Physics 6H **Quantum Extra Problems**

EP-1. Balls in a Box - Suppose the box contains $N = 50$ balls, each bearing an integer between 1 and 8; Letting n_k be the number of balls showing the value $v_k = k$, suppose that

$$n_1 = 3, n_2 = 2, n_3 = 5, n_4 = 8, n_5 = 13, n_6 = 9, n_7 = 6, n_8 = 4$$

use the probability concepts we have developed to calculate the probability that the numbers found on two random samplings will sum to 6. [answer = $135/2500$]

EP-2. More Balls in a Box - Consider the collection of numbered balls described in EP-1.

(a) Calculate $\langle v \rangle$ and Δv [answer: $\langle v \rangle$= 4.94 and Δv=1.8]
(b) Sketch a "frequency bar-graph" of the expected results of $M = 100$ samplings [i.e., lay out the values v_k on the horizontal axis, and construct vertical "bars" to indicate the number of times each v_k-value should be obtained]. Show on the graph by means of a vertical line the value $\langle v \rangle$. Also, draw a horizontal line of length $2\Delta v$ in such a way that it indicates roughly the "spread" or "dispersion" of v-values about $\langle v \rangle$.

EP-3. Playing Cards - Two cards are drawn at random from a shuffled deck and laid aside without being examined. Then a third card is drawn. Show that the probability that the third card is a spade is 1/4 just as it was for the first card. HINT: Consider all the (mutually exclusive) possibilities (two discarded cards spades, third card spade or not spade, etc).

EP-4. Birthdays - What is the probability that you and a friend have different birthdays? (for simplicity let a year have 365 days). What is the probability that three people have different birthdays? Show that the probability that n people have n different birthdays is

$$p = \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right)\left(1 - \frac{3}{365}\right).....\left(1 - \frac{n-1}{365}\right)$$

Estimate this for $n << 365$ by calculating $\ell n(p)$ (use the fact that $\ell n(1 + x) \approx x$ for $x << 1$). Find the smallest integer n for which $p < 1/2$. Hence show that for a group of 23 people or more, the probability is greater than 1/2 that two of them have the same birthday.

EP-5. Bayes - Suppose that you have 3 nickels and 4 dimes in your right pocket and 2 nickels and a quarter in your left pocket. You pick a pocket at random and from it select a coin at random. If it is a nickel, what is the probability that it came from your right pocket? Use Baye's formula.

EP-6. Is There Life?

The number of stars in our galaxy is about $N = 10^{11}$. Assume that the probability that a star has planets is $p = 10^{-2}$, the probability that the conditions on the planet are suitable for life is $q = 10^{-2}$, and the probability of life evolving, given suitable conditions, is $r = 10^{-2}$. These numbers are rather arbitrary.

(a) What is the probability of life existing in an arbitrary solar system (a star and planets, if any)?
(b) What is the probability that life exists in at least one solar system?

NOTE: A naive argument against a purely natural origin of life is sometimes based on the smallness of the probability (a), whereas it is the probability (b) that is relevant!

EP-7. Law of Large Numbers

This problem illustrates the law of large numbers (use IDL).

(a) Assuming the probability of obtaining "heads" in a coin toss is 0.5, compare the probability of obtaining "heads" in 5 out of 10 tosses with the probability of obtaining "heads" in 50 out of 100 tosses.
(b) For a set of 10 tosses and for a set of 100 tosses, calculate the probability that the fraction of "heads" will be between 0.445 and 0.555.

EP-8. Complex Numbers - Prove the following properties of complex numbers:

(a) $\operatorname{Re} z = \dfrac{z + z^*}{2}$, $\operatorname{Im} z = \dfrac{z - z^*}{2i}$

(b) z is a pure real number if and only if $z^* = z$ and
z is a pure imaginary number if and only if $z^* = -z$

(c)
$$z^{**} = z$$
$$(z_1 + z_2)^* = z_1^* + z_2^*$$
$$(z_1 z_2)^* = z_1^* z_2^*$$

(d) $|z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2$

(e) $|z| \geq |\operatorname{Re} z|$ and $|z| \geq |\operatorname{Im} z|$

(f) $|z_1 + z_2| \leq |z_1| + |z_2|$

EP-9. Complex Exponentials - Prove the following relations:

(a) $\left(e^{ikx}\right)^* = e^{-ikx}$
(b) $e^{ik_1 x} e^{ik_2 x} = e^{i(k_1 + k_2)x}$
(c) $\left|e^{ikx}\right|^2 = 1$
(d) $\dfrac{d}{dx} e^{ikx} = ike^{ikx}$
(e) $\int e^{ikx} dx = \dfrac{1}{ik} e^{ikx} + C$

EP-10. Complex Expressions - Evaluate the following expressions with your final answers in the form $z = a + bi$.

