A Simple Model of Decoherence in the Measurement of a Spin-1/2 Particle

Robert Nicholas McFarland
rmcfarl1@swarthmore.edu

Swarthmore College Department of Physics and Astronomy
500 College Ave. Swarthmore, PA 19081

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Advisor: John R. Boccio

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Abstract

We discuss the nature and role of the measurement problem in quantum mechanics and present the concept of decoherence as a proposed solution to the measurement problem. We review important conceptual foundations for understanding both the problem and the proposed solution, including axiomatic formulations of quantum theory and the densit operator formalism. We then present a simple yet physically realistic model of a measurement interaction, in which the spin of a spin-1/2 particle is measured using a Stern-Gerlach type device in the presence of an ideal gas environment. The model is solved first in the absence of an environment and then with the bath present. We demonstrate decoherence in the latter case and discuss the numerical predictions, analyzing possible problems in the model or calculation. Finally, we discuss the implications of a decoherence-based view of quantum measurment and consider the impact of such a model on interpretations of quantum mechanics.

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I Introduction

The role of measurement in quantum mechanics (QM) has been the greatest source of conceptual problems associated with the theory. The inability of scientists to understand this issue has caused quantum mechanics to be viewed as "spooky" or evidence of the supernatural by some and unintelligible or incomplete by others. The essential issue with measurement is this: apart from measurement theory, QM asserts that often the exact description of a system with respect to a particular property is not a specific value for that property but a distribution of values. This distribution does not represent epistemic uncertainty, spanning a range of the particular values that *might* specify the system; the distribution as a whole *is* the exact specification of the system. Once this is understood, the measurement problem becomes puzzling. According to empirical evidence, when a system characterized by a distribution of values for some property is "measured" such that one particular value is obtained, the system is thereafter completely characterized by that one value, and not by the distribution. Crudely put, measurement seems to change the system to agree with the result of the measurement.

In order to accommodate this result in quantum theory, an additional axiom is generally required. In addition to the normal time evolution, we must postulate the effect of measurement on a system as a separate process. This results in a number of difficulties. First of all, the process by which this transformation occurs is completely unspecified. The change is simply postulated to occur, and thus is essentially treated as an instantaneous "collapse" from a distribution to a particular value. Secondly, there is generally not a clear specification of what constitutes a measurement. At what point has the critical probe occurred that makes a measurement and thus triggers the instantaneous collapse? What sort of process has to happen to get this effect? Often a measurement process is defined as one which induces such a collapse, but this circularity clearly does not aid out understanding.

Several decades ago, the concept of decoherence was introduced to answer a similar question: why does the macroscopic world not display quantum behavior, if QM is the fundamental theory of the universe? The decoherence idea suggests that quantum interference between macroscopically distinct states is destroyed by interaction with a complex random environment ([1]). Since any large-scale system cannot be isolated from its environment, a large scale object cannot maintain quantum coherence for even the smallest timescales. Microscopic quantum systems, on the other hand, are in general quite well isolated from their environments and thus are able to remain in nonclassical states for relatively long times. Proponents of the decoherence model have suggested that this can account for the measurement phenomenon. In order for a system to be measured, states of the microscopic system (which are not directly accessible) must be coupled to corresponding "pointer" states of a macroscopic system (which are directly observable). Under perfect coupling, the macrostates and the microstates should display the same quantum interference between distinct states. However, the interference between the distinct macroscopic pointer states is very quickly destroyed by environmental effects. Thus the interference between the microscopic states is destroyed as well, leaving the system in a new state characterized by one particular value, which corresponds to the value indicted by the macroscopically observed pointer state.

This formulation of quantum measurement has many advantages. First of all, the process of decoherence via environmental interactions can be explained by the ordinary evolution of a complex quantum system. Thus, there is no need to add an additional axiom ad hoc to explain the phenomenon of measurement. Secondly, the "observer" disappears from the discussion almost entirely. The system must be coupled to a macroscopic, observable pointer so that the interested physicist can see the result and make use of it, but the actual transformation of the quantum state occurs independently of whether the physicist is around or not. No conscious entity is needed, only a complex environment. Thus we eliminate the need to weave a theory of mind into our theory of physics. Furthermore, the decoherence approach places measurement on one end of a spectrum of ordinary many-body interactions: measurements are simply interactions in which the degrees of freedom not under examination (the "environment" or "bath") are sufficiently complex that their interactions with the system of interest cause that system to decohere, allowing particular values of a property to be distinguished. Certain classes of interactions work as measurements, certain classes can be used as imperfect measurements, and certain classes are not useful for measuring at all, but measurements are fundamentally the same type of processes as other quantum processes; they are no longer in a separate category unto themselves.

In this paper, we will attempt to examine the suitability of the decoherence model of measurement by considering a simple, analytically solvable model of a measurement interaction. This approach has been pursued elsewhere (e.g. [2],[3],[4]), typically using harmonic oscillator pointer states and/or oscillator baths. We consider a Stern-Gerlach type interaction, in which the spin of a spin-1/2 particle is coupled to its center of mass coordinate, which is in contact with an ideal gas bath. Using the density operator formalism (as suggested in [5]), we will show the initial coupling between the spin and pointer states, followed by the bath interaction and the resulting exponential decay of the off-diagonal matrix elements that represent quantum interference between states.

II Conceptual Foundations

II.1 Axioms of Quantum Mechanics

If we wish to study foundational principles of quantum theory, we would be well advised to examine the theory in a manner that makes the foundational principles as apparent as possible, i.e. as a formal axiomatic system. In an important sense this question of measurement is a question of whether quantum mechanics as a whole rests on a few clear, logical, and simple postulates or whether a complete description of quantum phenomena requires an additional seemly ad-hoc postulate that bears little conceptual resemblance to the others. The aim of this paper is to investigate the claims that decoherence methods demonstrate that the former case is correct with respect to measurement.

The exact number and form af the axioms of (nonrelativistic) quantum mechanics may vary (see, for example, [6],[7]), but most formulations are similar in the scope and character of the underlying assumptions. The set of axioms we present here is chosen for simplicity and to clearly illustrate the central concepts behind the state vector formulation of quantum mechanics. We state the axioms as follows:

1) To every physical system $\square$ there corresponds a vector (ket) $|\square\rangle$ in a complex linear Hilbert space.

2) To every observable property O of a state there corresponds a linear Hermitian operator $\hat{O}$, and the only values this property can take are the eigenvalues of $\hat{O}$.

3) The time dependence of a state vector $|\square\rangle$ is given by the equation $i\hbar \frac{d}{dt}|\square\rangle = \hat{H}|\square\rangle$ where $\hat{H}$ is the Hamiltonian, the Hermitian operator corresponding to the system's energy.

4) When a state $|\square\rangle$ is measured with respect to a set of states $\{|j\rangle\}$, then with

probability $|\langle j|\psi\rangle|^2$ the result is $|j\rangle$ and the system is transformed into that state.

This formulation is very simplistic and is intended not to provide the most rigorous mathematical framework but to emphasize the central character of the formalism. Qualitatively, the first three axioms achieve the following:

1) Define a mathematical representation of physical states.

2) Define a mathematical representation of observable quantities.

3) Define the time evolution of these mathematical objects.

The fourth axiom has a quite different character from the others. Whereas the first three axioms declare a particular mathematical formalism and stipulate a particular form of (continuous, reversible) time evolution, the last axiom proclaims that in the event of a "measurement" (which is not, in general, defined) the mathematical object representing the system is instantly replaced by another one, apparently without regard to the previous prescription for time evolution. In effect, this axiom has the form "The preceding axiom fails to apply in cases X. In such cases, do Y instead." This rule, though less satisfactory than having a single, consistently applicable formalism, might be acceptable if the cases X were well defined. Instead, such formulations often rely on intuitive everyday notions of "measurement," thus opening the door to all sorts of conceptual and interpretive difficulties in addition to maintaining a fair amount of ambiguity regarding how and when the fourth axiom is to be applied.

Proponents of decoherence methods claim that this fourth axiom is unnecessary, as the results of measurement can be derived from the ordinary unitary evolution described by the third axiom. If this is true, then not only can we reduce the number of axioms (assumptions) on which our theory rests, but we can eliminate the term "measurement" from our formalism altogether, thus sidestepping the conceptual and interpretive problems by moving them beyond the domain of physics.

II.2 The Density Operator Formalism

II.2.1 The Basic Formalism

In standard treatments of QM, a quantum state ψ is represented by a vector $|\psi\rangle$ in complex Hilbert space. It is easy to see that this is a more general representation than the wavefunction, which can be expressed as the projection of this vector onto the position basis

$\psi(x) = \langle x|\psi\rangle$. Because of this, the state vector can be used to represent states that bear no explicit relationship to position, such as spin or photon polarization.

For the problem under consideration, we will work in a third representation, called the density operator formulation, in which a quantum system is represented by a positive definite Hermitian operator of unit trace known as the density operator. We will see that while in many cases the density operator is completely equivalent to the state vector representation, certain classes of states can only be adequately represented in the former and not the latter.

The density operator ρ of a system described by the state vector $|\psi\rangle$ is simply the projection operator $|\psi\rangle\langle\psi|$. We can express ρ as a function in the position representation by simply taking the matrix elements in the position basis $\rho(x,x') = \rho_{x,x'} = \langle x|\psi\rangle\langle\psi|x'\rangle = \psi(x)\psi^*(x')$. Note that in any basis the jth diagonal element $\rho_{jj} = \langle j|\rho|j\rangle = \langle j|\psi\rangle\langle\psi|j\rangle = |\langle j|\psi\rangle|^2 = p(j)$ is the probability that the system will be found in state $|j\rangle$. In general, a density operator has the form $\rho = \sum_i p_i|i\rangle\langle i|$ in some particular basis $\{|i\rangle\}$. In this basis, it is immediately evident that the eigenstates of ρ are just the states $|i\rangle$ and the probabilities p_i are the corresponding eigenvalues. When there is only one term in this sum, we have the single state projector as above, and ρ represents a pure quantum state. In this case, the density operator ρ and the state vector $|\psi\rangle$ contain the same information and thus are completely interchangeable representations of the state. When there is more that one nonzero term in the sum, the state is said to be in a *mixed state* or *statistical mixture*. In a mixed state, the diagonal elements represent purely statistical probabilities and the system cannot be written as a state vector. To see that this is the case, consider the most general form of the state vector of an arbitrary two state system

$$|\psi\rangle = a|+\rangle + e^{i\phi}b|-\rangle$$

where, without loss of generality, a, b, and ϕ are all real. According to the translation outlined above, the density operator for this system is

$$\rho = \begin{pmatrix} a^2 & abe^{-i\phi} \\ abe^{i\phi} & b^2 \end{pmatrix}.$$

Upon inspection, we can easily see that there is no nontrivial choice of a, b, and ϕ that could cause ρ to be diagonal in this basis and still satisfy the requirements of a density operator (in particular the unit trace requirement). The only way to obtain zeros on the off diagonals is to specify that ϕ is completely undetermined, and that we must therefore average over all possible values of ϕ. This averaging turns the complex exponentials to zero, giving us a diagonal matrix. Unfortunately, there is no way to accurately specify ϕ as a "completely undetermined" quantity in a manner that allows for rigorous calculations. Therefore, when we seek to describe statistical mixtures, we must abandon the state vector representation in favor of the density matrix.

What do statistical mixtures have to do with measurement theory? It turns out that a mixed state has exactly the statistical properties of a post-measurement state, with the diagonal elements giving classical probabilities for each result having been obtained and no interference among the outcome states. Empirical evidence suggests that the post measurement state must be invariant under additional measurements of the same kind if the system is not allowed sufficient time to undergo ordinary evolution between measurements. Furthermore, a measurement distinguishes one outcome from all other outcomes so that possible results that do not obtain will have no physical effect on the system after the measurement. Both of the properties hold for diagonal density matrices ([8]). We hypothesize, therefore, that a properly defined measurement process transforms the initial density operator of a system into one that is diagonal in the basis of measurement. Our primary goal in this thesis is to show that a simple model of measurement that includes the effects of a complex environment accounts for this diagonalization using only the normal prescription for the unitary time evolution of a quantum system in contact with a bath containing infinitely many degrees of freedom. Additionally we wish to consider the implications of diagonalization as a sufficient representation of measurement. These implications will be discussed in section IV.2.

We can easily derive the (Schrödinger picture) time dependence of ρ from the time dependence of the state vector:

$$\begin{aligned}
\dot{\rho}(t) &= \frac{d}{dt}\left|\psi(t)\right\rangle\left\langle\psi(t)\right| = \frac{d}{dt}\left(e^{-\frac{i}{\hbar}\hat{H}t}\left|\psi(0)\right\rangle\left\langle\psi(0)\right| e^{\frac{i}{\hbar}\hat{H}t} \right) \\
&= -\frac{i}{\hbar}\hat{H} e^{-\frac{i}{\hbar}\hat{H}t}\left|\psi(0)\right\rangle\left\langle\psi(0)\right| e^{\frac{i}{\hbar}\hat{H}t} + e^{-\frac{i}{\hbar}\hat{H}t}\left|\psi(0)\right\rangle\left\langle\psi(0)\right| e^{\frac{i}{\hbar}\hat{H}t}\,\frac{i}{\hbar}\hat{H} \\
&= -\frac{i}{\hbar}\left[\hat{H},\rho(t)\right].
\end{aligned}$$

In the interaction picture, which we will use heavily in this analysis, the formula for $\dot{\rho}(t)$ is nearly identical: we merely replace the total Hamiltonian with the interaction term $\hat{V}$. In general, this equation of motion cannot be solved exactly, and we will need to use time-dependent perturbation theory to expand the commutator to second order to obtain a closed-form solution. The details of this approach will be covered in section III.2.

II.2.2 The Reduced Density Operator

Another advantage of the density operator formalism is its ability to handle complex systems comprising two or more subsystems. For simple product states, the density operator of the joint system mirrors the joint state vector

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \Rightarrow \rho_{total} = \rho_A \otimes \rho_B$$

as can be verified by inspection. Likewise we can determine the density operator of entangled pure states by using the outer product rule on the corresponding joint state vector

$$|\psi\rangle = \left(|\alpha_1\rangle|\beta_1\rangle + |\alpha_2\rangle|\beta_2\rangle\right)/\sqrt{2} \Rightarrow \rho_{total} = |\psi\rangle\langle\psi| \neq \rho_\alpha \otimes \rho_\beta .$$

Note that in this case, just as the joint ket cannot be factored as a direct product of two subsystem kets, the density operator will not factor into a direct product of subsystem density operators.

The real power of the density operator arises when we wish to know the properties of one subsystem independent of its connections to the rest of the joint system. In this case we 'remove' the rest of the system by tracing over it, creating a new object called the *reduced density operator* of the first system

$$\rho_A^{(R)} = Tr_B \rho_{total} .$$

If ρ_{total} is simply a direct product state, then the trace removes the second system and returns the first system as a pure state

$$Tr_B\left(\rho_{total}\right) = Tr_B\left(\rho_A \otimes \rho_B\right) = \rho_A .$$

However, this is not true in general. To illustrate the use and significance of reduced density operators, let us consider a few simple examples of density matrices for a system composed of two sin-1/2 particles. First, we consider the direct product state

$$|\psi\rangle = \left(|+\rangle + |-\rangle\right)/\sqrt{2} \otimes \left(|\uparrow\rangle + |\downarrow\rangle\right)/\sqrt{2}$$
$$\Rightarrow \rho_{total} = 1/4\left(|+\rangle|\uparrow\rangle + |-\rangle|\uparrow\rangle + |+\rangle|\downarrow\rangle + |-\rangle|\downarrow\rangle\right)\left(\langle +|\langle\uparrow| + \langle -|\langle\uparrow| + \langle +|\langle\downarrow| + \langle -|\langle\downarrow|\right)$$

where $+/-$ refer to the spin of the first particle and $\uparrow/\downarrow$ refer to the spin of the second. In matrix form, we can write this as

$$\rho_{total} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}_{(|\pm\rangle|\uparrow\downarrow\rangle)} .$$

Tracing over the spin states of the second particle, we obtain the reduced density matrix for the first particle

$$\rho_A^{(R)} = Tr_B\left(\rho_{total}\right) = \sum_{\sigma=\uparrow\downarrow} \langle\sigma|\rho_{total}|\sigma\rangle = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{(|\pm\rangle)}$$

which we recognize (by inspection or by finding the single eigenvector) as the density matrix of the pure quantum state

$$|\psi\rangle = \left(|+\rangle + |-\rangle\right)/\sqrt{2}\,.$$

Similarly, the reduced density operator of the second system is

$$\rho_B^{(R)} = Tr_A\left(\rho_{total}\right) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{(|\uparrow\downarrow\rangle)}$$

and the product of the two reduced density operators is the total density operator

$$\rho_A^{(R)} \otimes \rho_B^{(R)} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{(|\pm\rangle)} \otimes \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{(|\uparrow\downarrow\rangle)} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}_{(|\pm\rangle|\uparrow\downarrow\rangle)} = \rho_{total} .$$

Thus in this case the reduced density operators taken together completely describe the total system. In contrast, let us consider the density operator of a Bell state:

$$|\Phi\rangle = \left(|+\rangle|\uparrow\rangle + |-\rangle|\downarrow\rangle\right)/\sqrt{2}$$
$$\Rightarrow \rho_{total} = 1/2\left(|+\rangle|\uparrow\rangle + |-\rangle|\downarrow\rangle\right)\left(\langle+|\langle\uparrow| + \langle-|\langle\downarrow|\right).$$

The matrix form of this density operator is

$$\rho_{total} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}_{(|\pm\rangle|\uparrow\downarrow\rangle)}$$

and the reduced density matrix of the first particle is

$$\rho_A^{(R)} = Tr_B\left(\rho_{total}\right) = \sum_{\sigma=\uparrow\downarrow} \langle\sigma|\rho_{total}|\sigma\rangle = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{(|\pm\rangle)}$$

which is the density matrix of a statistical mixture of the states

$$\left(|+\rangle + |-\rangle\right)/\sqrt{2} \quad \text{and} \quad \left(|+\rangle - |-\rangle\right)/\sqrt{2}.$$

Similarly, the reduced density operator of the second system is

$$\rho_B^{(R)} = Tr_A\left(\rho_{total}\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{(|\uparrow\downarrow\rangle)}$$

but the product of the two reduced density operators does not produce the total density operator

$$\rho_A^{(R)} \otimes \rho_B^{(R)} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{(|\pm\rangle)} \otimes \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{(|\uparrow\downarrow\rangle)} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{(|\pm\rangle|\uparrow\downarrow\rangle)} \neq \rho_{total}.$$

Instead, the product represents a statistical mixture, implying that the total system is not in a pure quantum state, which we know is false. In this case the reduced density operators taken together do not completely describe the total system. In order to completely characterize the joint system, we must consider the correlations between the two subsystems which cannot be found by examining the subsystems separately.

