\rangle = i\sqrt{\frac{m\hbar\omega}{2}}(\alpha^*-\alpha)$$

$$(\Delta x)^2 = \frac{\hbar}{2m\omega}\langle\alpha|(\hat{a}+\hat{a}^+)^2|\alpha\rangle - \langle x\rangle^2 = \frac{\hbar}{2m\omega}\left[\left(\alpha+\alpha^*\right)^2+1\right] - \frac{\hbar}{2m\omega}\left(\alpha+\alpha^*\right)^2$$

$$= \frac{\hbar}{2m\omega} \to \Delta x = \sqrt{\frac{\hbar}{2m\omega}}$$

$$(\Delta p)^2 = -\frac{m\hbar\omega}{2}\langle\alpha|(\hat{a}-\hat{a}^+)^2|\alpha\rangle - \langle p\rangle^2 = -\frac{m\hbar\omega}{2}\left[\left(\alpha-\alpha^*\right)^2-1\right] + \frac{m\hbar\omega}{2}\left(\alpha^*-\alpha\right)^2$$

$$= \frac{m\hbar\omega}{2} \to \Delta p = \sqrt{\frac{m\hbar\omega}{2}}$$

Therefore, the Heisenberg inequality becomes in this case an equality

$$\Delta x\Delta p = \frac{\hbar}{2}$$

independent of the value of α.

We can find the wave functions corresponding to $|\alpha\rangle$ using the earlier method. We have in the position representation:

$$\langle X|\hat{a}|\alpha\rangle = \alpha\langle X|\alpha\rangle = \frac{1}{\sqrt{2}}\langle X|\left(\hat{X}+i\hat{P}\right)|\alpha\rangle = \frac{1}{\sqrt{2}}X\langle X|\alpha\rangle + \frac{i}{\sqrt{2}}\left(-i\frac{\partial}{\partial X}\right)\langle X|\alpha\rangle$$

$$\frac{1}{\sqrt{2}}\left(X+\frac{\partial}{\partial X}\right)\langle X|\alpha\rangle = \alpha\langle X|\alpha\rangle \to \langle X|\alpha\rangle = Ae^{-(X-\alpha\sqrt{2})^2/2} = \psi_\alpha(X)$$

and in the momentum representation:

$$\langle P|\hat{a}|\alpha\rangle = \alpha\langle P|\alpha\rangle = \frac{1}{\sqrt{2}}\langle P|\left(\hat{X}+i\hat{P}\right)|\alpha\rangle = \frac{1}{\sqrt{2}}i\frac{\partial}{\partial P}\langle P|\alpha\rangle + \frac{i}{\sqrt{2}}P\langle p|\alpha\rangle$$

$$\frac{1}{\sqrt{2}}\left(P+\frac{\partial}{\partial P}\right)\langle P|\alpha\rangle = \alpha\langle P|\alpha\rangle \to \langle P|\alpha\rangle = A'\,e^{-(P+i\alpha\sqrt{2})^2/2} = \varphi_\alpha(P)$$

Suppose that at time $t=0$, the oscillator is in a quasi-classical state $|\psi(0)\rangle = |\alpha_0\rangle$ with $\alpha_0 = \rho e^{i\phi}$ where ρ is a real positive number. Then at a later time t

$$\left|\psi(t)\right\rangle = \left|\alpha_0, t\right\rangle = e^{-\frac{1}{2}|\alpha_0|^2} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} \left|n, t\right\rangle = e^{-\frac{1}{2}|\alpha_0|^2} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} e^{-i\frac{E_n}{\hbar}t} \left|n\right\rangle$$

$$= e^{-\frac{1}{2}|\alpha_0|^2} e^{-i\omega t/2} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} e^{-in\omega t} \left|n\right\rangle = e^{-i\omega t/2} \left|\alpha(t)\right\rangle$$

where $\alpha(t) = \alpha_0 e^{-i\omega t} = \rho e^{-i(\omega t - \phi)}$.