(a) $z = (2+3i)^3$
(b) $z = e^{i\pi/3}$
(c) $z = i^i$
(d) $z = \dfrac{1}{i-1}$
(e) $z = \sin\left(\dfrac{\pi}{2} + i\ell n(2)\right)$
(f) $z = (1+i)^{1-i}$

EP-10a. Series - Write the first five terms in a series expression for the following:
(a) $\dfrac{e^x}{1-x}$
(b) $\dfrac{\sin\sqrt{x}}{\sqrt{x}}$
(c) $\ell n(1+x)$
(d) $\sin(\ell n(1+x))$

EP-10b. ODEs - Find solutions to the following ODEs:
(a) $\dfrac{dy}{dt} = -12y \quad , \quad y(3) = 1$
(b) $4\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} - 48y = 0 \quad , \quad y(0) = 0 \, , \, \dfrac{dy}{dx}(0) = 7$
(c) $\dfrac{d^2y}{dx^2} + 16y = 0 \quad , \quad y(0) = 2 \, , \, \dfrac{dy}{dx}(0) = 8$

Rewrite the answer to (c) in terms of sines and cosines.

EP-11. Vectors - A vector 3 inches long points due west. What is its projection on the axis that points:

(a) due west
(b) due east
(c) due north
(d) straight up
(e) half-way between straight up and due west
(f) half-way between due east and due north
(g) half-way between straight up and due north.

EP-12. More Vectors - Given two vectors

$$\vec{A} = 7\hat{e}_1 + 6\hat{e}_2 - 13\hat{e}_3 \quad , \quad \vec{B} = -2\hat{e}_1 + 16\hat{e}_2 + 4\hat{e}_3$$

(a) Determine $\vec{A} \pm \vec{B}$
(b) Determine $\vec{A} \cdot \vec{B}$
(c) Determine a unit vector in the same direction as $\vec{A}$

EP-13. Bases - Given two vectors $\vec{A} = 7\hat{e}_1 + 6\hat{e}_2 \quad , \quad \vec{B} = -2\hat{e}_1 + 16\hat{e}_2$ written in the $\{\hat{e}_1, \hat{e}_2\}$ basis set and given another basis set

$$\hat{e}_q = \frac{1}{2}\hat{e}_1 + \frac{\sqrt{3}}{2}\hat{e}_2 \quad , \quad \hat{e}_p = -\frac{\sqrt{3}}{2}\hat{e}_1 + \frac{1}{2}\hat{e}_2$$

(a) Show that $\hat{e}_q$ and $\hat{e}_p$ are orthonormal

(b) Draw a diagram showing $\hat{e}_q$, $\hat{e}_p$, $\vec{A}$ and $\vec{B}$.

(c) Determine the new components of $\vec{A}$ and $\vec{B}$ in the $\{\hat{e}_q, \hat{e}_p\}$ basis set

EP-14. More Vectors - For the points A(3,1,1), B(4,2,0) and C(0,4,3) answer the following questions where O denotes the origin

(a) Determine vectors $\vec{Q}$ (A to B), $\vec{P}$ (B to C) and $\vec{R}$ (O to A)

(b) What is the distance from A to B?