Finally, let us consider a third case, somewhere between these two examples. We consider the following state

$$|\tilde{\Psi}\rangle = \left(\sqrt{1-\varepsilon^2}\,|+\rangle|\uparrow\rangle + \varepsilon|-\rangle|\uparrow\rangle + \varepsilon|+\rangle|\downarrow\rangle + \sqrt{1-\varepsilon^2}\,|-\rangle|\downarrow\rangle\right)/\sqrt{2}$$

where, without loss of generality, $\varepsilon^2 < 1/2$. It is clear that as $\varepsilon^2 \to 1/2$ $|\tilde{\Psi}\rangle$ approaches the direct product state $|\Psi\rangle$, and as $\varepsilon^2 \to 0$ $|\tilde{\Psi}\rangle$ approaches the Bell state $|\Phi^+\rangle$. The density matrix for this system is

$$\rho_{total} = \frac{1}{2}\begin{pmatrix} 1-\varepsilon^2 & \tilde{\varepsilon} & \tilde{\varepsilon} & 1-\varepsilon^2 \\ \tilde{\varepsilon} & \varepsilon^2 & \varepsilon^2 & \tilde{\varepsilon} \\ \tilde{\varepsilon} & \varepsilon^2 & \varepsilon^2 & \tilde{\varepsilon} \\ 1-\varepsilon^2 & \tilde{\varepsilon} & \tilde{\varepsilon} & 1-\varepsilon^2 \end{pmatrix}_{(|\pm\rangle|\uparrow\downarrow\rangle)}$$

where $\gamma \equiv \alpha\sqrt{1-\alpha^2} > \alpha^2$. Again, we can see that this approaches the other two example systems in the appropriate limits, but what can we say about this system for arbitrary α? Taking the reduced density matrix of the first particle, we get

$$\rho_A^{(R)} = Tr_B\left(\rho_{total}\right) = \frac{1}{2}\begin{pmatrix} 1 & 2\gamma \\ 2\gamma & 1 \end{pmatrix}_{(|\pm\rangle)}.$$

What does this matrix represent? We calculate the eigenstates of this operator to find

$$\left|\tilde{\phi}_1\right\rangle = \left(\left|+\right\rangle + \left|-\right\rangle\right)/\sqrt{2} \quad \text{with eigenvalue (probability)} \quad p_1 = \frac{1}{2}\left(1 + 2\alpha\sqrt{1-\alpha^2}\right)$$

$$\left|\tilde{\phi}_2\right\rangle = \left(\left|+\right\rangle - \left|-\right\rangle\right)/\sqrt{2} \quad \text{with eigenvalue (probability)} \quad p_2 = \frac{1}{2}\left(1 - 2\alpha\sqrt{1-\alpha^2}\right).$$

Thus as $\alpha^2 \to 1/2$ the second eigenvalue vanishes, and the density matrix describes the pure state $\left|\tilde{\phi}_1\right\rangle$. Likewise, as $\alpha^2 \to 0$ the eigenvalues become equal, and we have a completely mixed state. It is worth noting here that unless $\alpha = 0$ the reduced density operator has more than one nonzero eigenstate, and thus the subsystem appears to be in a mixed state which cannot be expressed as a ket. This appearance, however, is still only an artifact of our decision to ignore the correlations between system A and system B. The joint system remains in a pure quantum state.

Reduced density operators are important in measurement theory because they enable us to partially describe a system under observation apart from the complex environment with which it interacts. In the simplest form, a measurement consists of a system to be measured, the pointer of some measuring device, and an environment which interact in such a way as to entangle them (similar treatments found in [9]):

$$\left|S\right\rangle\left|P\right\rangle\left|E\right\rangle = \left(\sum_i \left|s_i\right\rangle\right)\left(\sum_j \left|p_j\right\rangle\right)\left(\sum_k \left|e_k\right\rangle\right) \to \sum_k \left|s_k\right\rangle\left|p_k\right\rangle\left|e_k\right\rangle.$$

At this point we have a state that is very much like a Bell state, but the environment system is so large and complex that we cannot completely describe its exact state. We therefore must trace (average) over all its degrees of freedom to obtain a reduced density operator for the system-pointer subsystem. Just as tracing over one particle in our Bell state example resulted in an apparent diagonalization of the density operator of the other particle, in this situation, the reduced density operator for the system-pointer will be that of a mixed state.

$$\rho_{sp}^{(R)} = Tr_E\left[\left(\sum_k |s_k\rangle|p_k\rangle|e_k\rangle\right)\left(\sum_j \langle s_j|\langle p_j|\langle e_j|\right)\right] = \sum_k |s_k\rangle|p_k\rangle\langle s_k|\langle p_k|.$$

When we just look at the reduced system apart from the environment, we find a statistical mixture of system-state/pointer-state pairs, with no quantum interference between them. This is what we expect from a measurement interaction. However, the total quantum system (system, pointer, and environment) is still in a pure quantum state. The appearance of a mixed state results from our ignoring the complex correlations that exist between the system-pointer and the rest of the universe. In the remainder of this thesis, we will attempt to account for this process of entanglement and diagonalization and determine whether this simple model of measurement holds for a physically realistic system.

III The Model and Its Solution

III.1 Pointer-system interaction

Before we examine the full system-pointer-bath problem, we will first consider the coupling between the system of interest and the observable quantum state that will serve as the pointer for the measurement interaction in the absence of any environmental effects. We consider a system in which the spin of a particle is coupled to the center of mass coordinate of some pointer particle, which may be the same particle. Such coupling can be realized in a number of different ways, the most common being the use of a nonhomogeneous magnetic field in a Stern-Gerlach apparatus, but for our purposes the particular implementation is irrelevant. The Hamiltonian of the total system (from [10], where it is used for a completely different purpose) is given by

$$\hat{H} = \frac{\hat{P}^2}{2M} + V(t)\hat{P}_x\sigma_z$$

where $P^2/2m$ is the pointer particle's kinetic energy and V(t) is an arbitrary (possibly) time-dependent coupling factor with units of velocity. We will assume that V(t) is well-behaved so that $\Phi(t) \equiv \int_0^t V(t')dt' \to 0$ as $t \to 0$. In the momentum representation, the Schrödinger equation for this system is

$$\frac{\hat{P}^2}{2M}\Phi(\vec{p},\sigma,t) + V(t)\hat{P}_x\hat{\sigma}_z\Phi(\vec{p},\sigma,t) = i\hbar\frac{\partial}{\partial t}\Phi(\vec{p},\sigma,t).$$

We stipulate that initially the pointer is in a neutral state, centered at the origin, that is,

$$\phi(\vec{r},0) = \left(\frac{1}{2\pi\Delta^2}\right)^{3/4} e^{-\frac{r^2}{4\Delta^2}}$$

for the pointer and the spin is initially in some arbitrary superposition of spin states

$$\left|\chi_z(0)\right\rangle = a_+ e^{i\theta}\left|+\right\rangle + a_-\left|-\right\rangle$$

(where, w.l.o.g., a_+, a_- and θ are positive and real) so that the total initial state is

$$\Psi(\vec{r},\sigma,0) = \left(\frac{1}{2\pi\Delta^2}\right)^{3/4} e^{-\frac{r^2}{4\Delta^2}}\left(a_+e^{i\theta}\chi_+ + a_-\chi_-\right).$$

where $\chi_\pm(\sigma) \equiv \left\langle\sigma\middle|\pm\right\rangle$ is the 'wavefunction' associated with the spin ket $\left|\pm\right\rangle$. Taking the Fourier transform of this expression, we find the initial state in the momentum representation

$$\Phi(\vec{k},\sigma,0) = \left(2\pi\right)^{3/2}\left(2\pi\right)^{3/4} e^{-k^2\Delta^2}\left(a_+e^{i\theta}\chi_+ + a_-\chi_-\right) \equiv N e^{-k^2\Delta^2}\left(a_+e^{i\theta}\chi_+ + a_-\chi_-\right).$$

We then solve the Schrödinger equation for the time dependence

$$\Phi(\vec{k},\sigma,t) = \Phi(\vec{k},\sigma,0)U(t) = \Phi(\vec{k},\sigma,0)e^{-i\nu k^2 t}e^{-ik_x\sigma_z\xi(t)}$$

(where $\nu = \hbar/2M$) and thus find the momentum-space wave function

$$\Phi(\vec{k},\sigma,t) = N e^{-k^2\Delta^2}\left(a_+e^{i\theta}\chi_+ + a_-\chi_-\right)e^{-i\nu k^2 t}e^{-ik_x\sigma_z\xi(t)}.$$

Making the substitutions

$$\alpha = \Delta^2 + i\nu t, \quad \tau = \Delta^2/\nu, \quad \text{and} \quad \eta(t) = 1 + it/\tau$$

and taking the inverse Fourier transform, we obtain the position-space wavefunction

$$\Psi(\vec{r},\sigma,t) = \frac{N}{8\pi^{3/2}}\alpha^{-3/2}e^{-\frac{\left(\vec{r}-\sigma_z\xi(t)\hat{e}_x\right)^2}{4\Delta^2\eta(t)}}\left(a_+e^{i\theta}\chi_+ + a_-\chi_-\right)$$

$$= \frac{N}{8\pi^{3/2}}\alpha^{-3/2}\left[a_+e^{i\theta}\chi_+e^{-\frac{\left(\vec{r}-\vec{\xi}\right)^2}{4\Delta^2\eta(t)}} + a_-\chi_-e^{-\frac{\left(\vec{r}+\vec{\xi}\right)^2}{4\Delta^2\eta(t)}}\right]$$

where $\vec{\xi} \equiv \xi\cdot\hat{e}_x$. From the wavefunction, we can easily find the elements of the density matrix

$$\rho(r,\sigma_z;r',\sigma_z';t) = \Psi(\vec{r},\sigma_z,t)\Psi^*(\vec{r}',\sigma_z',t) = \frac{N^2}{(4\pi)^3}\left(\Delta^4+\nu^2t^2\right)^{-3/2}a_\sigma a_{\sigma'}\left[e^{-\frac{\left(\vec{r}-\sigma_z\xi(t)\hat{e}_x\right)^2}{4\Delta^2\eta(t)}-\frac{\left(\vec{r}'-\sigma_z'\xi(t)\hat{e}_x\right)^2}{4\Delta^2\eta^*(t)}}\right]$$

Finally, by tracing over the spins and considering the diagonal elements ($r = r\square$), we obtain an expression for the probability density of the pointer in position space:

$$\Pr(\vec{r},t) = \frac{\square^2}{(4\square)^3}\left(\square^4 + \square^2 t^2\right)^{\square 3/2} \square$$

$$\left[a_+^{\;2} e^{\square\frac{(\vec{r}\square\vec{\square})^2}{2\square^2\square}} + a_\square^{\;2} e^{\square\frac{(\vec{r}+\vec{\square})^2}{2\square^2\square}} + 2a_+a_\square e^{\square\frac{(r^2\square\vec{\square}^2)}{2\square^2\square}} \cos\left[\square/\square\frac{\vec{\square}xt}{\square(\square^2+t^2)}\right]\right]$$

where $\square = \square(t)\square^{\square}(t) = 1 + (t/\square)^2$.

There are several key features to notice in this expression. First, we note the time dependent part of the coefficient clearly results in a broadening of the entire wavepacket over time. This broadening is a standard feature of the time evolution of any wavepacket governed by the Schrödinger equation. The first two terms in the brackets are Gaussians displaced in the positive and negative x-direction, respectively, with relative magnitudes determined by the amplitudes of the spin up/spin down components of the original spin state. This represents the coupling between the pointer position and the spin state, forming an entangled state. The particle's position is now strongly (though not completely) correlated to the z component of its spin. The final term represents the interference between the two wavepackets, indicating that the system remains in a superposition of the two states. It is this term that we expect to vanish due to decoherence in the full calculation, indicating a complete separation of the two distinct measurement outcomes.

III.2 The Master Equation

Having determined the "bathless" behavior of the spin-pointer system, we now examine the effects of the environment on the measurement process. We assume that the interaction potential is weak compared to the non-interaction energies and that, as a result, the spin does not directly interact with the bath. We note from our previous result that the spin will remain coupled to the pointer throughout the interaction. Thus, the spin evolution is entirely determined by the evolution of the pointer and therefore we need only consider the evolution of the pointer as a result of the environmental effects. We will begin by deriving the equation of motion for a general pointer-bath system and then refine this equation by adding in the particular characteristics of our interaction. Throughout this work we will attempt to be as general as possible, narrowing our scope to the particular model of interest only as it becomes necessary to make further progress. Our goal is to derive an equation of motion for the reduced density operator of the pointer, obtained by tracing out the environmental degrees of freedom and thus looking at the pointer alone, independent of its entanglement to the bath. Such an equation is called a *master equation*. The derivation presented here closely follows that found in [11], though other derivations of master equations appear in [12],[13].

The weak interaction assumption also enables us to use time-dependent perturbation theory to calculate the dynamic of the pointer-bath system. In the interaction picture, we have the equation of motion for the total density operator

$$\dot{\rho}(t) = \frac{-i}{\hbar}\left[\hat{V}(t)\ ,\ \rho(t)\right].$$

From this we can describe the evolution of ρ in the time interval $(t_0, t_0+\Delta t)$ by

$$\Delta\rho(t_0) = \rho(t_0+\Delta t) - \rho(t_0) = \int_{t_0}^{t_0+\Delta t} \frac{d\rho}{dt'}dt' = \frac{-i}{\hbar}\int_{t_0}^{t_0+\Delta t}\left[\hat{V}(t')\ ,\ \rho(t')\right]dt'$$

which can be iterated twice to give a second-order expression for $\Delta\rho$

$$\Delta\rho(t_0) = \frac{-i}{\hbar}\int_{t_0}^{t_0+\Delta t}\left[\hat{V}(t')\ ,\ \rho(t_0)\right]dt' - \frac{1}{\hbar^2}\int_{t_0}^{t_0+\Delta t}\int_{t_0}^{t'}\left[\hat{V}(t')\ ,\ \left[\hat{V}(t'')\ ,\ \rho(t'')\right]\right]\ dt''dt'.$$

We are concerned with the behavior of the pointer apart from the bath. Thus we need to trace out over the bath degrees of freedom to obtain the reduced density matrix of the pointer

$$\sigma(t) \equiv Tr_B\left(\rho(t)\right).$$

Similarly we will denote the reduced density matrix of the bath with the subscript B

$$\sigma_B(t) = Tr_P\left(\rho(t)\right).$$

We may now express the time-development equation above in terms of σ:

$$\Delta\sigma(t) = \frac{-i}{\hbar}\int_t^{t+\Delta t} Tr_B\left[\hat{V}(t')\ ,\ \rho(t')\right]dt' - \frac{1}{\hbar^2}\int_t^{t+\Delta t}\int_t^{t'} Tr_B\left[\hat{V}(t')\ ,\ \left[\hat{V}(t'')\ ,\ \rho(t'')\right]\right]\ dt''dt'.$$

III.2.1 Assumptions and Approximations

At this point, we introduce several assumptions about the environment and the relevant timescales of the interaction. These assumptions are quite general and for the most part model-independent.

Assumption 0: *Changes in* ρ *on the interval (t, t'') are negligible.* $\rho(t) = \rho(t'')$

This assumption is implicit in the discussion to stop the iteration above at second order. If Δt is small compared to the evolution time of $\rho(t)$, we can assume that the second-order change from $\rho(t)$ to $\rho(t'')$ is small.

Assumption 1: $\rho_B(t) = \rho_B$ ***is constant on the timescales of interest.***

This assumption can be made because a bath, by definition, is not significantly affected by interaction with the pointer, and we are ignoring interactions between the bath and *its* environment.

Assumption 2: ***The bath is in a stationary state.*** $\left[\hat{H}_B, \rho_B\right] = 0$**.**

This is equivalent to the assumption that the system is in a classical state of thermal equilibrium. It implies that, if $\left\{\left|\phi_i\right\rangle\right\}$ are the energy eigenstates,

$$0 = \left\langle\phi_i\right|\left[\hat{H}_B, \rho_B\right]\left|\phi_j\right\rangle = \left\langle\phi_i\right|\hat{H}_B\rho_B\left|\phi_j\right\rangle - \left\langle\phi_i\right|\rho_B\hat{H}_B\left|\phi_j\right\rangle = E_i\left\langle\phi_i\right|\rho_B\left|\phi_j\right\rangle - E_j\left\langle\phi_i\right|\rho_B\left|\phi_j\right\rangle$$
$$= (E_i - E_j)\left\langle\phi_i\right|\rho_B\left|\phi_j\right\rangle \Rightarrow \left\langle\phi_i\right|\rho_B\left|\phi_j\right\rangle = E_j\delta_{ij}.$$

Thus, we do not require that the whole bath be in a pure energy eigenstate (an unreasonable assumption) but merely that it be a mixture of energy eigenstates. This is reasonable because the bath is a very large system that does in fact interact with its environment, and thus would very rapidly evolve into such a mixture on a timescale much shorter than those of interest. The statistical distribution of this mixture is given by the ordinary partition function. Thus:

$$\rho_B = \sum_\alpha p_\alpha \left|\alpha\right\rangle\left\langle\alpha\right| = \frac{1}{Z}\sum_\alpha e^{-E_\alpha\beta}\left|\alpha\right\rangle\left\langle\alpha\right|$$

where

$$Z = \sum_\alpha e^{-E_\alpha\beta}$$

is the partition function of statistical mechanics.

Assumption 3: *The interaction takes the form* $\hat{V}(t) = -\sum_j \hat{P}_j(t)\hat{B}_j(t)$*, where P and B are observables of the pointer and bath, respectively.*

We will see that this does in fact hold for our system. For this derivation we will consider the sum to contain only one term, but the result will apply to the more general case.

Assumption 4: *The expectation value of any bath observable B averages to zero on timescales of interest.* $\langle \hat{B}(t) \rangle = 0$.

We can set the zero-point of the potential so that this is true at t=0, and since the bath is in a stationary state, the expectation values will not change. This eliminates the single integral in the expression for $\rho(t)$ above. This is equivalent to assuming that $\langle \hat{V}(t) \rangle_B = 0$ on the bath since

$$\langle \hat{V}(t) \rangle_B = Tr_B\left(\lambda \hat{P}(t)\hat{B}(t)\rho_B\right) = \lambda \hat{P}(t) Tr_B\left(\hat{B}(t)\rho_B\right) = \lambda \hat{P}(t)\langle \hat{B}(t) \rangle$$

It will be useful to consider the quantity $g(t',t'') = \langle \hat{B}(t')\hat{B}(t'') \rangle$, which is the *two time average* or *correlation function* of the observable B. We can easily see that

$$\langle \hat{B}(t')\hat{B}(t'') \rangle = Tr_B\left[\rho_B e^{\frac{i}{\hbar}\hat{H}_B t'}\hat{B}e^{\frac{-i}{\hbar}\hat{H}_B t'}e^{\frac{i}{\hbar}\hat{H}_B t''}\hat{B}e^{\frac{-i}{\hbar}\hat{H}_B t''}\right] = Tr_B\left(\rho_B \hat{B}(\tau)\hat{B}(0)\right) = g(\tau)$$

where $\tau = t' - t''$. Furthermore

$$g(\tau) = \langle \hat{B}(\tau)\hat{B}(0) \rangle = \sum_n p_n \langle n|\hat{B}(\tau)\hat{B}(0)|n\rangle = \sum_n \sum_m p_n \langle n|\hat{B}(\tau)|m\rangle\langle m|\hat{B}(0)|n\rangle$$

$$= \sum_n \sum_m p_n \langle n|e^{\frac{i}{\hbar}\hat{H}_B \tau}\hat{B}(0)e^{\frac{-i}{\hbar}\hat{H}_B \tau}|m\rangle\langle m|\hat{B}(0)|n\rangle = \sum_n \sum_m p_n e^{i\omega_n \tau}e^{-i\omega_m \tau}\left|\langle n|\hat{B}(0)|m\rangle\right|^2$$

$$= \sum_n \sum_m p_n e^{i\omega_{nm}\tau}\left|B_{nm}\right|^2$$

where $B_{nm} = \langle n|\hat{B}(0)|m\rangle$, $\omega_n = E_n / \hbar$, and $\omega_{nm} = \omega_n - \omega_m$. Because the energy levels of the bath are dense, the exponentials in g(t) will interfere destructively for large t. We define t_c to be the characteristic width of this decay in t. Thus over timescales larger than t_c the bath properties are completely uncorrelated.