Finally, we have

$$\left\langle \alpha, t\right|\hat{x}\left|\alpha, t\right\rangle = \sqrt{\frac{\hbar}{2m\omega_0}} b_0^2 \sum_k \frac{1}{k!} |\alpha|^{2k} \left(\alpha e^{-i\omega_0 t} + \alpha^* e^{i\omega_0 t}\right)$$

$$= 2\sqrt{\frac{\hbar}{2m\omega_0}} |\alpha| \cos(\omega_0 t - \varphi) (b_0^2 \sum_k \frac{|\alpha|^{2k}}{k!}) \quad , \quad x_0 = \sqrt{\frac{\hbar}{2m\omega_0}}$$

$$= x_0 \cos(\omega_0 t - \varphi) \quad , \quad x_0 = \rho \sqrt{\frac{2\hbar}{m\omega_0}}$$

and

$$\left\langle \alpha, t\right|\hat{p}\left|\alpha, t\right\rangle = -p_0 \sin(\omega_0 t - \varphi) \quad , \quad p_0 = \rho\sqrt{2m\hbar\omega}$$

In addition, we have (for $\rho >> 1$)

$$\frac{\Delta x}{x_0} = \frac{1}{2\rho} << 1 \quad , \quad \frac{\Delta p}{p_0} = \frac{1}{2\rho} << 1$$

This says that the relative uncertainties in the position and momentum of the oscillator are quite accurately defined at any time. Hence the name "quasi-classical state".

Let us look at some numbers. We consider a pendulum of length 1 meter and of mass 1 gram and assume that the state of this pendulum can be described by a quasi-classical state. At time $t = 0$ we assume that the pendulum is at $\langle x(0)\rangle = 1$ micrometer from its classical equilibrium position, with zero mean velocity.

An appropriate choice is $\langle x(0)\rangle = x_0$, $\langle p(0)\rangle = 0 \rightarrow \varphi = 0$. We also have

$$\omega = 2\pi\nu = \sqrt{\frac{g}{\ell}} = 3.13 s^{-1} \rightarrow \alpha(0) = 3.9 \times 10^9$$

The relative uncertainty in the position is

$$\frac{\Delta x}{x_0} = \frac{1}{2\rho} = \frac{1}{2\alpha(0)} = 1.3 \times 10^{-10}$$

We note that after $1/4$ period of oscillation,

$$T = period = \frac{2\pi}{\omega} \rightarrow \alpha(T/4) = \alpha(0) e^{i\omega T/4} = \alpha(0) e^{i\pi/2} = i\alpha(0) = -3.9i \times 10^9$$

2. Construction of a Schrodinger-Cat State

Suppose that during the interval $[0,T]$ we add to the harmonic potential, the coupling (interaction)

$$\hat{W} = \hbar g\left(\hat{a}^{+}\hat{a}\right)^{2} = \hbar g\hat{N}^{2}$$

We will assume that $g >> \omega$, $\omega T << 1$. Under these conditions, we can make the approximation that, during the interval $[0,T]$, the Hamiltonian of the system is simply $\hat{W}$. Assume that at time $t=0$, the system is in a quasi-classical state $|\psi(0)\rangle = |\alpha\rangle$.

The eigenvectors of $\hat{W}$ are $\{|n\rangle\}$ with $\hat{W}|n\rangle = \hbar g n^{2}|n\rangle$. This implies that for $|\psi(0)\rangle = |\alpha\rangle$,

$$|\psi(T)\rangle = e^{-\frac{1}{2}|\alpha|^{2}}\sum_{n=0}^{\infty}\frac{\alpha^{n}}{\sqrt{n!}}e^{-ign^{2}t}|n\rangle$$

Some special cases will be of interest later.