(c) What is the angle between the vectors $\vec{Q}$ and $\vec{P}$

EP-15. Brackets - Prove that

$$\langle V|W\rangle = \langle W|V\rangle^* \qquad \text{(equation (02\_37)}$$

EP-16. Soft State - Prove $|soft\rangle = |s\rangle = \frac{1}{\sqrt{2}}|g\rangle - \frac{1}{\sqrt{2}}|m\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

EP-17. Operators Acting - Given two operators

$$\hat{O}_1 = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \quad \text{and} \quad \hat{O}_2 = \begin{pmatrix} 10 & 4 \\ -5 & -2 \end{pmatrix}$$

and two vectors

$$|X\rangle = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad \text{and} \quad |Y\rangle = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Determine $\hat{O}_1|X\rangle$, $\hat{O}_2|X\rangle$, $\hat{O}_1|Y\rangle$, and $\hat{O}_2|Y\rangle$

EP-18. Eigenvectors - Here is an operator and four vectors

$$\hat{O} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \ , \ |A\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ |B\rangle = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \ |C\rangle = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \ |D\rangle = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Which of the four vectors are eigenvectors of the operator $\hat{O}$ and what are their eigenvalues?

EP-19. Color Operator - If we know what an operator does, then we can construct it. Let us do this for the color operator. We work in the hardness basis:

$|hard\rangle = |h\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|soft\rangle = |s\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with hardness operator $\hat{O}_h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Using this basis, the color states are given by

$$|green\rangle = |g\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{and} \quad |magenta\rangle = |m\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

All we need to remember in order to construct the color operator is how it acts on the color states

$$\hat{O}_c|g\rangle = |g\rangle \quad \text{and} \quad \hat{O}_c|m\rangle = -|m\rangle$$

Write out these last two equations treating the four elements of the color operator as unknowns. You should get four equations that allow you to calculate the four elements of the color operator. Do you get the same operator as in the textbook?

EP-20. Linear Functional - We call $|\psi\rangle$ a ket-vector (state) and $\langle\psi|$ its dual vector or bra-vector (state). Technically the dual vector is a linear functional, that is, a new mathematical object which has the property of turning ket-vectors into numbers (complex). An "operational" way to define the linear functional or bra-vector is as follows.

In a 2-dimensional vector space, if $|\psi\rangle = a_1|1\rangle + a_2|2\rangle$, where $(|1\rangle,|2\rangle)$ form an orthonormal basis, then $\langle\psi| = a_1^*\langle 1| + a_2^*\langle 2|$. It then acts on a ket-vector to produce a scalar. Show that $\langle\psi|\psi\rangle = |a_1|^2 + |a_2|^2$.

EP-21. Probabilities - We stated that if a particle is in the state $|\psi\rangle$ and you measure its color, for example, the outcome of the measurement is either $|green\rangle$ with probability $|\langle g|\psi\rangle|^2$ or $|magenta\rangle$ with probability $|\langle m|\psi\rangle|^2$. Let us try this out on a few states.

What are the possible outcomes and with what probability do they occur if the color is measured of a particle in

(a) the state $|hard\rangle$
(b) the state $|soft\rangle$
(c) the state $|\psi\rangle = \sqrt{\frac{3}{4}}|hard\rangle + \frac{1}{2}|soft\rangle$ (use the (hard,soft) basis for the calculation).

EP-22. Bases and Matrices - Suppose we have an orthonormal set of basis vectors $|1\rangle, |2\rangle$, and $|3\rangle$

(a) Express the orthonormality using scalar products between the basis vectors.

Suppose we have some operator $\hat{G}$ with the following properties with respect to this basis set

$$\hat{G}|1\rangle = 2|1\rangle - 4|2\rangle + 7|3\rangle$$
$$\hat{G}|2\rangle = -2|1\rangle + 3|3\rangle$$
$$\hat{G}|3\rangle = 11|1\rangle + 2|2\rangle - 6|3\rangle$$

(b) Write out the matrix representing the operator $\hat{G}$ in the $\{|1\rangle,|2\rangle,|3\rangle\}$ basis.

EP-23. Eigenvectors - Find the eigenvalues and eigenvectors of the operator

$$\hat{O}=\begin{pmatrix}0 & 2\\ 2 & 3\end{pmatrix}$$

EP-24. Spectral Decomposition - Suppose an operator $\hat{K}$ has the following eigenvectors and eigenvalues

$$\hat{K}|1\rangle = 2|1\rangle$$
$$\hat{K}|2\rangle = 3|2\rangle$$
$$\hat{K}|3\rangle = -6|3\rangle$$

(a) Write an expression for $\hat{K}$ in terms of its eigenvalues and eigenvectors (projection operators). Use this expression to derive the matrix representing $\hat{K}$ in the $\{|1\rangle,|2\rangle,|3\rangle\}$ basis.