Assumption 5: *Correlations between the pointer and bath are damped out on timescales of* Δt. *Thus* $\rho(t) = \sigma(t) \otimes \rho_B + \rho_{correl.}(t) \approx \sigma(t) \otimes \rho_B$.

Although in general entangled evolution produces correlations that render such factorization impossible, here such correlations average to zero on timescales larger than t_c. If $t_c << \Delta t$, these correlations can be neglected.

Taking all of these assumption into account, we obtain the expression

$$\frac{\Delta \rho(t)}{\Delta t} = -\frac{1}{\hbar^2 \Delta t} \int_t^{t+\Delta t} \int_t^{t'} Tr_B\left[\hat{V}(t'),\left[\hat{V}(t''),\rho(t) \otimes \rho_B\right]\right] dt'' dt'.$$

Changing variables to t and t' ($t' \to t' - t'' \equiv \tau$) so that

$$\int_t^{t+\Delta t} dt' \int_t^{t'} dt'' = \int_0^{\Delta t} d\tau \int_{t+\tau}^{t+\Delta t} dt'$$

gives

$$\frac{\Delta \rho(t)}{\Delta t} = -\frac{1}{\hbar^2 \Delta t} \int_0^{\Delta t} \int_{t+\tau}^{t+\Delta t} Tr_B\left[\hat{V}(t'),\left[\hat{V}(t'-\tau),\rho(t) \otimes \rho_B\right]\right] dt' d\tau$$

Assumption 6: *The integrand above is negligible for* $t > \Delta t$ and $t < t' < \tau$.
The time dependence of the traced-over commutator comes from g(t), which, as we have already seen, vanishes on timescales that are small compared Δt, so that extending the integral over t to infinity makes only a negligible change. In fact, since $t_c << \Delta t$ the second integral can be extended to (t,t+Δt), also with negligible error.

At this point we must say something about the form of our interaction potential. We assume that the environment-pointer potential can be expressed as a sum of radially-dependent particle-pointer interactions. Such a potential can, in turn, be expressed as the product of collision-momentum dependent operators of the form specified in assumption 3

$$\hat{V} = \sum_j \mathcal{V}(\vec{R} - \vec{r}_j) = \sum_j \left[\frac{1}{v} \sum_{\vec{q}} e^{i\vec{q}\cdot(\vec{R}-\vec{r}_j)} \mathcal{V}(\vec{q})\right] = \frac{1}{v} \sum_{\vec{q}} e^{i\vec{q}\cdot\vec{R}} \left(\sum_j e^{-i\vec{q}\cdot\vec{r}_j}\right) \mathcal{V}(\vec{q})$$
$$= \frac{1}{v} \sum_{\vec{q}} A_{\vec{q}}\left(n(\vec{q})\right) \mathcal{V}(\vec{q}).$$

where we have defined $A_{\vec{q}} \equiv e^{i\vec{q}\cdot\vec{R}}$ and $n(\vec{q}) \equiv \sum_j e^{-i\vec{q}\cdot\vec{r}_j}$. The time dependence of this potential is thus

$$\hat{V}(t) = e^{\frac{i}{\hbar}\hat{H}_B t}\hat{V}e^{-\frac{i}{\hbar}\hat{H}_B t} = \frac{1}{v}\sum_{\vec{q}} e^{\frac{i}{\hbar}\hat{H}_B t}A_{\vec{q}}e^{-\frac{i}{\hbar}\hat{H}_B t}\left(e^{\frac{i}{\hbar}\hat{H}_B t}n(\vec{q})e^{-\frac{i}{\hbar}\hat{H}_B t}\right)\left(e^{\frac{i}{\hbar}\hat{H}_B t}\gamma(\vec{q})e^{-\frac{i}{\hbar}\hat{H}_B t}\right)$$

$$= \frac{1}{v}\sum_{\vec{q}} A_{\vec{q}}(t)n(\vec{q},t)\gamma(\vec{q})$$

(since $\gamma(\vec{q},t) = \gamma(\vec{q},0) = \gamma(\vec{q})$). Altogether, this gives the equation

$$\frac{\Delta\rho(t)}{\Delta t} = -\frac{1}{\hbar^2\Delta t}\int_0 d\tau\int_t^{t+\Delta t} dt'\, Tr_B\left[\hat{V}(t'),\left[\hat{V}(t'-\tau),\rho(t)\otimes\rho_B\right]\right]$$

$$= -\frac{1}{\hbar^2\Delta t}\int_0 d\tau\int_t^{t+\Delta t} dt' \times$$

$$Tr_B\left[\frac{1}{v}\sum_{\vec{q}}\gamma(\vec{q})\cdot n(\vec{q},t')A_q(t'),\left[\frac{1}{v}\sum_{q'}\gamma(\vec{q}')\cdot n(\vec{q}',t'-\tau)A_{q'}(t'-\tau),\rho(t)\otimes\rho_B\right]\right]$$

Expanding the commutators and taking the trace over the bath, we get

$$\frac{\Delta\rho(t)}{\Delta t} = -\frac{1}{\hbar^2\Delta t}\int_0 d\tau\int_t^{t+\Delta t} dt'\frac{1}{v^2}\sum_{\vec{q}}\gamma(\vec{q})\sum_{q'}\gamma(\vec{q}')\times\Big[A_q(t')A_{q'}(t'-\tau)\rho(t)\left\langle n(\vec{q},t')n(\vec{q}',t'-\tau)\right\rangle$$
$$-A_q(t')\rho(t)A_{q'}(t'-\tau)\left\langle n(\vec{q}',t'-\tau)n(\vec{q},t')\right\rangle - A_{q'}(t'-\tau)\rho(t)A_q(t')\left\langle n(\vec{q},t')n(\vec{q}',t'-\tau)\right\rangle$$
$$+\rho(t)A_{q'}(t'-\tau)A_q(t')\left\langle n(\vec{q}',t'-\tau)n(\vec{q},t')\right\rangle\Big]$$

Finally, we note that

$$\left\langle n(\vec{q},t')n(\vec{q}',t'-\tau)\right\rangle = g_{\vec{q}}(\tau)\delta_{-\vec{q}\vec{q}'} \qquad \text{and} \qquad \left\langle n(\vec{q}',t'-\tau)n(\vec{q},t')\right\rangle = g_{\vec{q}}(-\tau)\delta_{-\vec{q}\vec{q}'}$$

and obtain the equation of motion

$$\frac{\Delta\rho(t)}{\Delta t} = -\frac{1}{v^2\hbar^2}\int_0\frac{d\tau}{\Delta t}\int_t^{t+\Delta t} dt'\sum_{\vec{q}}\left|\gamma(\vec{q})\right|^2\times\Big[g_{\vec{q}}(\tau)\left(A_q(t')A_{-q}(t'-\tau)\rho(t) - A_{-q}(t'-\tau)\rho(t)A_q(t')\right)+$$
$$g_{-\vec{q}}(-\tau)\left(\rho(t)A_{-q}(t'-\tau)A_q(t') - A_q(t')\rho(t)A_{-q}(t'-\tau)\right)\Big].$$

Since this equation is only valid for $\Delta t >> t_c$ this is not a true differential relation. This assumption that this equation gives the full time dependence of $\rho(t)$ (i.e. that $\Delta\rho(t)/\Delta t \approx \partial\rho(t)$) is called a *coarse-grained* approximation since it neglects the fine scale time dependence.

III.3 The C-Number Equation

The equation above is an operator-valued equation, which is not easy to deal with. We now wish to write the master equation as a c-number equation, since there are many well-known techniques for solving c-number differential equations. The treatment here presents the key points of the derivation. A more detailed derivation may be found in Appendix A.

Surrounding the operator equation with arbitrary momentum-state bras and kets, we get an equation for the matrix elements of the density operator.

$$\frac{\partial\rho_{kk'}}{\partial t}=\langle k|\frac{\partial\rho}{\partial t}|k'\rangle=\frac{-1}{v^2\hbar^2}\int_0 \frac{d\tau}{\Delta t}\int_t^{t+\Delta t} dt'\sum_{\vec{q}}|\phi(\vec{q})|^2\times[g_{\vec{q}}(\tau)\left(\langle k|A_{\vec{q}}(t')A_{-\vec{q}}(t'-\tau)\rho(t)|k'\rangle-\langle k|A_{-\vec{q}}(t'-\tau)\rho(t)A_{\vec{q}}(t')|k'\rangle\right)$$
$$+g_{-\vec{q}}(-\tau)\left(\langle k|\rho(t)A_{-\vec{q}}(t'-\tau)A_{\vec{q}}(t')\rho(t)|k'\rangle-\langle k|A_{\vec{q}}(t')\rho(t)A_{-\vec{q}}(t'-\tau)\rho(t)|k'\rangle\right)].$$

For the A's the matrix elements are

$$\langle l|A_{\vec{q}}(t')|m\rangle=e^{i\omega_{kk'}t'}\delta_{\vec{k}\vec{k}_-}\qquad\text{and}\qquad\langle l|A_{-\vec{q}}(t'-\tau)|m\rangle=e^{-i\omega_{kk'}(t'-\tau)}\delta_{\vec{k}\vec{k}_+}$$

where $\vec{k}_\pm=\vec{k}\pm\vec{q}$ and $\omega_{kk'}=\omega_k-\omega_{k'}=\left(E_k-E_{k'}\right)/\hbar=(k^2-k'^2)\hbar/2M$.

We then integrate with respect to t' and calculate the two-time averages $g_{\vec{q}}(\tau)=\left\langle n_{\vec{q}}(\tau)n_{-\vec{q}}(0)\right\rangle$ by first finding $n_{\vec{q}}(\tau)$, which we note is a (momentum state) number density operator

$$n_{\vec{q}}(\tau)=\sum_{\vec{p}}\hat{a}^\dagger_{\vec{p}-\vec{q}}\hat{a}_{\vec{p}}e^{-i\tilde{\omega}_{\vec{p}\vec{p}_-}\tau}.$$

Thus

$$g_{\vec{q}}(\tau)=\sum_{\vec{p}}e^{-i\tilde{\omega}_{\vec{p}\vec{p}_-}\tau}f_{\vec{p}_-}\left(1+f_{\vec{p}}\right)\qquad\text{and}\qquad g_{-\vec{q}}(-\tau)=\sum_{\vec{p}}e^{i\tilde{\omega}_{\vec{p}\vec{p}_+}\tau}f_{\vec{p}_+}\left(1+f_{\vec{p}}\right)$$

where $f_{\vec{p}}\equiv\left\langle\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}}\right\rangle$ and $\tilde{\omega}_{kk'}=\omega_k-\omega_{k'}=\left(E_k-E_{k'}\right)/\hbar=(k^2-k'^2)\hbar/2m$ pertains to bath energies. From this we can calculate the integrals with respect to τ and get (converting the sum over p to an integral)

$$\dot{\rho}_{kk'}=\frac{-1}{v\hbar^2}\frac{1}{(2\pi)^3}\int d^3p\sum_{\vec{q}}|\phi(\vec{q})|^2\frac{m}{\hbar q}\times[\rho_{\vec{k}\vec{k}'}f_{\vec{p}_-}\left(1+f_{\vec{p}}\right)\delta\left(\frac{q}{2}(1-\beta)-\left(\vec{p}-\beta\vec{k}\right)\cdot\hat{q}\right)$$
$$-\rho_{\vec{k}_+\vec{k}'_+}f_{\vec{p}_-}\left(1+f_{\vec{p}}\right)\delta\left(\left(\vec{p}-\beta\vec{k}\right)\cdot\hat{q}-\frac{q}{2}(\beta+1)\right)e^{-\frac{i\hbar}{M}\vec{q}\cdot\underline{P}t}+\rho_{\vec{k}\vec{k}'}f_{\vec{p}_+}\left(1+f_{\vec{p}}\right)\delta\left(\frac{q}{2}(1-\beta)+\left(\vec{p}-\beta\vec{k}\right)\cdot\hat{q}\right)$$
$$-\rho_{\vec{k}_-\vec{k}'_-}f_{\vec{p}_+}\left(1+f_{\vec{p}}\right)\delta\left(\left(\vec{p}-\beta\vec{k}\right)\cdot\hat{q}-\frac{q}{2}(\beta+1)\right)e^{+\frac{i\hbar}{M}\vec{q}\cdot\underline{P}t}]$$

where $\beta=m/M$.

At this point, we must again make several assumptions. First, we **assume** that the decoherence time will be small enough that the time-dependent exponentials will not deviate significantly from unity on the timescale of interest. Second, we **assume** that $f_p << 1$ so that $f_{\vec p_\pm}\left(1+f_{\vec p}\right) = f_{\vec p_\pm} + f_{\vec p_\pm} f_{\vec p} \approx f_{\vec p_\pm}$. This gives us

$$\dot\rho_{kk'} = \frac{-1}{v\hbar^2}\frac{1}{(2\pi)^3}\int d^3p \sum_{\vec q} \left|\tilde V(\vec q)\right|^2 \frac{m}{\hbar} q \left[\rho_{\vec k\vec k'} f_{\lambda\vec k - \frac{\vec q}{2} - \frac{\vec q\lambda}{2}} - \rho_{\vec k_+\vec k'_+} f_{\lambda\vec k - \frac{\vec q}{2} + \frac{\vec q\lambda}{2}} + \rho_{\vec k\vec k'} f_{\lambda\vec k + \frac{\vec q}{2} + \frac{\lambda\vec q}{2}} - \rho_{\vec k_-\vec k'_-} f_{\lambda\vec k + \frac{\vec q}{2} - \frac{\vec q\lambda}{2}}\right]$$

We further **assume** that the bath follows a Maxwellian distribution

$$f_p = n_0\left(\frac{a}{\pi}\right)^{3/2} e^{-ap^2} \qquad \text{where} \qquad a = \frac{\hbar^2}{2mk_BT} = \frac{\hbar^2}{2m}\beta .$$

We will also **assume** that the pointer is much heavier than the gas particles ($\lambda = m/M << 1$). This allows us to expand the square of the argument of f to second order in λ. Changing the sum over q to an integral and making the further substitution $\Gamma = \frac{-1}{\hbar^2}\frac{m}{\hbar}\frac{1}{(2\pi)^6} n_0\left(\frac{a}{\pi}\right)^{3/2}$, we obtain

$$\dot\rho_{kk'} = \Gamma\int d^3q\left|\tilde V(\vec q)\right|^2 q e^{-a\frac{q^2}{4}}\left[\rho_{\vec k\vec k'} e^{a\lambda\vec k\cdot\vec q} e^{-\frac{a\lambda q^2}{2}} - \rho_{\vec k_+\vec k'_+} e^{a\lambda\vec k\cdot\vec q} e^{\frac{a\lambda q^2}{2}} + \rho_{\vec k\vec k'} e^{-a\lambda\vec k\cdot\vec q} e^{-\frac{a\lambda q^2}{2}} - \rho_{\vec k_-\vec k'_-} e^{-a\lambda\vec k\cdot\vec q} e^{\frac{a\lambda q^2}{2}}\right]$$

Finally, to progress further we will **assume** that the momentum change due to scattering is much smaller than the pointer momentum ($q << k$). This allows us to expand $\rho_{\vec k_+\vec k'_+} = \rho(\vec k+\vec q,\vec k'+\vec q)$ as a double Taylor series about $\rho_{kk'}$:

$$\rho_{\vec k_+\vec k'_+} = \rho_{\vec k\vec k'} + q\cdot\left(\frac{\partial}{\partial\vec k}+\frac{\partial}{\partial\vec k'}\right)\rho_{\vec k\vec k'} + \frac{q^2}{2}\left(\frac{\partial}{\partial\vec k}+\frac{\partial}{\partial\vec k'}\right)^2\rho_{\vec k\vec k'} + O(q^3).$$

It will be useful to change variables to $\underline{P} = \vec k - \vec k'$ and $\underline{K} = \left(\vec k+\vec k'\right)/2$. Note that

$$\frac{\partial}{\partial\underline{K}} = \frac{\partial}{\partial\vec k}\frac{\partial\vec k}{\partial\underline{K}} + \frac{\partial}{\partial\vec k'}\frac{\partial\vec k'}{\partial\underline{K}} = \frac{\partial}{\partial\vec k}+\frac{\partial}{\partial\vec k'}$$

which means that

$$\rho_{\vec k_\pm\vec k'_\pm} \approx \rho_{\vec k\vec k'} + \vec q\cdot\frac{\partial}{\partial\underline{K}}\rho_{\vec k\vec k'} + \frac{1}{2}\left(\vec q\cdot\frac{\partial}{\partial\underline{K}}\right)^2\rho_{\vec k\vec k'}.$$

We can write this in terms of $\underline{K}$ and $\underline{P}$, making the further substitution $j \equiv a\lambda = \hbar^2\beta/2M$

$$\dot\rho_{kk'} = -\lambda\int d^3q\left|V(\vec q)\right|^2 q e^{-a\frac{q^2}{4}} e^{\frac{j}{2}\underline{P}\cdot\vec q}\left[\rho_{\vec k\vec k'}\left(e^{j\underline{K}\cdot\vec q} + e^{-j\underline{K}\cdot\vec q}\right)\left(e^{-\frac{jq^2}{2}} - e^{\frac{jq^2}{2}}\right)\right.$$

$$\left. - e^{\frac{jq^2}{2}}\left[\left(\vec q\cdot\frac{\partial}{\partial\underline{K}}\right)\rho_{\vec k\vec k'}\left(e^{j\underline{K}\cdot\vec q} - e^{-j\underline{K}\cdot\vec q}\right) + \frac{1}{2}\left(\vec q\cdot\frac{\partial}{\partial\underline{K}}\right)^2\rho_{\vec k\vec k'}\left(e^{j\underline{K}\cdot\vec q} + e^{-j\underline{K}\cdot\vec q}\right)\right]\right]$$

We **assume** that the pointer-gas collisions have a uniform angular distribution and that therefore q is isotropic, so that first-order terms of the form $\underline{K}\cdot\vec q$ average to zero when we integrate over all q-space. This allows us to carry out the angular integration and expand everything to lowest nonvanishing order in j:

$$\dot\rho_{kk'} = -\lambda\frac{4\pi}{3}\int dq\left|V(\vec q)\right|^2 e^{-a\frac{q^2}{4}}\left[-6jq^5\rho_{\vec k\vec k'} - 6jKq^5\frac{\partial\rho_{\vec k\vec k'}}{\partial K} - q^5\frac{\partial^2\rho_{\vec k\vec k'}}{\partial K^2}\right]$$