$$|\psi(T=2\pi/g)\rangle = e^{-\frac{1}{2}|\alpha|^{2}}\sum_{n=0}^{\infty}\frac{\alpha^{n}}{\sqrt{n!}}e^{-i2\pi n^{2}}|n\rangle = e^{-\frac{1}{2}|\alpha|^{2}}\sum_{n=0}^{\infty}\frac{\alpha^{n}}{\sqrt{n!}}|n\rangle = |\alpha\rangle$$

$$|\psi(T=\pi/g)\rangle = e^{-\frac{1}{2}|\alpha|^{2}}\sum_{n=0}^{\infty}\frac{\alpha^{n}}{\sqrt{n!}}e^{-i\pi n^{2}}|n\rangle = e^{-\frac{1}{2}|\alpha|^{2}}\sum_{n=0}^{\infty}\frac{\alpha^{n}}{\sqrt{n!}}(-1)^{n}|n\rangle = |-\alpha\rangle$$

$$\begin{aligned}|\psi(T=\pi/2g)\rangle &= e^{-\frac{1}{2}|\alpha|^{2}}\sum_{n=0}^{\infty}\frac{\alpha^{n}}{\sqrt{n!}}e^{-i\pi n^{2}/2}|n\rangle = e^{-\frac{1}{2}|\alpha|^{2}}\sum_{n=0}^{\infty}\frac{\alpha^{n}}{\sqrt{n!}}\frac{1}{2}\left[1-i+(1+i)(-1)^{n}\right]|n\rangle \\ &= e^{-\frac{1}{2}|\alpha|^{2}}\sum_{n=0}^{\infty}\frac{\alpha^{n}}{\sqrt{n!}}\frac{1}{\sqrt{2}}\left[e^{-i\pi/4}+e^{i\pi/4}(-1)^{n}\right]|n\rangle \qquad (2.1) \\ &= \frac{1}{\sqrt{2}}\left[e^{-i\pi/4}|\alpha\rangle + e^{i\pi/4}|-\alpha\rangle\right]\end{aligned}$$

Now, suppose that α is pure imaginary, that is, $\alpha = i\rho$. In this case, in the state $|\alpha\rangle$, the oscillator has a zero mean position and a positive velocity.

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\alpha+\alpha^{*}) = 0$$

$$\langle p\rangle = i\sqrt{\frac{m\hbar\omega}{2}}(\alpha^{*}-\alpha) = \sqrt{2m\hbar\omega}\,\rho$$

Similarly, in the state $|-\alpha\rangle$, the oscillator also has a zero mean position, but a negative velocity.

If $|\alpha| >> 1$, the states $|\alpha\rangle$ and $|-\alpha\rangle$ are macroscopically different. The state (2.1) is a quantum superposition of such states. It therefore constitutes a (harmless) version of Schrodinger's cat, where we represent "dead" and "alive" cats by simple vectors in Hilbert space.

3. Quantum Superposition Versus Statistical Mixture

We now consider the properties of the state (2.1) in a "macroscopic" situation $|\alpha| >> 1$. We will choose $\alpha = i\rho$ pure imaginary and we set $p_0 = \sqrt{2m\hbar\omega}\,\rho$.

The probability distributions for position and momentum are given by

$$\begin{aligned}
\Pr(X) &\propto \left| e^{-i\pi/4}\langle X|\alpha\rangle + e^{i\pi/4}\langle X|-\alpha\rangle \right|^2 \\
&\propto \left| e^{-i\pi/4} e^{-(X - i\rho\sqrt{2})^2/2} + e^{i\pi/4} e^{-(X + i\rho\sqrt{2})^2/2} \right|^2 \\
&\propto e^{-X^2}\cos^2\left(\sqrt{2}X\rho - \frac{\pi}{4}\right) \\
\Pr(P) &\propto \left| e^{-i\pi/4}\langle P|\alpha\rangle + e^{i\pi/4}\langle P|-\alpha\rangle \right|^2 \\
&\propto \left| e^{-i\pi/4} e^{-(P - \rho\sqrt{2})^2/2} + e^{i\pi/4} e^{-(P + \rho\sqrt{2})^2/2} \right|^2 \\
&\approx e^{-(P-\rho\sqrt{2})^2} + e^{-(P+\rho\sqrt{2})^2}
\end{aligned}$$

where in the last expression we have used the fact that, for $\rho >> 1$, the two Gaussians centered at $\rho\sqrt{2}$ and $-\rho\sqrt{2}$ have a negligible overlap.