(b) What is the expectation value of $\hat{K}$ in the state

$$|\alpha\rangle = \frac{1}{\sqrt{83}}\left(-3|1\rangle + 5|2\rangle + 7|3\rangle\right)$$

EP-25. Multiply Matrices

(a) Multiply the two matrices

$$\hat{A}=\begin{pmatrix}1 & 5 & 4\\ 7 & 2 & 1\\ 9 & 2 & 3\end{pmatrix} \quad , \quad \hat{B}=\begin{pmatrix}6 & 9 & -2\\ 5 & 5 & -3\\ -3 & -5 & 1\end{pmatrix}$$

(b) Determine the commutator $\left[\hat{A},\hat{B}\right]$

EP-26. Matrix Properties

$$\hat{B}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \hat{C}=\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$$

(a) Show that $\hat{C}=\hat{I}+2\hat{B}$
(b) Show that $\left[\hat{B},\hat{C}\right]=0$
(c) Find the eigenvectors and eigenvalues of $\hat{B}$ and $\hat{C}$

EP-27. Measurement Results - Given particles in state $|\alpha\rangle = \frac{1}{\sqrt{83}}\left(-3|1\rangle + 5|2\rangle + 7|3\rangle\right)$ where $\{|1\rangle,|2\rangle,|3\rangle\}$ form an orthonormal basis, what are the possible experimental results for a measurement of $\hat{Y}$ and with what probabilities do they occur if the operator $\hat{Y}$ is (in this basis)

$$\hat{Y} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

EP-28. Eigenvalue and Eigenvectors

Find the eigenvalues and normalized eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 0 \\ 5 & 0 & 3 \end{pmatrix}$$

Are the eigenvectors orthogonal? Comment on this.

EP-29. Orthogonal Basis Vectors

Compute the eigenvectors of the matrix operator

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Construct an orthonormal basis set from the eigenvectors of this operator.

EP-30. Expectation values

Let $R = \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}$ represent an observable, and $|\Psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$ be an arbitrary state vector(with $|a|^2 + |b|^2 = 1$). Calculate $\langle R^2 \rangle$ in two ways:

(a) Evaluate $\langle R^2 \rangle = \langle \Psi | R^2 | \Psi \rangle$ directly.

(b) Find the eigenvalues and eigenvectors of R, $R|r_n\rangle = r_n|r_n\rangle, n = 1,2$. Expand the state vector as a linear combination of the eigenvectors $|\Psi\rangle = c_1|r_1\rangle + c_2|r_2\rangle$ and evaluate $\langle R^2 \rangle = r_1^2|c_1|^2 + r_2^2|c_2|^2$. Do these results agree with the general results we derived earlier?

EP-31 Bases and Probabilities

The initial state $|\psi_{init}\rangle$ of a quantum system is given in an orthonormal basis consisting of the three states $|\alpha\rangle, |\beta\rangle$ and $|\gamma\rangle$ by the components

$$\langle \alpha | \psi_{init} \rangle = \frac{i}{\sqrt{3}} \quad , \quad \langle \beta | \psi_{init} \rangle = \sqrt{\frac{2}{3}} \quad , \quad \langle \gamma | \psi_{init} \rangle = 0$$

Calculate the probability of finding the system in the state $|\psi_{final}\rangle$ which has the components

$$\langle\alpha|\psi_{final}\rangle = \frac{1+i}{\sqrt{3}} \quad , \quad \langle\beta|\psi_{final}\rangle = \sqrt{\frac{1}{6}} \quad , \quad \langle\gamma|\psi_{final}\rangle = \sqrt{\frac{1}{6}}$$

in the same basis.

EP-32. Projection Operator Representation

Let the states $\{|1\rangle, |2\rangle |3\rangle\}$ form an orthonormal basis. We consider the operator given by $\hat{P}_2 = |2\rangle\langle 2|$. What is the matrix representation of this operator? What are its eigenvalues and eigenvectors. For the arbitrary state $|A\rangle = \frac{1}{\sqrt{83}}\left(-3|1\rangle + 5|2\rangle + 7|3\rangle\right)$, what is the result of $\hat{P}_2|A\rangle$?