We now have a general equation for any interaction potential subject to the form constraints placed earlier. To progress in our particular case, we will choose a particular interaction potential. We will choose a gaussian potential:

$$V(\vec r) = V_0 e^{-\frac{r^2}{d^2}}$$

so that

$$V(\vec q) = V_0\pi^{3/2}d^3 e^{-\frac{q^2d^2}{4}}.$$

This gives

$$\dot\rho_{kk'} = -\lambda V_0^2\frac{4\pi^4}{3}d^6\int dq e^{-\frac{q^2}{4}\left(2d^2+a\right)}\left[-6jq5\rho_{\vec k\vec k'} - 2jKq^5\frac{\partial\rho_{\vec k\vec k'}}{\partial K} - q^5\frac{\partial^2\rho_{\vec k\vec k'}}{\partial K^2}\right]$$

$$= -\lambda\frac{V_0^2 16(2\pi)^4 d^6}{3\left(2d^2+a\right)^3}\left[-6j\rho_{\vec k\vec k'} - 6jK\frac{\partial\rho_{\vec k\vec k'}}{\partial K} - \frac{\partial^2\rho_{\vec k\vec k'}}{\partial K^2}\right]$$

Finally, recalling that $\lambda = \frac{\sigma 1}{\hbar^2}\frac{m}{\hbar}\frac{1}{(2\pi)^6}n_0\left(\frac{a}{\pi}\right)^{3/2} = \frac{\sigma n_0\pi^{3/2}}{(2\pi)^{15/2}\sqrt{m}}$ we get

$$\dot{\rho}_{kk'} = \frac{16\square_0^{\,2} n_0 \pi^{3/2} d^6}{3(2\pi)^{7/2}\left(2d^2+a\right)^3\sqrt{m}}\left[6\frac{\hbar^2\square}{M}\rho_{\vec{k}\vec{k}'} + 6\frac{\hbar^2\square}{M}K\frac{\partial\rho_{\vec{k}\vec{k}'}}{\partial K} + \frac{\partial^2\rho_{\vec{k}\vec{k}'}}{\partial K^2}\right]$$

$$\equiv Z\left[6j\rho_{\vec{k}\vec{k}'} + 6jK\frac{\partial\rho_{\vec{k}\vec{k}'}}{\partial K} + \frac{\partial^2\rho_{\vec{k}\vec{k}'}}{\partial K^2}\right]$$

This gives us the equation of motion for the reduced density operator in the interaction picture. In order to compare it to our earlier pointer-system results, we need to convert this into the Schrödinger picture. We know the equation of motion is given by

$$i\hbar\dot{\rho}_s = \left[\hat{H},\rho_s\right] = \left[\hat{H}_P,\rho_s\right] + \left[\hat{H}_B+\hat{V},\rho_s\right]$$

so that the elements of the reduced density matrix in the Schrödinger picture are

$$\langle k|Tr_B(i\hbar\dot{\rho}^{Sch.})|k'\rangle = \langle k|Tr_B\left(\left[\hat{H}_P,\rho^{Sch.}\right]+\left[\hat{H}_B+\hat{V},\rho^{Sch.}\right]\right)|k'\rangle$$

$$i\hbar\dot{\rho}^{(R)\,Sch.}_{kk'} = \langle k|\left[\hat{H}_P,\rho_s\right]|k'\rangle + i\hbar\dot{\rho}^{Int.}_{kk'}$$

$$= \frac{-i\hbar}{M}KP\rho_{kk'} + Z\left[6j\rho_{kk'} + 6jK\frac{\partial\rho_{kk'}}{\partial K} + \frac{\partial^2\rho_{kk'}}{\partial K^2}\right]$$

We now write this in terms of the Wigner distribution function

$$\rho(P,K,t) = \int dX e^{-iPX} W(X,K,t)$$

to get

$$\frac{\partial}{\partial t}W + \frac{\hbar}{M}K\frac{\partial}{\partial X}W = Z\left[6jW \quad + 6jK\frac{\partial}{\partial K}W + \frac{\partial^2}{\partial K^2}W\right]$$

which is the fundamental equation governing this model.

We may note at this point that although we have selected a particular potential for this model, the form of the resulting PDE is in fact independent of this choice. If we define the quantity

$$\overline{\square} \equiv \frac{4\pi}{3}\int dq\left|\square(\vec{q})\right|^2 e^{-a\frac{q^2}{4}}q^5$$

we can write the differential equation as

$$\frac{\partial}{\partial t}W + \frac{\hbar}{M}K\frac{\partial}{\partial X}W = \tilde{Z}\left[6jW \quad + 6jK\frac{\partial}{\partial K}W + \frac{\partial^2}{\partial K^2}W\right]$$

where $\tilde{Z} \equiv \square\square = \dfrac{n_0 \square^{3/2}\square}{(2\square)^{15/2}\sqrt{m}}$ is the generalization of the Z coefficient for any radially-dependent potential. The advantage of choosing a particular potential is that it provides particular parameters (in this case d and $\square_0$) that characterize the scale of the problem and allows us to see the dependence of the final answer upon such parameters. We should keep in mind, however, that the general results of this model are not dependent upon our particular choice of potential.

III.4 Solving the Differential Equation

III.4.1 Finding the General Solution

We now have the differential equation

$$\frac{\partial}{\partial t}W + \frac{\hbar}{M}K\frac{\partial}{\partial X}W = Z\left[6jW + 6jK\frac{\partial}{\partial K}W + \frac{\partial^2}{\partial K^2}W\right]$$

which can be written as

$$\frac{\partial}{\partial t}W + \frac{\hbar}{M}K\frac{\partial}{\partial X}W = \beta W + \beta K\frac{\partial}{\partial K}W + Z\frac{\partial^2}{\partial K^2}W$$

by defining $\beta \equiv 6jZ$ .To solve this equation, we will use a method that examines the characteristic curves of the first order part in order to find an appropriate set of coordinates in which to work. We rewrite the first order part of the equation in the form

$$\left[\frac{\partial}{\partial t} + \left(\frac{\hbar}{M}K\right)\frac{\partial}{\partial X} + (-\beta K)\frac{\partial}{\partial K}\right]W = \beta W$$

and note that this has the form of a total differential when

$$\frac{\partial K}{\partial t} = -\beta K \Rightarrow K = Ce^{-\beta t} \qquad \text{and} \qquad \frac{\partial X}{\partial t} = \frac{\hbar}{M}K \Rightarrow X = -gCe^{-\beta t} + \xi$$

where $g \equiv \hbar / M\beta$.

These constraints define characteristic curves (see [14]), parameterized by the constants of integration C and ξ, with respect to which we can write the total derivative

$$\frac{dW}{dt} = \beta W \Rightarrow W = \varphi(\xi, C)e^{\beta t}.$$

Inverting these equations, we can express the new parameters of integration C and ξ in terms of the old variables K and X:

$$C = Ke^{\square} \quad \text{and} \quad \square = X + gK$$

From these we can determine the differential operators and write the full equation above in terms of these new variables determined by the characteristic curves (details in Appendix B)

$$\frac{\partial \square}{\partial t} = Z\left[e^{2\square}\frac{\partial^2 \square}{\partial C^2} + 2e^{\square} g \frac{\partial^2 \square}{\partial C \partial \square} + g^2 \frac{\partial^2 \square}{\partial \square^2}\right].$$

To simplify this further, we transform $\square$ according to the rule

$$\square(\square, C, t) = \int_{\square} dh e^{i\square h} \int_{\square} dz e^{\square iCz} f(z,h,t)$$

to get

$$\frac{\partial}{\partial t} \iint dh e^{i\square h} dz e^{\square iCz} f(z,h,t) = Z\left[e^{2\square} \frac{\partial^2}{\partial C^2} \iint dh e^{i\square h} dz e^{\square iCz} f + 2e^{\square} g \frac{\partial^2}{\partial C \partial \square} \iint dh e^{i\square h} dz e^{\square iCz} f \right.$$

$$\left. + g^2 \frac{\partial^2}{\partial \square^2} \iint dh e^{i\square h} dz e^{\square iCz} f \right]$$

$$= \square \iint dh e^{i\square h} dz e^{\square iCz} Z z^2 e^{2\square} f + 2 \iint dh e^{i\square h} dz e^{\square iCz} gZzhe^{\square} f$$

$$\square \iint dh e^{i\square h} dz e^{\square iCz} g^2 h^2 f$$

Now we can equate the integrands, giving us

$$\frac{\partial}{\partial t} f(z,h,t) = \square \left(Zz^2 e^{2\square} \square 2gZzhe^{\square} + Zg^2h^2 \right) f(z,h,t)$$

which we integrate to get

$$f(z,h,t) = f(z,h,0) \exp\left[\square Z\left(z^2 \square_2 \square 2gzh \square_1 + g^2 h^2 t\right)\right]$$

where $\square_1 \equiv \left(e^{\square} \square 1\right)/\square$ and $\square_2 \equiv \left(e^{2\square} \square 1\right)/2\square$.

We now, in principle, have the solution to the equation. Our task at this point is to transform this solution for f into a solution for $\square$ and then insert the initial condition. Writing this result for f in terms of $\square$, we get

$$\square(\square, C, t) = \int_{\square} dh e^{i\square h} \int_{\square} dz e^{\square iCz} \left(\int_{\square} d\square e^{\square i\square h} \int_{\square} dC \square e^{iC\square z} \square(\square, C\square, 0) \right) \square$$

$$\exp\left[\square Z\left(z^2 \square_2 \square 2gzh \square_1 + g^2 h^2 t\right)\right].$$

We can now change variables once again from $(\square, C, t)$ back to (X, K, t) and write an equation for

$W(X,K,t)$, remembering that $W(t=0)=\rho(t=0)$.

$$W(X,K,t)=\frac{e^{\gamma}}{(2\pi)^2}\int_{-\infty}^{\infty} dh e^{i(gK+X)h}\int_{-\infty}^{\infty} dz e^{-iKe^{\gamma}z}\left(\int_{-\infty}^{\infty} dX' e^{-i(gK'+X')h}\int_{-\infty}^{\infty} dK' e^{iK'z}W(X',K',0)\right)\times$$
$$\exp\left[-Z\left(z^2\Gamma_2-2gzh\Gamma_1+g^2h^2t\right)\right]$$

By collecting terms into a single propagator $J(X,K,t;X',K',0)$ we can write this as

$$W(X,K,t)=\frac{1}{(2\pi)}\int_{-\infty}^{\infty} dX'\int_{-\infty}^{\infty} dK' J(X,K,t;X',K',0)W(X',K',0).$$

We therefore wish to solve for J:

$$J=\frac{e^{\gamma}}{(2\pi)}\int_{-\infty}^{\infty} dz\int_{-\infty}^{\infty} dh\exp\left[i\left(\left(g(K-K')-(X-X')\right)h+\left(K'-Ke^{\gamma}\right)z\right)-Z\left(z^2\Gamma_2+2gzh\Gamma_1+g^2h^2t\right)\right]$$

$$=\frac{e^{\gamma}}{\left(2\sqrt{\pi Z\Gamma_2}\right)}\int_{-\infty}^{\infty} dh e^{\frac{-\left(K'-Ke^{\gamma}\right)^2}{4Z\Gamma_2}}\exp\left[-h^2g^2Z\left(t-\frac{\Gamma_1^2}{\Gamma_2}\right)\right.$$
$$\left.-ih\left(g(K-K')-(X-X')+g\frac{\Gamma_1}{\Gamma_2}\left(Ke^{\gamma}-K'\right)\right)\right]$$

$$=e^{\gamma}\sqrt{\frac{u}{Z\alpha}}e^{-u\left(K'-Ke^{\gamma}\right)^2}\exp\left[-\left(g(K-K')-(X-X')+g\frac{\Gamma_1}{\Gamma_2}\left(Ke^{\gamma}-K'\right)\right)^2/(4Z\alpha)\right]$$

where $\alpha\equiv g^2\left(t-\Gamma_1^{\,2}/\Gamma_2\right)$ and $u\equiv 1/(4Z\Gamma_2)$. From W we can easily find ρ:

$$\rho(P,K,t)=\frac{1}{(2\pi)}\int_{-\infty}^{\infty} dXe^{-iXP}W(X,K,t)$$
$$=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dX'\int_{-\infty}^{\infty} dK' W(X',K',0)\int_{-\infty}^{\infty} dXe^{-iXP}J(X,K,t;X',K',0).$$

This prompts us to define a new propagator by transforming J

$$\begin{aligned}
\tilde{J}(P,K,t;X\square,K\square,0) &= \int_{\square} dX e^{\square iXP} J(X,K,t;X\square,K\square,0) \\
&= e^{\square}\sqrt{\frac{u}{Z\square}}\, e^{\square u\left(K\square Ke^{\square}\right)^2} \int_{\square} dX e^{\square iXP} \exp\left[\square\left(\square\square\left(X\square X\square\right)\right)^2/(4Z\square)\right] \\
&= e^{\square}\sqrt{\frac{u}{Z\square}}\, e^{\square u\left(K\square Ke^{\square}\right)^2} \sqrt{4\square Z\square}\exp\left[\square iP\left(X\square+\square\right)\square P^2 Z\square\right] \\
&= \sqrt{4\square u}\, e^{\square} e^{\square iX\square P} \exp\left[\square iP\square\square P^2 Z\square\square u\left(K\square Ke^{\square}\right)^2\right]
\end{aligned}$$

where $\square \equiv g\frac{\square_1}{\square_2}\left(Ke^{\square}\square K\square\right)\square g\left(K\square K\square\right) = g\left(K+K\square\right)\frac{\square_1^{\,2}}{\square_2}\frac{\square}{2}$.

Since the $X\square$ dependence is completely contained in the single exponential term, we can immediately perform the $X\square$ integral in the expression for $\square$.

$$\begin{aligned}
\square(P,K;\square,\square\square;t) &= \frac{\sqrt{4\square u}}{(2\square)^2} e^{\square} \int_{\square} dK\square \exp\left[\square iP\square\square P^2 Z\square\square u\left(K\square\square Ke^{\square}\right)^2\right] \int_{\square} dX\square W\left(X\square,K\square,0\right) e^{\square iPX\square} \\
&= \frac{\sqrt{4\square u}}{(2\square)^2} e^{\square} \int_{\square} dK\square \exp\left[\square iP\square\square P^2 Z\square\square u\left(K\square\square Ke^{\square}\right)^2\right] \square(P,K\square;\square,\square\square;0)
\end{aligned}$$

We now turn to our initial condition, determined by the bathless state a short time after the coupling interaction begins.

$$\square(P,K;\square,\square\square;0) = a^*_{\square\square} a_{\square}\, 2\left(2\square\square^2\right)^{1/2} \exp\left[\square 2\square^2 K^2 \square iK\square\left(\square\square\square\square\right)\right] \exp\left[\square\square^2 P^2/2\square iP\square\left(\square+\square\square\right)/2\right].$$

We can therefore solve for each matrix element by integration with respect to $K\square$. Detailed treatment of the remaining calculations appears in Appendix C.

III.4.2 Initial State: The Diagonal Terms

We first consider the diagonal terms ($\square = \square\square$). Based on the bathless case and the predictions of the decoherence model, we expect the same-spin terms to have spatial probability distributions that follow a displaced Gaussian shape. Furthermore, while we may expect some increased spreading of the wavepacket due to stronger dissipative effects, such spreading should remain relatively slow so that the packet maintains its general shape for a long time. We begin by inserting the initial state of the diagonal terms into the equation of motion and integrating with respect to K' to get an expression in terms of P and K, the relative and center-of-mass momenta:

$$\square(P,K;\pm,\pm;t) = a_{\pm}^{\;2}\frac{\sqrt{4\square}}{(2\square)}\sqrt{\frac{2\square^2 u}{u+2\square^2}}\,e^{\square}\exp\left[\frac{\square 2\square^2 u}{\left(u+2\square^2\right)}K^2 e^{2\square}\right]\exp\left[\square iP\square/2 \mp iP\square\right]\square$$

$$\exp\left[\square P^2\left(\frac{\square^2}{2}+Z\square+\frac{\square^2}{16\left(u+2\square^2\right)}\right)\square iP\frac{\square}{2}\left(1+\frac{ue^{\square}}{2\left(u+2\square^2\right)}\right)K\right]$$

In order to compare this with our earlier result for the bathless case, we transform this back into the position representation by a pair of Fourier transforms. The first such transform produces

$$\square(P,r;\pm,\pm;t) = a_{\pm}^{\;2}\exp\left[\frac{\square e^{\square 2\square}r^2\left(u+2\square^2\right)}{8\square^2 u}\right]\exp\left[\square P^2 A + P\left(Br \mp i\square\right)\right]$$

where r is the center-of-mass coordinate (corresponding to $r \square r'$ in the coordinates of the bathless case) and A and B are given in Appendix C. We are here concerned only with the limiting cases of A, and B will disappear from our final answer entirely. The second transformation gives

$$\square(R,r;\pm,\pm;t) = a_{\pm}^{\;2}\exp\left[\frac{\square e^{\square 2\square}r^2\left(u+2\square^2\right)}{8\square^2 u}\right]\sqrt{\frac{\square}{A}}\exp\left[\square\frac{\left(R\mp\square\square iBr\right)^2}{4A}\right].$$

We now wish to find the probability distribution, which is given by the diagonal elements in the position basis. In our derivation of the bathless case, this mean that $r = r'$. Since we are now working with relative and center-of-mass coordinates, this condition means that $r = 0$. Thus

$$\square(R,0;\pm,\pm;t) = a_{\pm}^{\;2}\sqrt{\frac{\square}{A}}\exp\left[\square\frac{\left(R\mp\square\right)^2}{4A}\right].$$

When $r = 0$, R is equivalent to r from our bath-free solution. It is clear that this solution has the form expected, which is very similar to the bathless case. In the limit as $t \square 0$ $A \square \square^2/2$, so the initial state is a simple displaced Gaussian, as we would hope. For large timescales ($\square t >> 1$) the time dependence of A is linear, governed by the term $Zg^2 t$. Thus the wavepacket has linear spread.