These probability distributions are plotted in the figure below for $\alpha = 5i$.

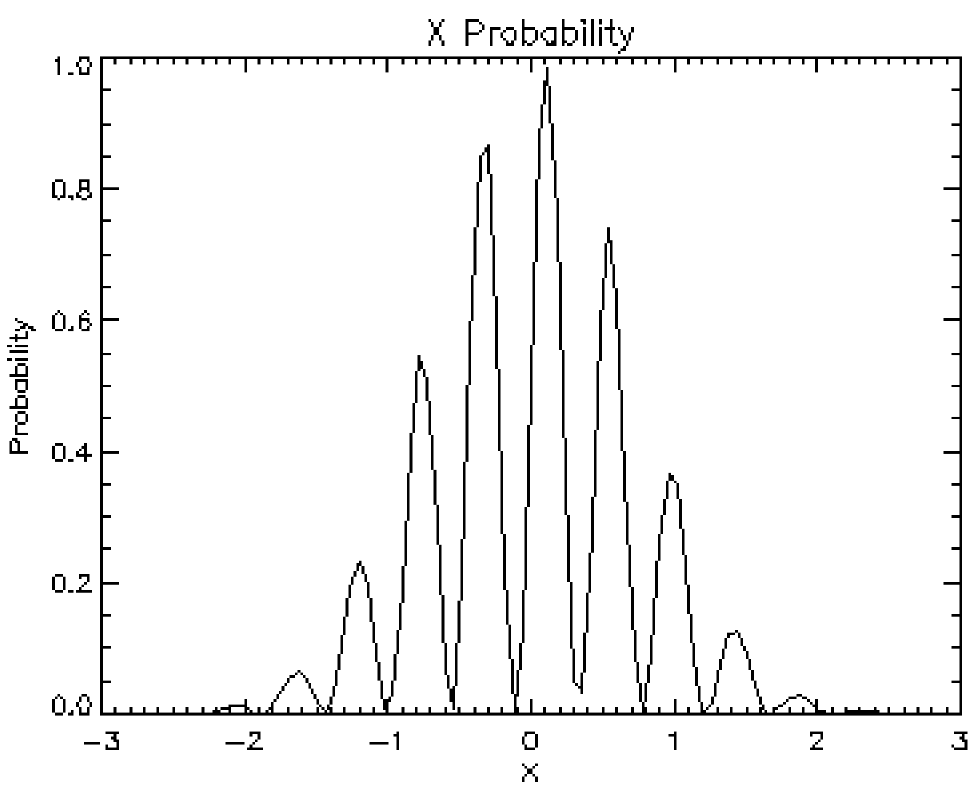

Figure 1

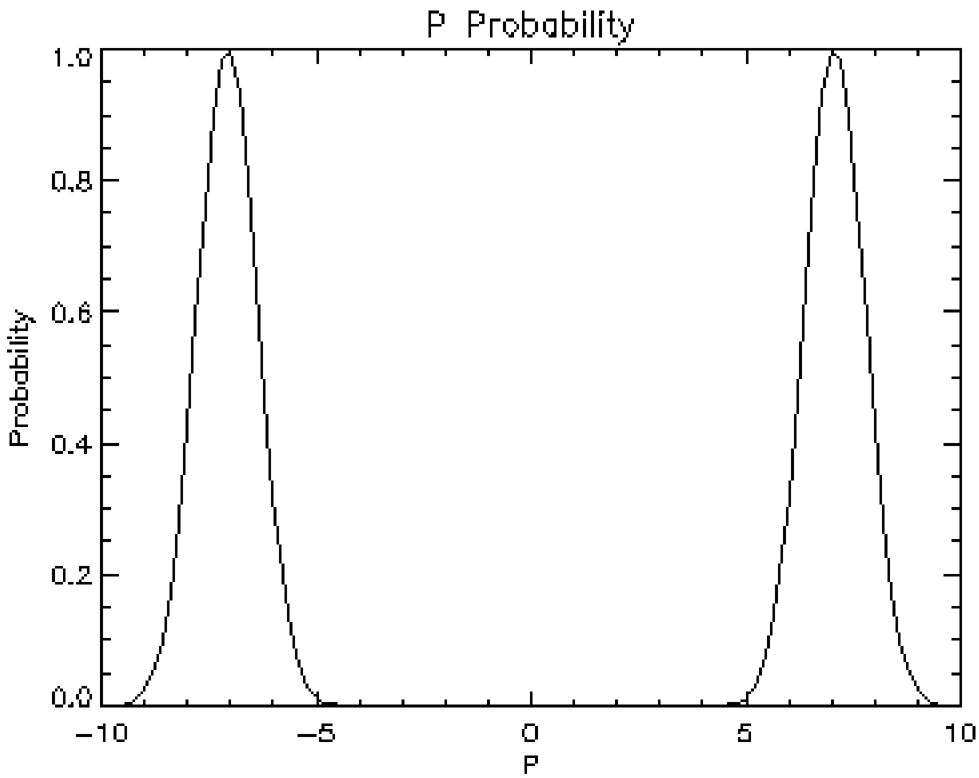

Figure 2

Suppose that a physicist (Alice) prepares N independent systems all in the state (2.1) and measures the momentum of each of these systems. Suppose the measuring apparatus has a resolution δp such that:

$$\sqrt{m\hbar\omega} << \delta p << p_0$$

For $N >> 1$, the results of these measurements is that Alice (plotting a histogram) will find two peaks, each of which contains roughly half of the events, centered respectively at p_0 and $-p_0$(resembling figure 2)

The state (2.1) represents the quantum superposition of two states which are macroscopically different, and therefore leads to the paradoxical situations mentioned earlier.

Another physicist (Bob) claims that the measurements done by Alice have not been performed on N quantum systems in the state (2.1), but that Alice is actually dealing with a nonparadoxical "statistical mixture", that is, half of the N systems are in the state $|\alpha\rangle$ and the other half in the state $|-\alpha\rangle$.

Assuming the this is true, the statistical mixture of Bob leads (after N momentum measurements) to the same momentum distribution as that measured by Alice: the $N/2$ oscillators in the state $|\alpha\rangle$ all lead to a mean momentum $+p_0$ and the $N/2$ oscillators in the state $|-\alpha\rangle$ all lead to a mean momentum $-p_0$. Up to this point, there is therefore no difference and no paradoxical behavior related to the quantum superposition (2.1).

In order to settle the matter, Alice now measures the position of each of the N independent systems, all prepared in the state (2.1). Assuming that the resolution δx of the measuring apparatus is such that

$$\delta x << \frac{1}{|\alpha|}\sqrt{\frac{\hbar}{m\omega}} \rightarrow \delta X << \frac{1}{|\alpha|} = \frac{1}{\rho}$$

Alice has sufficient resolution to observe the oscillations of the function $\cos^2\left(\sqrt{2}X\rho - \frac{\pi}{4}\right)$ in the distribution $\Pr(X)$. The shape of the distribution for x will therefore reproduce the probability law for X as drawn in figure 1 above, that is a modulation of period $\left[\hbar\pi^2/(2m\alpha^2\omega)\right]^{1/2}$, with a Gaussian envelope.

We continue with the assumption that Bob is dealing with a statistical mixture. If Bob performs a position measurement on the $N/2$ systems in the state $|\alpha\rangle$, he will find a Gaussian distribution corresponding to the probability law

$$\Pr(X) \propto |\langle X|\alpha\rangle|^2 \propto e^{-X^2}$$

He will find the same distribution for $N/2$ systems in the state $|-\alpha\rangle$. The sum of his results will be a Gaussian distribution, which is quite different from the result expected by Alice.