EP-33. Operator Algebra

An operator for a two-state system is given by

$$\hat{H} = a\left(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|\right)$$

where a is a number. Find the eigenvalues and the corresponding eigenkets (linear combinations of $|1\rangle$ and $|2\rangle$, which are eigenkets).

EP-34. Spectral Decomposition

Find the eigenvalues and eigenvectors of the matrix

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Construct the corresponding projection operators, and verify that the matrix can be written in terms of its eigenvalues and eigenvectors. This is the **spectral decomposition** for this matrix.

EP-35. Powers - Suppose that we have some operator $\hat{Q}$ such that $\hat{Q}|q\rangle = q|q\rangle$ i.e., $|q\rangle$ is an eigenvector of $\hat{Q}$ with eigenvalue q.

(a) Show that $|q\rangle$ is also an eigenvector of the operators $\hat{Q}^2, \hat{Q}^n$ and $e^{\hat{Q}}$
(b) What are the corresponding eigenvalues?

EP-36. Eigenket properties

Consider a 3-dimensional ket space. If a certain set of orthonormal kets, say $|1\rangle$, $|2\rangle$, and $|3\rangle$, are used as the basis kets, the operators $\hat{A}$ and $\hat{B}$ are represented by

$$\hat{A} \rightarrow \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \quad , \quad \hat{B} \rightarrow \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$$

where a and b are both real numbers.

(a) Obviously $\hat{A}$ exhibits a degenerate spectrum. Does $\hat{B}$ also exhibit a degenerate spectrum?
(b) Show that $\hat{A}$ and $\hat{B}$ commute.
(c) Find a new set of orthonormal kets which are simultaneous eigenkets of both $\hat{A}$ and $\hat{B}$. Specify the eigenkets of $\hat{A}$ and $\hat{B}$. Does your specification of eigenvalues completely characterize each eigenket?

EP-37. Hardness World - Let us define a state using the hardness basis $(|h\rangle, |s\rangle)$

$$|A\rangle = \cos\theta |h\rangle + e^{i\phi} \sin\theta |s\rangle$$

where θ and ϕ are constants.

(a) Is this state normalized? Show your work.
(b) Find the state $|B\rangle$ that is orthogonal to $|A\rangle$. Make sure $|B\rangle$ is normalized.
(c) Express $|h\rangle$ and $|s\rangle$ in the $(|A\rangle, |B\rangle)$ basis.
(d) What are the possible outcomes of a hardness measurement on state $|A\rangle$ and with what probability will each occur?
(e) Express the hardness operator in the $(|A\rangle, |B\rangle)$ basis.

EP-38. Eigenvalues - Determine the eigenvalues and eigenstates of the following matrix

$$\begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

EP-39. Two-Path Experiment - Consider the following situations based on the two path color-hardness experiment in the notes:

(a) just as in the figure
(b) a wall in one path (x_3, y_1)
(c) a wall in one path (x_2, y_2)

Make a chart with each row representing one of these situations. In the columns of the chart, give

(1) the state of the particles that emerge at (x_5, y_4)
(2) the probability that a hardness box at (x_5, y_4) measures the hardness to be hard
(3) the probability that a hardness box at (x_5, y_4) measures the hardness to be soft
(4) the probability that a color box at (x_5, y_4) measures the color to be green
(5) the probability that a color box at (x_5, y_4) measures the color to be magenta

EP-40. Two-Path Experiment redux - Imagine now that a hardness box is placed in the one path at (x_2, y_2). The hard output of this box is blocked, but the soft output of this box sends the particles along the same path they were on before they entered this additional box.

(a) What is the state of the particles that emerge at (x_5, y_4)? What fraction of the particles that enter the apparatus emerge at (x_5, y_4), and what would be the results of measurements of the hardness and color at (x_5, y_4)?
(b) Answer the same questions if instead a color box is placed in the one path at (x_3, y_1) with its magenta output blocked and the green output directing the particles along their original direction.