III.4.3 Initial State: The Off-Diagonal Terms

Now we look at the off-diagonal terms ($\sigma = -\sigma'$). While we expect these to have a Gaussian shape as well, but they should decay away very rapidly, much faster than the spread of the diagonal terms. Following the same procedure as before, we first integrate with respect to K' to get an expression in the momentum representation.

$$\rho(P,K;\pm,\mp;t) = a_{\pm}a_{\mp}e^{\pm i\phi}\frac{\sqrt{4\pi}}{(2\pi)}\sqrt{\frac{2\alpha^2 u}{u+2\alpha^2}}e^{\gamma t}\exp\left[-\frac{2\alpha^2 u}{\left(u+2\alpha^2\right)}K^2 e^{2\gamma t}\right]\exp\left[-\frac{\left(P\mp 4\bar{\sigma}\right)^2}{16\left(u+2\alpha^2\right)}\right]\times$$

$$\exp\left[-P^2\left(\frac{\beta^2}{2}+Z\right)-iK\left(\frac{\beta}{2}P\pm\frac{2ue^{\gamma t}\bar{\sigma}}{\left(u+2\alpha^2\right)}\right)\right].$$

The first transformation yields

$$\rho(P,r;\pm,\mp;t) = a_{\pm}a_{\mp}e^{\pm i\phi}\exp\left[-\frac{e^{-2\gamma t}r^2}{4u}-\frac{\left(e^{-\gamma t}r\mp 2\bar{\sigma}\right)^2}{8\alpha^2}\right]\exp\left[-P^2A+P\left(Br\mp\beta\frac{\left(1+e^{-\gamma t}\right)\bar{\sigma}}{4\alpha^2}\right)\right]$$

(where A and B are the same as in the diagonal case) and the second

$$\rho(R,r;\pm,\mp;t) = a_{\pm}a_{\mp}e^{\pm i\phi}\exp\left[-\frac{e^{-2\gamma t}r^2}{4u}-\frac{\left(e^{-\gamma t}r\mp 2\bar{\sigma}\right)^2}{8\alpha^2}\right]\sqrt{\frac{\pi}{A}}\exp\left[\left(-iBr+R\pm i\beta\frac{\left(1+e^{-\gamma t}\right)\bar{\sigma}}{4\alpha^2}\right)^2/4A\right].$$

Again setting $r = 0$ we obtain

$$\rho(R,0;\pm,\mp;t) = a_{\pm}a_{\mp}e^{\pm i\phi}\exp\left[-\frac{\bar{\sigma}^2}{2\alpha^2}\right]\sqrt{\frac{\pi}{A}}\exp\left[\left(R\pm i\frac{g}{2\alpha^2}\left(1-e^{-\gamma t}\right)\bar{\sigma}\right)^2/4A\right].$$

Further algebraic manipulation (shown in the appendix) allows us to write this as

$$\rho(R,0;\pm,\mp;t) = a_{\pm}a_{\mp}e^{\pm i\phi}\sqrt{\frac{\pi}{A}}\exp\left[\mp\frac{ig\left(1-e^{-\gamma t}\right)\bar{\sigma}}{4A\alpha^2}\right]\exp\left[-\frac{R^2+\bar{\sigma}^2}{4A}\right]\exp\left[-\frac{\bar{\sigma}^2}{2\alpha^2}\Gamma(t)\right]$$

Note the presence of the time-dependent term $\Gamma(t)$ (defined explicitly in Appendix C). Initially, $\Gamma(0) = 0$, and this term has no effect on the system. In the large time limit however, ($\gamma t >> 1$) $\Gamma(t) \to 2Zg^2t/\alpha^2$ and we have

$$\rho(R,0;\pm,\mp;t) = a_{\pm}a_{\mp}e^{\pm i\phi}\sqrt{\frac{\pi}{A}}\exp\left[\mp\frac{igR\bar{\sigma}}{4A\alpha^2}\right]\exp\left[-\frac{R^2+\bar{\sigma}^2}{4A}\right]\exp\left[-\frac{\bar{\sigma}^2}{\alpha^2}\frac{Zg^2}{\alpha^2}t\right].$$

Now that we have the solutions for all the elements of the density matrix, we can trace over the spins to get the probability distribution for the pointer.

$$\Pr(R,t) = \frac{\square}{A}\left[a_+^{\,2} e^{-\frac{(R-\bar{\square})^2}{4A}} + a_-^{\,2} e^{-\frac{(R+\bar{\square})^2}{4A}} + \; 2a_\pm a_\mp e^{-\frac{\bar{\square}^2 Zg^2}{\square^4}t} e^{-\frac{(R^2+\bar{\square}^2)}{4A}} \cos\left(\square + \frac{gx\square}{4A\square^2}\right)\right].$$

Note that this is very similar to the bath-free solution, except for the additional decaying exponential on the interference term. This is the decoherence factor that destroys the quantum interference, producing a mixed state. The first two terms, corresponding to contributions from the diagonal terms, have the same form as in the bathless case. The strength of the decoherence is given by the factor

$$\frac{\bar{\square}^2}{\square^2}\frac{Zg^2}{\square^2}$$

which defines a characteristic time of this decay

$$\square_d \equiv \frac{\square^4}{\bar{\square}^2 Zg^2}.$$

In general, the quantity $Zg^2/\square^2$ is small, and in fact has to be in order for our assumptions about the time dependence of $\square(t) \,\square\, 2Zg^2 t/\square^2$ to hold. However, the relative smallness of this factor is more than canceled by the $\bar{\square}^2/\square^2$ term, which is necessarily quite large. What does this factor mean? We have already established that $\square$ is the characteristic width of the unperturbed wavepacket, while $\bar{\square}$ is the characteristic separation between the spin-up and spin-down packets induced by the measuring apparatus. In order to have an adequate measurement, the separation must be much larger than the packet widths. In this case, the exponent of the decoherence term, which goes as the square of the ratio of the separation to the width, is quite large (despite the size of the other factor), and the exponential will in general be small enough to render the entire term negligibly small. The only case in which the term is not negligibly small is when $\bar{\square}$ is on the order of $\square$ or smaller. In this case, the packets have not been sufficiently separated to reliably distinguish between them, and thus the action of our device cannot properly be termed a measurement. We therefore have the off-diagonal terms of the reduced density operator becoming negligibly small as the result of our measurements, as desired. Furthermore, the magnitude of this decay is exponential in the "goodness" of our measurement.

IV Analysis and Discussion

IV.1 Numbers and Scale

Having solved for the behavior of this system, let us examine the typical scale of the parameters involved to obtain quantitative predictions and assess whether this model is physically realistic. If we take the pointer to be a silver atom and the gas to be diatomic nitrogen then

$$M = 108au, \quad m = 28au, \quad \square = 0.259, \quad \hbar / M = 5.8 \times 10^{-10} m^2/s, \quad d = 10^{-10} m$$

We will assume standard temperature of 273 K, so that

$$\beta = 1/(k_b 273K) = 2.65 \times 10^{20} J^{-1}.$$

Similarly at standard pressure of 1 atm, the number density of an ideal gas is

$$n_0 = 2.69 \times 10^{25} m^{-3}.$$

In order for the interaction potential to be significant, the typical interaction energy must be much larger than the typical thermal energy. Therefore

$$\square_0 = 50 \cdot (\frac{3}{2}kT) = \frac{75}{\beta} = 2.83 \times 10^{-19} J.$$

From these parameters, we can calculate several terms of interest:

$$j = \frac{\hbar^2}{M}\beta = 1.63 \times 10^{-23} m^2, \qquad a = \frac{\hbar^2}{m}\beta = 5.5 \times 10^{-23} m^2,$$

$$d^2 = 10^{-20} m^2 >> a \quad || \quad \frac{d^6}{\left(2d^2 + a\right)^3} \approx \frac{1}{8},$$

$$\frac{16\beta^{3/2}}{3(2\square)^{7/2}\sqrt{m}} \frac{d^6}{\left(2d^2 + a\right)^3} = 2.00 \times 10^{40} \frac{s^3}{kg^2 \cdot m^3}$$

and finally

$$Z = 4.31 \times 10^{28} m^{-2} s^{-1},$$

$$\square = 6jZ = 4.21 \times 10^{6} s^{-1}.$$

This allows us to define a characteristic timescale of the bath

$$t_d \equiv 1/\square = 2.5 \times 10^{-5} s.$$

Next, let us examine the term $\overline{\square}^2/\square^2$. If the pointer is a silver atom, the the characteristic packet width $\square \square d = 10^{-10} m$. It is realistic to assume that the separation induced by the measuring device is macroscopic, on the order of $\overline{\Delta x} \approx 10^{-2} m$. This gives us a ratio

$$\overline{\Delta x}/\lambda \approx 10^{-2} m/10^{-10} m = 10^{8} \Rightarrow \left(\overline{\Delta x}/\lambda\right)^{2} = 10^{16}.$$

Now let us examine the other factor Zg^2/λ^2. From the values above, this ratio is

$$Zg^2/\lambda^2 \approx 10^{-2} m^2 s^{-1}/10^{-20} m^2 = 10^{18} s^{-1}$$

which gives a decoherence time of

$$\tau_d = \frac{\lambda^4}{\overline{\Delta x}^2 Zg^2} \approx 10^{-34} s.$$

Here we immediately notice a problem. The term $Zg^2 t/\lambda^2$ is much larger than unity, which indicates that our approximation $\Gamma(t) \equiv Zg^2 t/\left(\lambda^2 + Zg^2 t\right) \approx Zg^2 t/\lambda^2$ is not valid. This is primarily due to t_d being much larger than expected for atomic processes. Increasing the 'strength' of the bath via the interaction potential or density will increase Λ (and thus decrease t_d and make the approximation valid) but the value of t_d is so far off from the critical value (estimated to be about $t_d = 10^{-12} s$) that such adjustments would have to be unrealistically large. The fact that such adjustments could be made, however, indicates that in principle this model provides at least a qualitative demonstration of decoherence. Given that the physical constraints on this model have been both reasonable and relatively lax, a fundamentally unphysical flaw in the model seems unlikely. We therefore presume that some insidious numerical error is the source of this problem, and we shall subject the model to continued and greater scrutiny to determine whether this is the case.

In spite of the problem with the linear behavior of the decoherence exponent, we note that for large times, when the suspect approximation clearly does not hold, the term $\Gamma(t) \equiv Zg^2 t/\left(\lambda^2 + Zg^2 t\right)$ converges to unity, causing the exponential to converge to

$$\exp\left[-\overline{\Delta x}^2/\lambda^2\right] \approx \exp\left[-10^{16}\right].$$

This process takes place on timescales large enough to render the troubled approximation invalid $t > \tilde{t}_d = 10^{-12} s$), and therefore is quite rapid. Unlike standard decoherence terms, this factor does not approach zero, and therefore the off-diagonal terms never fully vanish. However, the asymptotic value is extremely small and will in general be negligible compared to nearly any other effect on the system. Furthermore the term approaches this miniscule value very quickly. From an experimental perspective, this has all the properties of decoherence, as it would be extraordinarily difficult to experimentally distinguish this process from total decoherence. Therefore we consider this behavior, and the demonstrated of the potential for a standard decay term, to indicate that the system has in fact decohered as a result of the measurement process.

IV.2 Implications and Interpretations

We have seen that the diagonal elements of ρ give the probabilities associated with each basis state. The off-diagonal terms indicate the degree of quantum interference between different basis states. Roughly speaking, they represent amplitudes for transitions from one basis state to another. For a measurement to be meaningful, the post measurement state must be robust, meaning that such transitions out of the measured state must be forbidden except as prescribed by ordinary time evolution. On infinitesimal timescales, in which such evolution is negligible, the system must remain well defined in the observed state. Thus all off-diagonal interference terms must be zero, and there are no physical correlations among the basis states.

To understand how to interpret the diagonalization model of measurement, we will consider the diagonalized density matrix of a simple two state system. Note that we can separate the diagonal density matrix into pure state projectors:

$$\rho = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} = a^2|+\rangle\langle+| + b^2|-\rangle\langle-|$$

which we can interpret to say simply that with (classical) probability a^2 the pure state $|+\rangle\langle+|$ is the appropriate representation, and with (classical) probability b^2 the state $|-\rangle\langle-|$ is correct. In this way we see ρ as the mean operator of the system with respect to the prior probabilities. If we condition the probabilities on a particular observed outcome (in the exact same way that we adjust classical probabilities as our knowledge of outcomes becomes more complete), we get a single projector for the system, but this is merely a classical probabilistic conditioning. Without such a conditioning, we could continue to calculate with this ρ, perhaps performing another measurement. If we measure with respect to the same basis, the density operator remains unchanged, as required. If we measure with respect to some new basis $\{|0\rangle, |1\rangle\}$ such that

$$|+\rangle = c|0\rangle + d|1\rangle$$
$$|-\rangle = d|0\rangle - c|1\rangle$$

then we have the process

$$\rho = a^2|+\rangle\langle+| + b^2|-\rangle\langle-| = a^2\begin{pmatrix} c^2 & cd \\ dc & d^2 \end{pmatrix} + b^2\begin{pmatrix} d^2 & -cd \\ -dc & c^2 \end{pmatrix}$$

$$\rightarrow\ a^2\begin{pmatrix} c^2 & 0 \\ 0 & d^2 \end{pmatrix} + b^2\begin{pmatrix} d^2 & 0 \\ 0 & c^2 \end{pmatrix} = a^2\left(c^2|0\rangle\langle0| + d^2|1\rangle\langle1|\right) + b^2\left(d^2|0\rangle\langle0| + c^2|1\rangle\langle1|\right)$$

$$= \left(a^2c^2 + b^2d^2\right)|0\rangle\langle0| + \left(a^2d^2 + b^2c^2\right)|1\rangle\langle1|$$

where again the probabilities for each final pure state projector are simply classical probabilities. We could easily condition this probability distribution on an observed outcome from one or both experiments. Because these are classical probabilities, we are not required to condition the distribution at each measurement but instead have the option of considering the probabilities relative to any conditioning we desire. We have this freedom because the probabilities are epistemic at this point.

The point of this argument is that the diagonalized density operator, without further collapse, suffices as a complete description of the post-measurement quantum system. A much more thorough proof is given in [8], where it is also argued that questions such as which of these outcomes is actually observed and why do we only perceive only one outcome are problems of the philosophy of consciousness and not of physics. This view is consistent with our aim, stated at the outset, to remove the idea of ‘measurement’ from physics entirely, except in as much as it refers to a well-defined *physical* process. One may also note at this point that the nondisappearance of the ‘unobserved’ diagonal terms forms the mathematical beginnings of what a lot of additional interpretation has developed into the Everett interpretation of quantum mechanics and other ‘many-worlds’ views like it. From the empiricist’s perspective, statements about the so-called ‘other universes’ (unobserved terms) are meaningful only if they can effect observable differences in the observed term. Under complete diagonalization of the density operator, this cannot happen, since the quantum interference between different terms has been totally destroyed. Under near-diagonalization of the type observed in this model, the interference is not completely destroyed, and in principle this could be seen as an avenue for experimental testing of the Everett model. In practice, however, the interference terms, though technically nonzero, are far too small to be observed by foreseeable techniques.

A. Derivation of the C-Number Equation

Here is the full derivation of the c-number differential equation, which is too lengthy and detailed to be placed in the body of Section III.3. Our goal is to convert the operator-valued master equation derived in Section III.2 into a c-number differential equation. We begin by surrounding the operator equation with arbitrary momentum-state bras and kets, we get an equation for the matrix elements of the density operator.

$$\frac{\Delta\sigma_{kk'}}{\Delta t} = \langle k|\frac{\Delta\sigma}{\Delta t}|k'\rangle = \frac{-1}{v^2\hbar^2}\int_0^\infty \frac{d\tau}{\Delta t}\int_t^{t+\Delta t} dt' \sum_{\vec{q}} |\phi(\vec{q})|^2 \times [g_{\vec{q}}(\tau)\Big(\langle k|A_{\vec{q}}(t')A_{-\vec{q}}(t'-\tau)\sigma(t)|k'\rangle - \langle k|A_{-\vec{q}}(t'-\tau)\sigma(t)A_{\vec{q}}(t')|k'\rangle\Big)$$
$$+g_{-\vec{q}}(-\tau)\Big(\langle k|\sigma(t)A_{-\vec{q}}(t'-\tau)A_{\vec{q}}(t')\sigma(t)|k'\rangle - \langle k|A_{\vec{q}}(t')\sigma(t)A_{-\vec{q}}(t'-\tau)\sigma(t)|k'\rangle\Big).$$

For the A's the matrix elements are

$$\langle l|A_{\vec{q}}(t')|m\rangle = \langle l|e^{\frac{i}{\hbar}\hat{H}_P t'} e^{i\vec{q}\cdot\vec{R}} e^{\frac{-i}{\hbar}\hat{H}_P t'}|m\rangle = \int dx dx' \langle l|x\rangle\langle x|e^{\frac{i}{\hbar}\hat{H}_P t'} e^{i\vec{q}\cdot\vec{R}} e^{\frac{-i}{\hbar}\hat{H}_P t'}|x'\rangle\langle x'|m\rangle = e^{i\Omega_{kk'}t'}\delta_{\vec{k}'\vec{k}_-}$$

and

$$\langle l|A_{-\vec{q}}(t'-\tau)|m\rangle = \langle l|e^{\frac{i}{\hbar}\hat{H}_P (t'-\tau)} e^{-i\vec{q}\cdot\vec{R}} e^{\frac{-i}{\hbar}\hat{H}_P (t'-\tau)}|m\rangle = \int dx dx' \langle l|x\rangle\langle x|e^{\frac{i}{\hbar}\hat{H}_P (t'-\tau)} e^{-i\vec{q}\cdot\vec{R}} e^{\frac{-i}{\hbar}\hat{H}_P (t'-\tau)}|x'\rangle\langle x'|m\rangle = e^{-i\Omega_{kk'}(t'-\tau)}\delta_{\vec{k}'\vec{k}_-}$$

where $\vec{k}_\pm = \vec{k} \pm \vec{q}$ and $\Omega_{kk'} = \omega_k - \omega_{k'} = \left(E_k - E_{k'}\right)/\hbar = (k^2 - k'^2)\hbar/2M$.