The position measurement should, in principle, allow one to discriminate between the quantum superposition and the statistical mixture.

In our earlier discussion of numbers for a pendulum we found that $\alpha = 3.9\times10^9$. Therefore, the resolution δx which is necessary in order to tell the difference between a set of N systems in a quantum superposition (2.1), and a statistical mixture consisting of $N/2$ systems in the state $|\alpha\rangle$ and $N/2$ systems in the state $|-\alpha\rangle$ is given by

$$\delta x << \frac{1}{|\alpha|}\sqrt{\frac{\hbar}{m\omega}} \approx 5\times10^{-26}m$$

Clearly, it is impossible to attain such a resolution in practice!

4. The Fragility of a Quantum Superposition

In a realistic physical situation, one must take into account the coupling of the oscillator with its environment, in order to estimate how long one can discriminate between the quantum superposition (2.1), that is, the "Schrodinger cat" which is "alive and dead", and a simple statistical mixture, that is, a set of cats (systems), half of which are alive, the other half beginning dead; each cat being **either** alive or dead.

If the oscillator is initially in the quasi-classical state $|\alpha_0\rangle$ and if the environment is in a state $|\chi_e(0)\rangle$, the wave function of the total system is the product of the individual wave functions, and the state vector of the total system can be written as the (tensor) product of the state vectors of the two subsystems:

$$|\Phi(0)\rangle = |\alpha_0\rangle|\chi_e(0)\rangle$$

The coupling is responsible for the damping of the oscillator's amplitude.

At a later time t, the state vector of the total system becomes

$$|\Phi(t)\rangle = |\alpha_1\rangle|\chi_e(t)\rangle$$

where $\alpha_1 = \alpha(t)e^{-\gamma t}$. The number $\alpha(t)$ corresponds to the quasi-classical state one would find in the absence of damping (evaluated earlier as $\alpha(t) = \alpha_0 e^{-i\omega t}$) and γ is a real positive number.

From earlier

$$E(t) = \hbar\omega\left(|\alpha(t)|^2 + 1/2)\right) = \hbar\omega\left(|\alpha_0|^2 e^{-2\gamma t} + 1/2)\right)$$

The energy decreases with time. After a time much longer than γ^{-1}, the oscillator is in its ground state. This dissipation model corresponds to a zero temperature environment. The mean energy acquired by the environment is

$$E(0) - E(t) = \hbar\omega|\alpha_0|^2(1 - e^{-2\gamma t}) \approx 2\hbar\omega|\alpha_0|^2\gamma t \quad , \quad 2\gamma t << 1$$

For initial states of the "Schrodinger cat" type for the oscillator, the state vector of the total system, at $t = 0$,

$$|\Phi(0)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\pi/4}|\alpha_0\rangle + e^{i\pi/4}|-\alpha_0\rangle\right)|\chi_e(0)\rangle$$

and, at a later time t,

$$|\Phi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\pi/4}|\alpha_1\rangle|\chi_e^{(+)}(t)\rangle + e^{i\pi/4}|-\alpha_1\rangle|\chi_e^{(-)}(t)\rangle\right)$$

still with $\alpha_1 = \alpha(t)e^{-\gamma t}$. We assume that t is chosen such that α_1 is pure imaginary, $|\alpha_1| >> 1$, and $|\chi_e^{(+)}(t)\rangle$ and $|\chi_e^{(-)}(t)\rangle$ are two normalized states of the environment that are a priori different (but not orthogonal).

The probability distribution of the oscillator's position, measured independently of the state of the environment, is then

$$\Pr(x) = \frac{1}{2}\left[|\langle x|\alpha_1\rangle|^2 + |\langle x|-\alpha_1\rangle|^2 + 2\mathrm{Real}\left(i\langle x|\alpha_1\rangle^*\langle x|-\alpha_1\rangle\langle\chi_e^{(+)}(t)|\chi_e^{(-)}(t)\rangle\right)\right]$$

Let $\eta = \langle\chi_e^{(+)}(t)|\chi_e^{(-)}(t)\rangle$. We then have $0 \le \eta \le 1, \eta$ real.