Now we can solve for the four sets of matrix elements:

1:
$$\langle k|A_{\vec{q}}(t')A_{-\vec{q}}(t'-\tau)\sigma(t)|k'\rangle = \sum_{l,m}\langle k|A_{\vec{q}}(t')|l\rangle\langle l|A_{-\vec{q}}(t'-\tau)|m\rangle\langle m|\sigma(t)|k'\rangle = \sum_{\vec{l},\vec{m}}\left(e^{i\Omega_{\vec{k}\vec{l}}t'}\delta_{\vec{l}\vec{k}_-}\right)\left(e^{i\Omega_{\vec{l}\vec{m}}(t'-\tau)}\delta_{\vec{m}\vec{l}_+}\right)\sigma_{\vec{k}\vec{k}'}$$
$$= e^{i\Omega_{\vec{k}\vec{k}_-}\tau}\sigma_{\vec{k}\vec{k}'}$$

2:
$$\langle k|A_{-\vec{q}}(t'-\tau)\sigma(t)A_{\vec{q}}(t')|k'\rangle = \sum_{l,m}\langle k|A_{-\vec{q}}(t'-\tau)|l\rangle\langle l|\sigma(t)|m\rangle\langle m|A_{\vec{q}}(t')|k'\rangle = \sum_{\vec{l},\vec{m}}\left(e^{i\Omega_{\vec{k}\vec{l}}(t'-\tau)}\delta_{\vec{l}\vec{k}_+}\right)\sigma_{\vec{l}\vec{m}}\left(e^{i\Omega_{\vec{m}\vec{k}'}t'}\delta_{\vec{k}'\vec{m}_-}\right)$$
$$= e^{i\Omega_{\vec{k}\vec{k}_+}(t'-\tau)}e^{-i\Omega_{\vec{k}'\vec{k}'_+}t'}\sigma_{\vec{k}_-\vec{k}'_-}$$

3:
$$\langle k|\sigma(t)A_{-\vec{q}}(t'-\tau)A_{\vec{q}}(t')|k'\rangle = \sum_{l,m}\langle k|\sigma(t)|l\rangle\langle l|A_{-\vec{q}}(t'-\tau)|m\rangle\langle m|A_{\vec{q}}(t')|k'\rangle = \sum_{\vec{l},\vec{m}}\sigma_{\vec{k}\vec{l}}\left(e^{i\Omega_{\vec{l}\vec{m}}(t'-\tau)}\delta_{\vec{m}\vec{l}_+}\right)\left(e^{i\Omega_{\vec{m}\vec{k}'}t'}\delta_{\vec{k}'\vec{m}_-}\right)$$
$$= e^{-i\Omega_{\vec{k}\vec{k}_+}\tau}\sigma_{\vec{k}\vec{k}'}$$

4:
$$\langle k|A_{\vec{q}}(t')\sigma(t)A_{-\vec{q}}(t'-\tau)|k'\rangle = \sum_{l,m}\langle k|A_{\vec{q}}(t')|l\rangle\langle l|\sigma(t)|m\rangle\langle m|A_{-\vec{q}}(t'-\tau)|k'\rangle = \sum_{\vec{l},\vec{m}}\left(e^{i\Omega_{\vec{k}\vec{l}}t'}\delta_{\vec{l}\vec{k}_-}\right)\sigma_{\vec{l}\vec{m}}\left(e^{i\Omega_{\vec{m}\vec{k}'}(t'-\tau)}\delta_{\vec{k}'\vec{m}_+}\right)$$
$$= e^{-i\Omega_{\vec{k}'\vec{k}'_-}(t'-\tau)}e^{i\Omega_{\vec{k}\vec{k}_-}t'}\sigma_{\vec{k}_-\vec{k}'_-}$$

It will be helpful to note that

$$\Omega_{\vec{k}\vec{k}_\pm} = \omega_k - \omega_{\vec{k}_\pm} = \frac{\hbar}{2M}\left(k^2 - (\vec{k} \pm \vec{q})^2\right) = \frac{\hbar}{2M}\left(\mp 2\vec{k}\cdot\vec{q} - q^2\right),$$

which leads to the identity

$$\Omega_{\vec{k}\vec{k}_\pm} - \Omega_{\vec{k}'\vec{k}'_\pm} = \frac{\hbar}{2M}\left(\mp 2\vec{k}\cdot\vec{q} - q^2 \pm 2\vec{k}'\cdot\vec{q} + q^2\right) = \mp\frac{\hbar}{M}\vec{q}\cdot\left(\vec{k} - \vec{k}'\right) \equiv \mp\frac{\hbar}{M}\vec{q}\cdot\underline{P}.$$

We then integrate with respect to t'

1, 3: $$\int_t^{t+\Delta t} \frac{dt'}{\Delta t} e^{\mp i\Omega_{\vec{k}\vec{k}_\mp}\tau}\sigma_{\vec{k}\vec{k}'} = e^{\mp i\Omega_{\vec{k}\vec{k}_\mp}\tau}\sigma_{\vec{k}\vec{k}'}$$

2, 4: $$\int_t^{t+\Delta t} \frac{dt'}{\Delta t} e^{\mp i\Omega_{\vec{k}\vec{k}_\pm}\tau}\sigma_{\vec{k}_\pm\vec{k}_\pm} e^{\pm i\left(\Omega_{\vec{k}\vec{k}_\pm} - \Omega_{\vec{k}\vec{k}_\mp}\right)t'} = \int_t^{t+\Delta t} \frac{dt'}{\Delta t} e^{\mp i\Omega_{\vec{k}\vec{k}_\pm}\tau}\sigma_{\vec{k}_\pm\vec{k}_\pm} e^{\mp\frac{i\hbar t'}{M}\vec{q}\cdot\underline{P}} \xrightarrow{\Delta t\to 0} \sigma_{\vec{k}_\pm\vec{k}_\pm} e^{\mp i\Omega_{\vec{k}\vec{k}_\pm}\tau} e^{\mp\frac{i\hbar}{M}\vec{q}\cdot\underline{P}t}$$

We now calculate the two-time averages $g_{\vec{q}}(\tau) = \left\langle n_{\vec{q}}(\tau) n_{-\vec{q}}(0)\right\rangle$ by first finding $n_{\vec{q}}(\tau)$, which we note is a (momentum state) number density operator. Such number operators for particle momentum are not well defined in our simple state vector formulation, we turn to ideas in second quantization to deal with this term. In the second quantized formalism, the coordinates x and p are not operators but parameters of the creation (annihilation) operators $\hat{\psi}^\dagger(x)$ and $\hat{a}^\dagger{}_p$ ($\hat{\psi}(x)$ and $\hat{a}_p$) that create (annihilate) a particle with position x and momentum p, respectively. As with harmonic oscillator creation and annihilation operators, we can generate a number operator $\hat{\psi}^\dagger(x)\hat{\psi}(x)$ (or $\hat{a}_p^\dagger\hat{a}_p$) which indicates the number of particles with position x (or momentum p). Thus, we can write $n_{\vec{q}}(\tau)$ as the Fourier transform of the position-state number operator, which is readily expressed in terms of these second-quantized creation and annihilation operators (as found in [15]).

$$n_{\vec q}(0)=\int e^{-i\vec q\cdot\vec x}n(\vec x)d^3x=\int e^{-i\vec q\cdot\vec x}\hat\psi^\dagger(x)\hat\psi(x)d^3x=\int e^{-i\vec q\cdot\vec x}\left(\frac{1}{\sqrt V}\sum_{\vec p'}e^{-i\vec p'\cdot\vec x}\hat a^\dagger{}_{\vec p'}\right)\left(\frac{1}{\sqrt V}\sum_{\vec p}e^{i\vec p\cdot\vec x}\hat a_{\vec p}\right)d^3x=\frac{1}{V}\sum_{\vec p,\vec p'}\hat a^\dagger{}_{\vec p'}\hat a_{\vec p}\int e^{i(\vec p-\vec p'-\vec q)\cdot\vec x}d^3x$$

$$=\sum_{\vec p,\vec p'}\hat a^\dagger{}_{\vec p'}\hat a_{\vec p}\delta(\vec p-\vec p'-\vec q)=\sum_{\vec p}\hat a^\dagger{}_{\vec p-\vec q}\hat a_{\vec p}.$$

$$n_{\vec q}(\tau)=\sum_{\vec p}\left(\hat a_{\vec p-\vec q}(\tau)\right)^\dagger\hat a_{\vec p}(\tau)=\sum_{\vec p}\left(\hat a_{\vec p-\vec q}e^{-i\omega_{\vec p-\vec q}\tau}\right)^\dagger\hat a_{\vec p}e^{-i\omega_{\vec p}\tau}=\sum_{\vec p}\hat a^\dagger{}_{\vec p-\vec q}\hat a_{\vec p}e^{-i\left(\omega_{\vec p}-\omega_{\vec p-\vec q}\right)\tau}=\sum_{\vec p}\hat a^\dagger{}_{\vec p-\vec q}\hat a_{\vec p}e^{-i\tilde\Omega_{\vec p\vec p_-}\tau}.$$

Thus

$$g_{\vec q}(\tau)=\left\langle n_{\vec q}(\tau)n_{-\vec q}(0)\right\rangle=\sum_{\vec p,\vec p'}e^{-i\tilde\Omega_{\vec p\vec p_-}\tau}\left\langle\hat a^\dagger_{\vec p_-}\hat a_{\vec p}\hat a^\dagger_{\vec p'_+}\hat a_{\vec p'}\right\rangle=\sum_{\vec p,\vec p'}e^{-i\tilde\Omega_{\vec p\vec p_-}\tau}\left\langle\hat a^\dagger_{\vec p_-}\left(\delta_{\vec p'\vec p_-}+\hat a^\dagger_{\vec p'_+}\hat a_{\vec p}\right)\hat a_{\vec p'}\right\rangle=\sum_{\vec p,\vec p'}e^{-i\tilde\Omega_{\vec p\vec p_-}\tau}\delta_{\vec p'\vec p_-}\left(\left\langle\hat a^\dagger_{\vec p_-}\hat a_{\vec p'}\right\rangle+\left\langle\hat a^\dagger_{\vec p_-}\hat a^\dagger_{\vec p'_+}\hat a_{\vec p}\hat a_{\vec p'}\right\rangle\right)$$

$$=\sum_{\vec p}e^{-i\tilde\Omega_{\vec p\vec p_-}\tau}\left(\left\langle\hat a^\dagger_{\vec p_-}\hat a_{\vec p_-}\right\rangle+\left\langle\hat a^\dagger_{\vec p_-}\hat a_{\vec p_-}\hat a^\dagger_{\vec p}\hat a_{\vec p}\right\rangle\right)=\sum_{\vec p}e^{-i\tilde\Omega_{\vec p\vec p_-}\tau}\left(\left\langle\hat a^\dagger_{\vec p_-}\hat a_{\vec p_-}\right\rangle+\left\langle\hat a^\dagger_{\vec p_-}\hat a_{\vec p_-}\right\rangle\left\langle\hat a^\dagger_{\vec p}\hat a_{\vec p}\right\rangle\right)=\sum_{\vec p}e^{-i\tilde\Omega_{\vec p\vec p_-}\tau}f_{\vec p_-}\left(1+f_{\vec p}\right)$$

and

$$g_{-\vec q}(-\tau)=\sum_{\vec p}e^{i\tilde\Omega_{\vec p\vec p_+}\tau}f_{\vec p_+}\left(1+f_{\vec p}\right)$$

where $f_{\vec p}\equiv\left\langle\hat a^\dagger_{\vec p}\hat a_{\vec p}\right\rangle$ and $\tilde\Omega_{kk'}=\omega_k-\omega_{k'}=\left(E_k-E_{k'}\right)/\hbar=(k^2-k'^2)\hbar/2m$ pertains to the bath energies.

From this we can calculate the integrals with respect to τ.

1: $$\int_0^\infty d\tau e^{i\Omega_{\vec k\vec k_-}\tau}g_{\vec q}(\tau)=\sum_{\vec p}f_{\vec p_-}\left(1+f_{\vec p}\right)\int_0^\infty d\tau e^{i\Omega_{\vec k\vec k_-}\tau}e^{-i\tilde\Omega_{\vec p\vec p_-}\tau}=\sum_{\vec p}f_{\vec p_-}\left(1+f_{\vec p}\right)\delta\left(\Omega_{\vec k\vec k_-}-\tilde\Omega_{\vec p\vec p_-}\right)$$

2: $$\int_0^\infty d\tau e^{-i\Omega_{\vec k\vec k_+}\tau}g_{\vec q}(\tau)=\sum_{\vec p}f_{\vec p_-}\left(1+f_{\vec p}\right)\int_0^\infty d\tau e^{-i\Omega_{\vec k\vec k_+}\tau}e^{-i\tilde\Omega_{\vec p\vec p_-}\tau}=\sum_{\vec p}f_{\vec p_-}\left(1+f_{\vec p}\right)\delta\left(-\Omega_{\vec k\vec k_+}-\tilde\Omega_{\vec p\vec p_-}\right)$$

3: $$\int_0^\infty d\tau e^{-i\Omega_{\vec k'\vec k'_+}\tau}g_{-\vec q}(-\tau)=\sum_{\vec p}f_{\vec p_+}\left(1+f_{\vec p}\right)\int_0^\infty d\tau e^{-i\Omega_{\vec k'\vec k'_+}\tau}e^{-i\tilde\Omega_{\vec p\vec p_+}\tau}=\sum_{\vec p}f_{\vec p_+}\left(1+f_{\vec p}\right)\delta\left(-\Omega_{\vec k'\vec k'_+}+\tilde\Omega_{\vec p\vec p_+}\right)$$

4: $$\int_0^\infty d\tau e^{i\Omega_{\vec k'\vec k'_-}\tau}g_{-\vec q}(-\tau)=\sum_{\vec p}f_{\vec p_+}\left(1+f_{\vec p}\right)\int_0^\infty d\tau e^{i\Omega_{\vec k'\vec k'_-}\tau}e^{i\tilde\Omega_{\vec p\vec p_+}\tau}=\sum_{\vec p}f_{\vec p_+}\left(1+f_{\vec p}\right)\delta\left(\Omega_{\vec k'\vec k'_-}+\tilde\Omega_{\vec p\vec p_+}\right).$$

Making the substitution $\mu=m/M$, we find

$$\Omega_{\vec k\vec k_\pm}+\tilde\Omega_{\vec p\vec p_\mp}=\frac{\hbar}{2}\left(\frac{1}{M}\left(\mp2\vec k\cdot\vec q-q^2\right)+\frac{1}{m}\left(\pm2\vec p\cdot\vec q-q^2\right)\right)=\frac{\hbar q}{m}\left(\pm\left(\vec p-\mu\vec k\right)\cdot\hat q-\frac{q}{2}(\mu+1)\right)$$

$$\Omega_{\vec k\vec k_\pm}-\tilde\Omega_{\vec p\vec p_\pm}=\frac{\hbar}{2}\left(\frac{1}{M}\left(\mp2\vec k\cdot\vec q-q^2\right)-\frac{1}{m}\left(\mp2\vec p\cdot\vec q-q^2\right)\right)=\frac{\hbar q}{m}\left(\frac{q}{2}(1-\mu)\pm\left(\vec p-\mu\vec k\right)\cdot\hat q\right).$$

Altogether, this gives (converting the sum over p to an integral)

$$\dot{\sigma}_{kk'} = \frac{-1}{v\hbar^2}\frac{1}{(2\pi)^3}\int d^3p\sum_{\vec{q}}|\phi(\vec{q})|^2\frac{m}{\hbar}q\times[\sigma_{\vec{k}\vec{k}'}f_{\vec{p}_-}\left(1+f_{\vec{p}}\right)\delta\left(\frac{q}{2}(1-\mu)-\left(\vec{p}-\mu\vec{k}\right)\cdot\hat{q}\right)-\sigma_{\vec{k}_+\vec{k}'_+}f_{\vec{p}_-}\left(1+f_{\vec{p}}\right)\delta\left(\left(\vec{p}-\mu\vec{k}\right)\cdot\hat{q}-\frac{q}{2}(\mu+1)\right)e^{-\frac{i\hbar}{M}\vec{q}\cdot\underline{Pt}}$$

$$\sigma_{\vec{k}\vec{k}'}f_{\vec{p}_+}\left(1+f_{\vec{p}}\right)\delta\left(\frac{q}{2}(1-\mu)+\left(\vec{p}-\mu\vec{k}\right)\cdot\hat{q}\right)-\sigma_{\vec{k}_-\vec{k}'_-}f_{\vec{p}_+}\left(1+f_{\vec{p}}\right)\delta\left(-\left(\vec{p}-\mu\vec{k}\right)\cdot\hat{q}-\frac{q}{2}(\mu+1)\right)e^{+\frac{i\hbar}{M}\vec{q}\cdot\underline{Pt}}]$$

At this point, we must again make several assumptions. First, we **assume** that the decoherence time will be small enough that the time-dependent exponentials will not deviate significantly from unity on the timescale of interest. Second, we **assume** that $f_p << 1$ so that $f_{\vec{p}_\pm}\left(1+f_{\vec{p}}\right)=f_{\vec{p}_\pm}+f_{\vec{p}_\pm}f_{\vec{p}}\approx f_{\vec{p}_\pm}$. Making this substitution and examining the deltas, we find the arguments of the f's.

1: $$\delta\left(\frac{q}{2}(1-\mu)-\left(\vec{p}-\mu\vec{k}\right)\cdot\hat{q}\right)\to\vec{p}=\mu\vec{k}+\frac{\vec{q}}{2}(1-\mu)\to\vec{p}-\vec{q}=\mu\vec{k}-\frac{\vec{q}}{2}(1+\mu)$$

2: $$\delta\left(\left(\vec{p}-\mu\vec{k}\right)\cdot\hat{q}-\frac{q}{2}(\mu+1)\right)\to\vec{p}=\mu\vec{k}+\frac{\vec{q}}{2}(\mu+1)\to\vec{p}-\vec{q}=\mu\vec{k}-\frac{\vec{q}}{2}(1-\mu)$$

3: $$\delta\left(\frac{q}{2}(1-\mu)+\left(\vec{p}-\mu\vec{k}\right)\cdot\hat{q}\right)\to\vec{p}=\mu\vec{k}-\frac{\vec{q}}{2}(1-\mu)\to\vec{p}+\vec{q}=\mu\vec{k}+\frac{\vec{q}}{2}(1+\mu)$$

4: $$\delta\left(-\left(\vec{p}-\mu\vec{k}\right)\cdot\hat{q}-\frac{q}{2}(\mu+1)\right)\to\vec{p}=\mu\vec{k}-\frac{\vec{q}}{2}(\mu+1)\to\vec{p}+\vec{q}=\mu\vec{k}+\frac{\vec{q}}{2}(1-\mu).$$

This gives us

$$\dot{\sigma}_{kk'}=\frac{-1}{v\hbar^2}\frac{1}{(2\pi)^3}\sum_{\vec{q}}|\phi(\vec{q})|^2\frac{m}{\hbar}q\times\left[\sigma_{\vec{k}\vec{k}'}f_{\mu\vec{k}-\frac{\vec{q}}{2}-\frac{\vec{q}\mu}{2}}-\sigma_{\vec{k}_+\vec{k}'_+}f_{\mu\vec{k}-\frac{\vec{q}}{2}+\frac{\vec{q}\mu}{2}}+\sigma_{\vec{k}\vec{k}'}f_{\mu\vec{k}+\frac{\vec{q}}{2}+\frac{\mu\vec{q}}{2}}-\sigma_{\vec{k}_-\vec{k}'_-}f_{\mu\vec{k}+\frac{\vec{q}}{2}-\frac{\vec{q}\mu}{2}}\right].$$

We further **assume** that the bath follows a Maxwellian distribution

$$f_p=n_0\left(\frac{a}{\pi}\right)^{3/2}e^{-ap^2}\qquad\text{where}\qquad a=\frac{\hbar^2}{2mk_BT}=\frac{\hbar^2}{2m}\beta.$$

We will also **assume** that the pointer is much heavier than the gas particles ($\mu=m/M<<1$). This allows us to expand the square of the argument of f to second order in μ:

1: $$(\vec{p}-\vec{q})^2=\left(\mu\vec{k}-\frac{\vec{q}}{2}(1+\mu)\right)^2=-\mu\vec{k}\cdot\vec{q}+\mu\frac{q^2}{2}+\frac{q^2}{4}+O(\mu^2)$$

2: $$(\vec{p}-\vec{q})^2 = \left(\mu\vec{k} - \frac{\vec{q}}{2}(1-\mu)\right)^2 = -\mu\vec{k}\cdot\vec{q} - \mu\frac{q^2}{2} + \frac{q^2}{4} + O(\mu^2)$$

3: $$(\vec{p}+\vec{q})^2 = \left(\mu\vec{k} + \frac{\vec{q}}{2}(1+\mu)\right)^2 = \mu\vec{k}\cdot\vec{q} + \mu\frac{q^2}{2} + \frac{q^2}{4} + O(\mu^2)$$

4: $$(\vec{p}+\vec{q})^2 = \left(\mu\vec{k} + \frac{\vec{q}}{2}(1-\mu)\right)^2 = \mu\vec{k}\cdot\vec{q} - \mu\frac{q^2}{2} + \frac{q^2}{4} + O(\mu^2).$$

Changing the sum over q to an integral $\left(\frac{1}{v}\sum_{\vec{q}} \to \frac{1}{(2\pi)^3}\int d^3q\right)$ and making the further substitution $\xi = \frac{-1}{\hbar^2}\frac{m}{\hbar}\frac{1}{(2\pi)^6}n_0\left(\frac{a}{\pi}\right)^{3/2}$, we obtain

$$\dot{\sigma}_{kk'} = \xi\int d^3q\left|\phi(\vec{q})\right|^2 q e^{-a\frac{q^2}{4}} \times \left[\sigma_{\vec{k}\vec{k}'}e^{a\mu\vec{k}\cdot\vec{q}}e^{-\frac{a\mu q^2}{2}} - \sigma_{\vec{k}_+\vec{k}'_+}e^{a\mu\vec{k}\cdot\vec{q}}e^{\frac{a\mu q^2}{2}} + \sigma_{\vec{k}\vec{k}'}e^{-a\mu\vec{k}\cdot\vec{q}}e^{-\frac{a\mu q^2}{2}} - \sigma_{\vec{k}_-\vec{k}'_-}e^{-a\mu\vec{k}\cdot\vec{q}}e^{\frac{a\mu q^2}{2}}\right].$$

Finally, to progress further we will **assume** that the momentum change due to scattering is much smaller than the pointer momentum ($q << k$). This allows us to expand $\sigma_{\vec{k}_+\vec{k}'_+} = \sigma(\vec{k}+\vec{q},\vec{k}'+\vec{q})$ as a double Taylor series about $\sigma_{kk'}$:

$$\sigma_{\vec{k}_+\vec{k}'_+} = \sigma_{\vec{k}\vec{k}'} + q\cdot\left(\frac{\partial}{\partial\vec{k}} + \frac{\partial}{\partial\vec{k}'}\right)\sigma_{\vec{k}\vec{k}'} + \frac{q^2}{2}\left(\frac{\partial}{\partial\vec{k}} + \frac{\partial}{\partial\vec{k}'}\right)^2\sigma_{\vec{k}\vec{k}'} + O(q^3).$$

It will be useful to change variables to $\underline{P} = \vec{k} - \vec{k}'$ (defined earlier) and $\underline{K} = \left(\vec{k}+\vec{k}'\right)/2$. Note that

$$\frac{\partial}{\partial\underline{K}} = \frac{\partial}{\partial\vec{k}}\frac{\partial\vec{k}}{\partial\underline{K}} + \frac{\partial}{\partial\vec{k}'}\frac{\partial\vec{k}'}{\partial\underline{K}} = \frac{\partial}{\partial\vec{k}} + \frac{\partial}{\partial\vec{k}'}$$

which means that

$$\sigma_{\vec{k}_\pm\vec{k}'_\pm} \approx \sigma_{\vec{k}\vec{k}'} + \vec{q}\cdot\frac{\partial}{\partial\underline{K}}\sigma_{\vec{k}\vec{k}'} + \frac{1}{2}\left(\vec{q}\cdot\frac{\partial}{\partial\underline{K}}\right)^2\sigma_{\vec{k}\vec{k}'}.$$

The equation now reads

$$\dot{\sigma}_{kk'} = \xi\int d^3q|\phi(\vec{q})|^2 qe^{-a\frac{q^2}{4}} \times\left[\sigma_{\vec{k}\vec{k}'}e^{a\mu\vec{k}\cdot\vec{q}}\left(e^{-\frac{a\mu q^2}{2}} - e^{\frac{a\mu q^2}{2}}\right) - \left(\vec{q}\cdot\frac{\partial}{\partial\underline{K}}\sigma_{\vec{k}\vec{k}'} + \frac{1}{2}\left(\vec{q}\cdot\frac{\partial}{\partial\underline{K}}\right)^2\sigma_{\vec{k}\vec{k}'}\right)e^{a\mu\vec{k}\cdot\vec{q}}e^{\frac{a\mu q^2}{2}}\right.$$

$$\left.+\sigma_{\vec{k}\vec{k}'}e^{-a\mu\vec{k}\cdot\vec{q}}\left(e^{-\frac{a\mu q^2}{2}} - e^{\frac{a\mu q^2}{2}}\right) - \left(-\vec{q}\cdot\frac{\partial}{\partial\underline{K}}\sigma_{\vec{k}\vec{k}'} + \frac{1}{2}\left(\vec{q}\cdot\frac{\partial}{\partial\underline{K}}\right)^2\sigma_{\vec{k}\vec{k}'}\right)e^{-a\mu\vec{k}\cdot\vec{q}}e^{\frac{a\mu q^2}{2}}\right]$$

$$= \xi\int d^3q|\phi(\vec{q})|^2 qe^{-a\frac{q^2}{4}} \times\left[\sigma_{\vec{k}\vec{k}'}\left(e^{a\mu\vec{k}\cdot\vec{q}} + e^{-a\mu\vec{k}\cdot\vec{q}}\right)\left(e^{-\frac{a\mu q^2}{2}} - e^{\frac{a\mu q^2}{2}}\right) - e^{\frac{a\mu q^2}{2}}\left(\left(\vec{q}\cdot\frac{\partial}{\partial\underline{K}}\right)\sigma_{\vec{k}\vec{k}'}\left(e^{a\mu\vec{k}\cdot\vec{q}} - e^{-a\mu\vec{k}\cdot\vec{q}}\right) + \frac{1}{2}\left(\vec{q}\cdot\frac{\partial}{\partial\underline{K}}\right)^2\sigma_{\vec{k}\vec{k}'}\left(e^{a\mu\vec{k}\cdot\vec{q}} + e^{-a\mu\vec{k}\cdot\vec{q}}\right)\right)\right]$$

We can write this in terms of $\underline{K}$ and $\underline{P}$ and make the further substitution $j = a\mu = \hbar^2\beta/2M$:

$$\dot{\sigma}_{kk'} = \xi\int d^3q|\phi(\vec{q})|^2 qe^{-a\frac{q^2}{4}}e^{\frac{j}{2}\underline{P}\cdot\vec{q}} \times\left[\sigma_{\vec{k}\vec{k}'}\left(e^{j\underline{K}\cdot\vec{q}} + e^{-j\underline{K}\cdot\vec{q}}\right)\left(e^{-\frac{jq^2}{2}} - e^{\frac{jq^2}{2}}\right) - e^{\frac{jq^2}{2}}\left(\left(\vec{q}\cdot\frac{\partial}{\partial\underline{K}}\right)\sigma_{\vec{k}\vec{k}'}\left(e^{j\underline{K}\cdot\vec{q}} - e^{-j\underline{K}\cdot\vec{q}}\right) + \frac{1}{2}\left(\vec{q}\cdot\frac{\partial}{\partial\underline{K}}\right)^2\sigma_{\vec{k}\vec{k}'}\left(e^{j\underline{K}\cdot\vec{q}} + e^{-j\underline{K}\cdot\vec{q}}\right)\right)\right]$$

Expanding the dot product exponentials to second order gives

$$\dot{\sigma}_{kk'} = \xi\int d^3q|\phi(\vec{q})|^2 qe^{-a\frac{q^2}{4}}\left(1 + \frac{j}{2}\underline{P}\cdot\vec{q} + \frac{j^2}{8}(\underline{P}\cdot\vec{q})^2\right)\left[\sigma_{\vec{k}\vec{k}'}\left(2 + j^2(\underline{K}\cdot\vec{q})^2\right)\left(e^{-\frac{jq^2}{2}} - e^{\frac{jq^2}{2}}\right)\right.$$

$$\left.-e^{\frac{jq^2}{2}}\left(\left(\vec{q}\cdot\frac{\partial}{\partial\underline{K}}\right)\sigma_{\vec{k}\vec{k}'}\left(2j\underline{K}\cdot\vec{q} + j^6(\underline{K}\cdot\vec{q})^6/3\right) + \frac{1}{2}\left(\vec{q}\cdot\frac{\partial}{\partial\underline{K}}\right)^2\sigma_{\vec{k}\vec{k}'}\left(2 + j^2(\underline{K}\cdot\vec{q})^2\right)\right)\right]$$

$$\dot{\sigma}_{kk'} = \xi\int d^3q|\phi(\vec{q})|^2 qe^{-a\frac{q^2}{4}}\left[\sigma_{\vec{k}\vec{k}'}\left(2 + j(\underline{P}\cdot\vec{q}) + \frac{j^2}{4}(\underline{P}\cdot\vec{q})^2 + j^2(\underline{K}\cdot\vec{q})^2\right)\left(e^{-\frac{jq^2}{2}} - e^{\frac{jq^2}{2}}\right)\right.$$

$$\left.-e^{\frac{jq^2}{2}}\left(\left(\vec{q}\cdot\frac{\partial}{\partial\underline{K}}\right)\sigma_{\vec{k}\vec{k}'}\left(2j(\underline{K}\cdot\vec{q}) + j^2(\underline{P}\cdot\vec{q})(\underline{K}\cdot\vec{q})\right) + \frac{1}{2}\left(\vec{q}\cdot\frac{\partial}{\partial\underline{K}}\right)^2\sigma_{\vec{k}\vec{k}'}\left(2 + j\underline{P}\cdot\vec{q} + \frac{j^2}{4}(\underline{P}\cdot\vec{q})^2 + j^2(\underline{K}\cdot\vec{q})^2\right)\right)\right]$$

We **assume** that q is isotropic, so that first-order terms of the form $\underline{K}\cdot\vec{q}$ average to zero when we integrate over all q-space.

$$\dot{\sigma}_{kk'} = \xi \int d^3q \left|\phi(\vec{q})\right|^2 q e^{-a\frac{q^2}{4}} \Bigg[\sigma_{\vec{k}\vec{k}'} \left(2 + \frac{j^2}{4}(\underline{P}\cdot\vec{q})^2 + j^2(\underline{K}\cdot\vec{q})^2 \right)\left(e^{-\frac{jq^2}{2}} - e^{\frac{jq^2}{2}} \right)$$

$$-e^{\frac{jq^2}{2}} \left(\left(\vec{q}\cdot\frac{\partial}{\partial\underline{K}} \right) \sigma_{\vec{k}\vec{k}'} \left(2j(\underline{K}\cdot\vec{q}) + j^2(\underline{P}\cdot\vec{q})(\underline{K}\cdot\vec{q}) \right) + \frac{1}{2}\left(\vec{q}\cdot\frac{\partial}{\partial\underline{K}} \right)^2 \sigma_{\vec{k}\vec{k}'} \left(2 + \frac{j^2}{4}(\underline{P}\cdot\vec{q})^2 + j^2(\underline{K}\cdot\vec{q})^2 \right) \right) \Bigg]$$

$$= \xi \int d^3q \left|\phi(\vec{q})\right|^2 q e^{-a\frac{q^2}{4}} \Bigg[\sigma_{\vec{k}\vec{k}'} \left(2 + \frac{j^2}{4}(\underline{P}\cdot\vec{q})^2 + j^2(\underline{K}\cdot\vec{q})^2 \right)\left(e^{-\frac{jq^2}{2}} - e^{\frac{jq^2}{2}} \right) - e^{\frac{jq^2}{2}} \left(2j\left(\vec{q}\cdot\frac{\partial\sigma_{\vec{k}\vec{k}'}}{\partial\underline{K}} \right)(\underline{K}\cdot\vec{q}) + \left(\vec{q}\cdot\frac{\partial\sigma_{\vec{k}\vec{k}'}}{\partial\underline{K}} \right)^2 \right) \Bigg]$$

Let us examine the quadratic term $(\underline{K}\cdot\vec{q})(\underline{P}\cdot\vec{q})$ by expanding it into components and examining the effect of integration over q:

$$\int d^3q(\underline{K}\cdot\vec{q})(\underline{P}\cdot\vec{q}) = \int dq_1dq_2dq_3 \sum_{l,m} K_l P_m q_l q_m = \int dq_1dq_2dq_3 \left(\sum_l K_l P_l q_l^{\,2} + \sum_{l\neq m} K_l P_m q_l q_m \right) = \sum_l \int dq_1dq_2dq_3 K_l P_l q_l^{\,2} + \sum_{l\neq m} \int dq_1dq_2dq_3 K_l P_m q_l q_m .$$

The other terms can be obtained by simply seting $\underline{K} = \underline{P}$ in the above derivation. If we consider that odd functions integrate to zero over all space, the second term vanishes. Taking the integrand of the nonvanishing term, we have

$$\dot{\sigma}_{kk'} = \xi \int d^3q \left|\phi(\vec{q})\right|^2 q e^{-a\frac{q^2}{4}} \Bigg[\sigma_{\vec{k}\vec{k}'} \left(2 + \frac{j^2}{4}\sum_l P_l^2 q_l^{\,2} + j^2 \sum_l K_l^{\,2} q_l^{\,2} \right)\left(e^{-\frac{jq^2}{2}} - e^{\frac{jq^2}{2}} \right) - e^{\frac{jq^2}{2}} \left(2j\sum_l K_l \frac{\partial\sigma_{\vec{k}\vec{k}'}}{\partial K_l} q_l^{\,2} + \sum_l \frac{\partial^2\sigma_{\vec{k}\vec{k}'}}{\partial K_l^{\,2}} q_l^{\,2} \right) \Bigg]$$

$$= \xi \int d^3q \left|\phi(\vec{q})\right|^2 q e^{-a\frac{q^2}{4}} \sum_l \Bigg[\sigma_{\vec{k}\vec{k}'} \left(2 + \frac{j^2}{4} P_l^2 q_l^{\,2} + j^2 K_l^{\,2} q_l^{\,2} \right)\left(e^{-\frac{jq^2}{2}} - e^{\frac{jq^2}{2}} \right) - e^{\frac{jq^2}{2}} \left(2jK_l \frac{\partial\sigma_{\vec{k}\vec{k}'}}{\partial K_l} q_l^{\,2} + \frac{\partial^2\sigma_{\vec{k}\vec{k}'}}{\partial K_l^{\,2}} q_l^{\,2} \right) \Bigg]$$

We choose coordinants such that K is in the z direction ($K_l = K\delta_{lz}$ and $q_l = q\cos\theta$) for all terms except the coefficient of the first derivative. For that term, we choose coordinants such that q is in the z direction ($q_l = q\delta_{lz}$ and $K_l = K\cos\theta$). This gives

$$\sum_l K_l \frac{\partial\sigma_{\vec{k}\vec{k}'}}{\partial K_l} q_l^{\,2} = q^2 K\cos\theta \frac{\partial\sigma_{\vec{k}\vec{k}'}}{\partial K\cos\theta} = q^2 K\cos\theta\left(\frac{1}{\cos\theta}\frac{\partial\sigma_{\vec{k}\vec{k}'}}{\partial K} - \frac{1}{K\sin\theta}\frac{\partial\sigma_{\vec{k}\vec{k}'}}{\partial\theta} \right) = \left(q^2 K \frac{\partial\sigma_{\vec{k}\vec{k}'}}{\partial K} - q^2\cot\theta\frac{\partial\sigma_{\vec{k}\vec{k}'}}{\partial\theta} \right)$$

The second term integrates to zero, so altogether we have

$$\dot{\sigma}_{kk'} = \xi \int q^3\, dq d\Omega \left|\phi(\vec{q})\right|^2 e^{-a\frac{q^2}{4}} \Bigg[\sigma_{\vec{k}\vec{k}'} \left(2 + \frac{j^2}{4} P_z^2 q^2\cos^2\theta + j^2 K^2 q^2 \cos^2\theta \right)\left(e^{-\frac{jq^2}{2}} - e^{\frac{jq^2}{2}} \right) - e^{\frac{jq^2}{2}} \left(2jK\frac{\partial\sigma_{\vec{k}\vec{k}'}}{\partial K} q^2 + \frac{\partial^2\sigma_{\vec{k}\vec{k}'}}{\partial K^2} q^2\cos^2\theta \right) \Bigg]$$

From this we can carry out the angular integration, noting

$$\int d\Omega \cos^2\theta = \frac{4\pi}{3} \qquad \text{and} \qquad \int d\Omega = 4\pi$$

$$\dot{\sigma}_{kk'} = \xi \frac{4\pi}{3}\int dq |\phi(\vec{q})|^2 e^{-a\frac{q^2}{4}}\left[\sigma_{\vec{k}\vec{k}'}\left(6q^3 + \frac{j^2}{4}P_z^{\,2}q^5 + j^2K^2q^5\right)\left(e^{-\frac{jq^2}{2}} - e^{\frac{jq^2}{2}}\right) - e^{\frac{jq^2}{2}}\left(6jK\frac{\partial\sigma_{\vec{k}\vec{k}'}}{\partial K}q^5 + \frac{\partial^2\sigma_{\vec{k}\vec{k}'}}{\partial K^2}q^5\right)\right]$$

We now expand everything to lowest nonvanishing order in j

$$\dot{\sigma}_{kk'} = \xi \frac{4\pi}{3}\int dq |\phi(\vec{q})|^2 e^{-a\frac{q^2}{4}}\left[\sigma_{\vec{k}\vec{k}'}\left(6q^3 + \frac{j^2}{4}P_z^{\,2}q^5 + j^2K^2q^5\right)\left(1 - \frac{jq^2}{2} - 1 - \frac{jq^2}{2}\right) - (1)\left(6jK\frac{\partial\sigma_{\vec{k}\vec{k}'}}{\partial K}q^5 + \frac{\partial^2\sigma_{\vec{k}\vec{k}'}}{\partial K^2}q^5\right)\right]$$

$$= \xi \frac{4\pi}{3}\int dq |\phi(\vec{q})|^2 e^{-a\frac{q^2}{4}}\left(-6jq^5\sigma_{\vec{k}\vec{k}'} - 6jKq^5\frac{\partial\sigma_{\vec{k}\vec{k}'}}{\partial K} - q^5\frac{\partial^2\sigma_{\vec{k}\vec{k}'}}{\partial K^2}\right)$$

We now have a general equation for any interaction potential subject to the form constraints placed earlier. To progress in our particular case, we must choose a particular interaction potential. We will choose a gaussian potential:

$$\phi(\vec{r}) = \phi_0 e^{-\frac{r^2}{d^2}}$$

so that

$$\phi(\vec{q}) = \int e^{i\vec{q}\cdot\vec{r}}\phi_0 e^{-\frac{r^2}{d^2}}d^3r = -2\pi\phi_0\int e^{iqr\cos\theta}e^{-\frac{r^2}{d^2}}r^2 dr d\cos\theta = -2\pi\phi_0\int\frac{1}{q}\sin(qr)2re^{-\frac{r^2}{d^2}}dr$$

$$= -2\pi\phi_0\left[\frac{d^2}{q}\sin(qr)\Big|_0^\infty - \int_0^\infty d^2\cos(qr)e^{-\frac{r^2}{d^2}}dr\right] = \phi_0 d^2\frac{\pi}{2}\int_{-\infty}^{\infty}\left(e^{iqr-\frac{r^2}{d^2}} + e^{-iqr-\frac{r^2}{d^2}}\right)dr = \phi_0\pi^{3/2}d^3e^{-\frac{q^2d^2}{4}}.$$