This says that the probability distribution of the position keeps its Gaussian envelope, but the contrast of the oscillations (cross term) is reduced by a factor η.

The probability distribution for the momentum is given by

$$\Pr(p) = \frac{1}{2}\Big[\left|\langle p|\alpha_1\rangle\right|^2 + \left|\langle p|-\alpha_1\rangle\right|^2 + 2\eta\mathrm{Real}\left(i\langle p|-\alpha_1\rangle^*\langle p|\alpha_1\rangle\right)\Big]$$

Since the overlap of the two Gaussians $\langle p|\alpha_1\rangle$ and $\langle p|-\alpha_1\rangle$ is negligible for $|\alpha_1| >> 1$, the crossed term, which is proportional to η does not contribute significantly. We recover two peaks centered at $\pm|\alpha_1|\sqrt{2m\hbar\omega}$. The distinction between a quantum superposition and a statistical mixture can be made by position measurements. The quantum superposition leads to a modulation of spatial period $\left[\hbar\pi^2/(2m\alpha^2\omega)\right]^{1/2}$ with a Gaussian envelope, whereas only the Gaussian is observed for a statistical mixture.

In order to see this modulation, the parameter η must not be too small, say $\eta \geq 1/10$.

In a very simple model, the environment is represented by a second oscillator, of the same mass and frequency as the first one. We will assume that this second oscillator is initially in its ground state $|\chi_e(0)\rangle = |0\rangle$. If the coupling between the two oscillators is quadratic, we can take for granted that

- the states $|\chi_e^{(\pm)}(t)\rangle$ are quasi-classical: $|\chi_e^{(\pm)}(t)\rangle = |\pm\beta\rangle$
- and that, for short times ($\gamma t << 1$): $|\beta|^2 = 2\gamma t|\alpha_0|^2$

A simple calculation then gives

$$\langle\beta|-\beta\rangle = e^{-|\beta|^2}\sum_n \frac{\beta^{*n}(-\beta)^n}{n!} = e^{-|\beta|^2}e^{-|\beta|^2} = e^{-2|\beta|^2}$$

From earlier considerations we must have $\eta = \langle\beta|-\beta\rangle = e^{-2|\beta|^2} \geq 1/10 \rightarrow |\beta| \leq 1$. For times shorter than γ^{-1}, the energy of the first oscillator is

$$E(t) = E(0) - 2\gamma t|\alpha_0|^2\hbar\omega$$

The energy of the second oscillator is

$$E'(t) = \hbar\omega(|\beta(t)|^2 + 1/2) = \hbar\omega/2 + 2\gamma t|\alpha_0|^2\hbar\omega$$

The total energy is conserved: the energy transferred during the time t is

$$\Delta E(t) = 2\gamma t|\alpha_0|^2\hbar\omega = \hbar\omega|(\beta(t)|^2$$

In order to distinguish between a quantum superposition and a statistical mixture, we must have $\Delta E \leq \hbar\omega$. In other words, if a single energy quantum $\hbar\omega$ is transferred, it becomes problematic to tell the difference.

If we return to the numerical example of the pendulum we have the following results: with $1/2\gamma = 1\,year = 3\times 10^{7} s$, the time it takes to reach $|\beta| = 1$ is $(2\gamma|\alpha_0|^2)^{-1} \approx 2\times 10^{-12} s$!

Conclusion

Even for a system as well protected from the environment as we have assumed for the pendulum, the quantum superpositions of macroscopic states are unobservable. After a very short time, all measurements one can make on a system initially prepared in such a state coincide with those made on a statistical mixture. It is therefore not possible, at present, to observe the effects related to the paradoxical character of a macroscopic quantum superposition. However, it is quite possible to observe "mesoscopic" kittens, for systems which have a limited number of degrees of freedom and are well isolated.