This gives

$$\dot{\sigma}_{kk'} = \xi\phi_0^{\,2}\frac{4\pi^4}{3}d^6\int dq e^{-\frac{q^2}{4}(2d^2+a)}\left(-6jq^5\sigma_{\vec{k}\vec{k}'} - 2jKq^5\frac{\partial\sigma_{\vec{k}\vec{k}'}}{\partial K} - q^5\frac{\partial^2\sigma_{\vec{k}\vec{k}'}}{\partial K^2}\right) = \xi\frac{\phi_0^{\,2}(4\pi)^4d^6}{3(2d^2+a)^3}\left(-6j\sigma_{\vec{k}\vec{k}'} - 6jK\frac{\partial\sigma_{\vec{k}\vec{k}'}}{\partial K} - \frac{\partial^2\sigma_{\vec{k}\vec{k}'}}{\partial K^2}\right)$$

Finally, recalling that $\xi = \frac{-1}{\hbar^2}\frac{m}{\hbar}\frac{1}{(2\pi)^6}n_0\left(\frac{a}{\pi}\right)^{3/2} = \frac{-n_0\beta^{3/2}}{(2\pi)^{15/2}\sqrt{m}}$ we get

$$\dot{\sigma}_{kk'} = \frac{16\phi_0^{\,2}n_0\beta^{3/2}d^6}{3(2\pi)^{7/2}(2d^2+a)^3\sqrt{m}}\left(6\frac{\hbar^2\beta}{M}\sigma_{\vec{k}\vec{k}'} + 6\frac{\hbar^2\beta}{M}K\frac{\partial\sigma_{\vec{k}\vec{k}'}}{\partial K} + \frac{\partial^2\sigma_{\vec{k}\vec{k}'}}{\partial K^2}\right) \equiv Z\left(6j\sigma_{\vec{k}\vec{k}'} + 6jK\frac{\partial\sigma_{\vec{k}\vec{k}'}}{\partial K} + \frac{\partial^2\sigma_{\vec{k}\vec{k}'}}{\partial K^2}\right)$$

This gives us the equation of motion for the reduced density operator in the interaction picture. In order to compare it to our earlier pointer-system results, we need to convert this into the Scrödinger picture. We know the equation of motion is given by

$$i\hbar\dot{\rho}_s = \left[\hat{H}, \rho_s\right] = \left[\hat{H}_P, \rho_s\right] + \left[\hat{H}_B + \hat{V}, \rho_s\right]$$

so that the elements of the reduced density matrix in the Scrödinger picture are

$$\begin{aligned}
\langle k|Tr_B(i\hbar\dot{\rho}^{Sch.})|k'\rangle &= \langle k|Tr_B\left(\left[\hat{H}_P, \rho^{Sch.}\right] + \left[\hat{H}_B + \hat{V}, \rho^{Sch.}\right]\right)|k'\rangle \\
i\hbar\dot{\rho}_{kk'}^{(R)Sch.} &= \langle k|\left[\hat{H}_P, \rho_s\right]|k'\rangle + i\hbar\dot{\rho}_{kk'}^{Int.} \\
\dot{\rho}_{kk'}^{(R)Sch.} &= \frac{-i\hbar}{2M}\left(k^2 - k'^2\right)\rho_{kk'} + \dot{\rho}_{kk'}^{Int.} = \frac{-i\hbar}{M}KP\rho_{kk'} + \frac{16\phi_0{}^2 n_0 \beta^{3/2} d^6}{3(2\pi)^{7/2}\left(2d^2 + a\right)^3\sqrt{m}}\left(6\frac{\hbar^2\beta}{M}\rho_{kk'} + 6\frac{\hbar^2\beta}{M}K\frac{\partial\rho_{kk'}}{\partial K} + \frac{\partial^2\rho_{kk'}}{\partial K^2}\right) \\
&= \frac{-i\hbar}{M}KP\rho_{kk'} + Z\left(6j\rho_{kk'} + 6jK\frac{\partial\rho_{kk'}}{\partial K} + \frac{\partial^2\rho_{kk'}}{\partial K^2}\right)
\end{aligned}$$

We now write this in terms of the Wigner distribution function

$$\rho(P,K,t) = \int_{-\infty}^{\infty} dXe^{-iPX}W(X,K,t)$$

which gives

$$\int_{-\infty}^{\infty} dXe^{-iPX}\dot{W}(X,K,t) = \frac{-i\hbar}{M}KP\int_{-\infty}^{\infty} dXe^{-iPX}W(X,K,t) + Z\left(6j\int_{-\infty}^{\infty} dXe^{-iPX}W(X,K,t) + 6jK\int_{-\infty}^{\infty} dXe^{-iPX}\frac{\partial}{\partial K}W(X,K,t) + \int_{-\infty}^{\infty} dXe^{-iPX}\frac{\partial^2}{\partial K^2}W(X,K,t)\right).$$

Note that

$$\int_{-\infty}^{\infty} dXPe^{-iPX}W = \int_{-\infty}^{\infty} dX\left(i\frac{\partial}{\partial X}e^{-iPX}\right)W = -i\int_{-\infty}^{\infty} dXe^{-iPX}\frac{\partial}{\partial X}W$$

so that

$$\int_{-\infty}^{\infty} dXe^{-iPX}\dot{W} = \int_{-\infty}^{\infty} dXe^{-iPX}\left[-\frac{\hbar}{M}K\frac{\partial}{\partial X}W + Z\left(6jW + 6jK\frac{\partial}{\partial K}W + \frac{\partial^2}{\partial K^2}W\right)\right]$$

and thus

$$\frac{\partial}{\partial t}W + \frac{\hbar}{M}K\frac{\partial}{\partial X}W = Z\left(6jW + 6jK\frac{\partial}{\partial K}W + \frac{\partial^2}{\partial K^2}W\right).$$

This PDE is the fundamental equation governing this model.

B. The Change of Variables Via Characteristics

We wish to solve the PDE

$$\frac{\partial}{\partial t}W \square \frac{\hbar}{M}K\frac{\partial}{\partial X}W = \square W + \square K\frac{\partial}{\partial K}W + Z\frac{\partial^2}{\partial K^2}W$$

by first finding a suitable change of variables. We choose the new variables to be the constants of integration obtained from the characteristic curves, as discussed in the body of the thesis. The change of variables we make in this section actually involves three variables:

$$W(K,X,t) \square\ W(C,\square,\square)$$

where

$$C = Ke^{\square}, \qquad \square = X \square gK, \qquad \text{and } \square = t.$$

The third variable change is trivial, but it is important that we treat it in the same manner as the other two in order to obtain the correct translations for the differentials. Once we obtain the correct translated equation, we immediately switch back to t. To translate out equation into the new variables, we first translate the partial derivatives with respect to the old coordinates.

$$\frac{\partial W}{\partial t} = \frac{\partial W}{\partial C}\frac{\partial C}{\partial t} + \frac{\partial W}{\partial \square}\frac{\partial \square}{\partial t} + \frac{\partial W}{\partial \square}\frac{\partial \square}{\partial t} = \frac{\partial W}{\partial C}\square Ke^{\square} + \frac{\partial W}{\partial \square}$$

and therefore

$$\frac{\partial W}{\partial t} = \frac{\partial \square}{\partial C}\square Ce^{\square} + \frac{\partial \square}{\partial \square}e^{\square} + \square\square e^{\square}$$

since $\frac{\partial W}{\partial \square} = \frac{\partial \square}{\partial \square}e^{\square} + \square\square e^{\square}$. Likewise

$$\frac{\partial W}{\partial X} = \frac{\partial W}{\partial C}\frac{\partial C}{\partial X} + \frac{\partial W}{\partial \square}\frac{\partial \square}{\partial X} + \frac{\partial W}{\partial \square}\frac{\partial \square}{\partial X} = \frac{\partial \square}{\partial \square}e^{\square}$$

$$\frac{\partial W}{\partial K} = \frac{\partial W}{\partial C}\frac{\partial C}{\partial K} + \frac{\partial W}{\partial \square}\frac{\partial \square}{\partial K} + \frac{\partial W}{\partial \square}\frac{\partial \square}{\partial K} = \frac{\partial W}{\partial C}e^{\square} \square g\frac{\partial W}{\partial \square}$$

and from the latter we can find the general relation

$$\frac{\partial}{\partial K} = e^{\square}\frac{\partial}{\partial C} \square g\frac{\partial}{\partial \square}.$$

From these terms, we can rewrite the entire equation

$$\frac{\partial \square}{\partial C}\square Ce^{\square} + \frac{\partial \square}{\partial \square}e^{\square} + \square\square e^{\square} \square \frac{\hbar}{M}K\frac{\partial \square}{\partial \square}e^{\square} = \square\square e^{\square} + \square K\left(\square e^{\square}\frac{\partial}{\partial C} \square g\frac{\partial}{\partial \square}\right)\square e^{\square} + Z\left(\square e^{\square}\frac{\partial}{\partial C} \square g\frac{\partial}{\partial \square}\right)^{2}\square e^{\square}$$

which simplifies to

$$\frac{\partial \square}{\partial \square}e^{\square} = Z\left(\square e^{\square}\frac{\partial}{\partial C} \square g\frac{\partial}{\partial \square}\right)^{2}\square e^{\square}.$$

C. Final Integrations

Nearly every integral in this thesis has the following form

$$\int_{\square} dx \;\; \exp\left[\square\left(ax^{2} + bx + c\right)\right].$$

The solution to this integral is well documented [3]:

$$\int_{\square} dx \;\; \exp\left[\square\left(ax^{2} + bx + c\right)\right] = \sqrt{\frac{\square}{a}}\exp\left[\frac{b^{2}}{4a}\square c\right].$$

C.1 The Diagonal Solutions

In order to obtain the final position representation expression for the elements of $\square$, we must preform three variable changes using integrals of this form. For the diagonal case, inserting the inital state gives us

$$\square(P,K;\pm,\pm;t) = a_{\pm}{}^{2}2\sqrt{2\square\square^{2}u}\,\frac{\sqrt{4\square}}{(2\square)^{2}}e^{\square}\int_{\square} dK\square\exp\left[\square iP\square \square P^{2}Z\square \square u\left(K\square \square Ke^{\square}\right)^{2}\right]\square$$
$$\exp\left[\square 2\square^{2}K\square^{2}\right]\exp\left[\square\square^{2}P^{2}/2 \mp iP\square\right]$$

which can be rewritten as

$$\square(P,K;\pm,\pm;t) = a_{\pm}{}^{2}2\sqrt{2\square\square^{2}u}\,\frac{\sqrt{4\square}}{(2\square)^{2}}e^{\square}\exp\left[\square ue^{2\square}K^{2}\right]\exp\left[\square\square^{2}P^{2}/2 \square P^{2}Z\square \square iPk\square/2 \mp iP\square\right]\square$$
$$\int_{\square} dK\square\exp\left[\square\left(u + 2\square^{2}\right)K\square^{2} \square\left(iP\square/2 \square 2e^{\square}Ku\right)K\square\right].$$

where $\square \equiv g\square\square_{1}^{2}/\square_{2}$. Integrating with respect to $K\square$ gives

$$\square(P,K;\pm,\pm;t) = a_{\pm}{}^{2}\frac{\sqrt{4\square}}{(2\square)}\sqrt{\frac{2\square^{2}u}{u+2\square^{2}}}e^{\square}\exp\square\frac{\square 2\square^{2}u}{\left(u+2\square^{2}\right)}K^{2}e^{2\square}\square\exp\left[\square iP\square/2\mp iP\overline{\square}\right]\square$$

$$\exp\square P^{2}\square\frac{\square^{2}}{2}+Z\square+\frac{\square^{2}}{16\left(u+2\square^{2}\right)}\square\square iP\frac{\square}{2}\square 1+\frac{ue^{\square}}{2\left(u+2\square^{2}\right)}\square K\square$$

This is the correct equation for the reduced density opperator of the pointer in momentum space, given in terms of the relative and center-of-mass momenta (recall that we earlier defined P and K from k and $k\square$ in this way). We now translate this equation into the position space via Fourier transforms. We will make the transformation from (P,K) to (R,r), the center-of-mass and relative position coordinates, respectively (we remember that the Fourier transform inverts the relative / center-of-mass status). We first perform the K integration to get

$$\square(P,r;\pm,\pm;t) = a_{\pm}{}^{2}\exp\square\frac{\square e^{\square 2\square}r^{2}\left(u+2\square^{2}\right)}{8\square^{2}u}\square\exp\left[\square P^{2}A+P\left(Br\mp i\overline{\square}\right)\right]$$

where we have defined

$$A \equiv \frac{\square^{2}}{2}+Z\square+\frac{e^{\square 2\square}\square^{2}}{16u}+\frac{\square^{2}}{32\square^{2}}\left(1+e^{\square\square}\right)^{2}$$

$$= \frac{\square^{2}}{2}+Zg^{2}\square t\square\frac{\left(e^{\square}\square 1\right)\left(3e^{\square}\square 1\right)}{2\square e^{2\square}}\square+\frac{g^{2}}{8\square^{2}}\left(1\square e^{\square\square}\right)^{2}$$

and

$$B \equiv \frac{e^{\square 2\square}\left(u+2\square^{2}\right)\square+ue^{\square\square}\square}{8u\square^{2}}.$$

With this substitution, the final integral is quite simple:

$$\square(R,r;\pm,\pm;t) = a_{\pm}{}^{2}\exp\square\frac{\square e^{\square 2\square}r^{2}\left(u+2\square^{2}\right)}{8\square^{2}u}\square\sqrt{\frac{\square}{A}}\exp\square\square\frac{\left(R\mp\overline{\square}\square iBr\right)^{2}}{4A}\square.$$

C.2 The Off-Diagonal Solutions

The solution for the off diagonal elements is very similar. We begin with the initial state

$$\square(P,K;\pm,\mp;t) = a_\pm a_\mp e^{\pm i\square} 2\sqrt{2\square\square^2 u}\,\frac{\sqrt{4\square}}{(2\square)^2} e^{\square} \int_{\square} dK'\exp\left[\square iP\square\square P^2 Z\square\square u\left(K\square\square K e^{\square}\right)^2\right]\square$$
$$\exp\left[\square 2\square^2 K'^2 \mp iK'\square\overline{\square}\right]\exp\left[\square\square^2 P^2/2\right]$$

which we rewrite as

$$\square(P,K;\pm,\mp;t) = a_\pm a_\mp e^{\pm i\square} 2\sqrt{2\square\square^2 u}\,\frac{\sqrt{4\square}}{(2\square)^2} e^{\square}\exp\left[\square u e^{2\square}K^2\right]\exp\left[\square\square^2P^2/2\square P^2 Z\square\square iPK\square/2\right]\square$$
$$\int_\square dK'\exp\left[\square\left(u+2\square^2\right)K'^2\square\left(iP\square/2\square 2e^{\square}Ku\pm 2i\overline{\square}\right)K'\right].$$

and integrate to get

$$\square(P,K;\pm,\mp;t) = a_\pm a_\mp e^{\pm i\square}\frac{\sqrt{4\square}}{(2\square)}\sqrt{\frac{2\square^2 u}{u+2\square^2}}\,e^{\square}\exp\left(\frac{\square 2\square^2 u}{\left(u+2\square^2\right)}K^2 e^{2\square}\right)\exp\left(\frac{\square\left(P\square\pm 4\overline{\square}\right)^2}{16\left(u+2\square^2\right)}\right)\square$$
$$\exp\left(\square P^2\left(\frac{\square^2}{2}+Z\right)\square iK\left(\frac{\square}{2}P\pm\frac{2ue^{\square}\overline{\square}}{\left(u+2\square^2\right)}\right)\right).$$

Once again, we perform a Fourier transform from K to r to get

$$\square(P,r;\pm,\mp;t) = a_\pm a_\mp e^{\pm i\square}\exp\left(\frac{\square e^{\square 2\square}r^2}{4u}\square\frac{\left(e^{\square\square}r\square 2\overline{\square}\right)^2}{8\square^2}\right)\exp\left(\square P^2 A + P\left(\square Br\mp\square\frac{\left(1+e^{\square\square}\right)\overline{\square}}{4\square^2}\right)\right)$$

where A and B are the same as for the diagonal case. The final integral then produces

$$\square(R,r;\pm,\mp;t) = a_\pm a_\mp e^{\pm i\square}\exp\left(\frac{\square e^{\square 2\square}r^2}{4u}\square\frac{\left(e^{\square\square}r\square 2\overline{\square}\right)^2}{8\square^2}\right)\sqrt{\frac{\square}{A}}\exp\left(\square\left(\square iBr + R\pm i\square\frac{\left(1+e^{\square\square}\right)\overline{\square}}{4\square^2}\right)^2/4A\right).$$

We now set $r = 0$ and expand the square to get

$$\square(R,0;\pm,\mp;t) = a_\pm a_\mp e^{\pm i\square}\exp\left(\square\frac{\overline{\square}^2}{2\square^2}\right)\sqrt{\frac{\square}{A}}\exp\left(\square\left(R^2\pm i\frac{g}{\square^2}\left(1\square e^{\square\square}\right)\overline{\square}\square\frac{g^2}{2\square^2}\left(1\square e^{\square\square}\right)^2\frac{\overline{\square}^2}{2\square^2}\right)/4A\right)$$
$$= a_\pm a_\mp e^{\pm i\square}\sqrt{\frac{\square}{A}}\exp\left(\square\left(R^2\pm i\frac{g}{\square^2}\left(1\square e^{\square\square}\right)\overline{\square}\right)/4A\right)\square$$
$$\exp\left(\square\frac{\overline{\square}^2}{4A}\right)\exp\left(\frac{\overline{\square}^2}{4A}\right)\exp\left(\square\frac{\overline{\square}^2}{2\square^2}\right)\exp\left(\square\frac{g^2}{2\square^2}\left(1\square e^{\square\square}\right)^2\frac{\overline{\square}^2}{2\square^2}\right)$$

We can combine the last three exponentials and expand A to get

$$\exp\left[-\frac{\bar{\beta}^2}{2\sigma^2}\left(1-\frac{2\sigma^2}{4A}-\frac{g^2}{8\gamma^2 A}\left(1-e^{-\gamma t}\right)^2\right)\right]=\exp\left[-\frac{\bar{\beta}^2}{2\sigma^2}\Gamma(t)\right]$$

where we have defined the new quantity

$$\Gamma(t)\equiv\left(\frac{A-\frac{\sigma^2}{2}-\frac{g^2}{8\gamma^2}\left(1-e^{-\gamma t}\right)^2}{A}\right)=\left(\frac{4Zg^2\left(t-\left(e^{\gamma t}-1\right)\left(3e^{\gamma t}-1\right)/2\gamma e^{2\gamma t}\right)}{2\sigma^2+4Zg^2\left(t-\left(e^{\gamma t}-1\right)\left(3e^{\gamma t}-1\right)/2\gamma e^{2\gamma t}\right)+\frac{g^2}{2\gamma^2}\left(1-e^{-\gamma t}\right)^2}\right).$$

The last two terms in the denominator are much smaller than the first term on any typical atomic timescale. Thus, they can be neglected on such timescales.

$$\Gamma(t)\approx\left(\frac{2Zg^2\left(t-\left(e^{\gamma t}-1\right)\left(3e^{\gamma t}-1\right)/2\gamma e^{2\gamma t}\right)}{\sigma^2}\right).$$

As $t\to 0$ this term vanishes. For $\gamma t >> 1$ the second term in the numerator becomes constant, so the term is dominated by the linear part.

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