EP-41. Optical Elements - Suppose three optical elements can be represented by the following three operators in the $(|x\rangle, |y\rangle)$ basis.

$$\hat{O}_{\lambda/4} = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \qquad \hat{O}_{CP} = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \qquad \hat{O}_{\lambda/2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(a) What is the probability of a photon in state $|45°\rangle$ getting through the element represented by $\hat{O}_{\lambda/4}$ and what state will it be in if it does?
(b) What is the probability of a photon in state $|0°\rangle$ getting through the element represented by $\hat{O}_{CP}$ and what state will it be in if it does?
(c) What is the probability of a photon in state $|135°\rangle$ getting through the element represented by $\hat{O}_{\lambda/2}$ and what state will it be in if it does?

EP-42. Polaroids - Imagine a situation in which a photon in the $|x\rangle$ state strikes a vertical polaroid. Clearly the probability of the photon getting through the vertical polaroid is 0. Now consider the case of two polaroids with the photon in the $|x\rangle$ state striking a polaroid at $45°$ and then striking a vertical polaroid.

(a) Show that the probability of the photon getting through both polaroids is 1/4.

Consider now the case of three polaroids with the photon in the $|x\rangle$ state striking a polaroid at $30°$ first, then a polaroid at $60°$ and finally a vertical polaroid.

(b) Show that the probability of the photon getting through all three polaroids is 27/64.

EP-43. Change the Basis - In examining light polarization, we have been working in the $(|x\rangle, |y\rangle)$ basis.

(a) Just to show how easy it is to work in other bases, express $|x\rangle$ and $|y\rangle$ in the $(|R\rangle, |L\rangle)$ and $(|45°\rangle, |135°\rangle)$ bases.

(b) If you are working in the $(|R\rangle, |L\rangle)$ basis, what would the operator representing a vertical polaroid look like?

EP-44. Calcite - A photon is polarized at an angle θ to the optic axis is sent in the z direction through a slab of calcite $10^{-2}\,cm$ thick in the z direction. Assume that the optic axis lies in the x-y plane. Calculate, as a function of θ, the transition probability for the photon to emerge left circularly polarized. Sketch the result.

Let the frequency of the light be given by $c/\omega = 5000\,\mathring{A}$, and let $n_e = 1.50$ and $n_o = 1.65$ for the calcite.

EP-45. Make a Filter - Using calcite and polaroid, devise a filter that will pass light of frequency $c/\omega = 5000\,\mathring{A}$ only if it is right circularly polarized. Use the same calcite parameters as in the previous problem.

EP-46. No Change - What is the condition on the length of a slab of calcite, for frequency ω, such that $|\psi_{out}\rangle$ is always, to within a phase factor, the same as $|\psi_{in}\rangle$?

EP-47. Turpentine - Turpentine is an "optically active" substance. If we send plane polarized light into turpentine then it emerges with its plane of polarization rotated. Specifically, turpentine induces a left-hand rotation of about 5° per cm of turpentine that the light traverses. Write down the transition matrix that relates the incident polarization state to the emergent polarization state. Show that this matrix is unitary. Why is that important? Find its eigenvectors and eigenvalues, as a function of the length of turpentine traversed.

EP-48. Unpolarized Light - Unpolarized light traveling in the z direction is sent through a block of calcite whose optic axis lies in the x-y plane. What is the effect of the calcite on the polarization properties of the beam? What will turpentine do to an unpolarized beam ?

EP-49. What QM is all about - Photons polarized at 30° to the x-axis are sent through a y-polaroid. An attempt is made to determine how frequently the photons that pass through the polaroid, pass through as "right circularly polarized photons" and how frequently they pass through as "left circularly polarized photons". This attempt is made as follows:

First, a prism that passes only right circularly polarized light is placed between the source of the 30° polarized photons and the y-polaroid, and it is determined how frequently the 30° photons pass through the y-polaroid. Then this experiment is repeated with a prism that passes only left circularly polarized photons instead of the one that passes only right.

Show by explicit calculation that the sum of the probabilities for passing through the y-polaroid measured in these two experiments is different from the probability that one would measure if there were no prism in the path of the photon and only the y-polaroid.

Relate this experiment to the two-slit diffraction experiment.

EP-50. More Optical Operators - Look again at the three operator in

Problem 41. The solution to that problem shows that in part (a) incoming $|45°\rangle$ photons come out as $|L\rangle$ photons with probability 1, that in part (b) incoming $|0°\rangle$ photons were changed to $|L\rangle$ photons with probability 1/2, and in part (c) $|135°\rangle$ photons come out as $|-135°\rangle$ photons with probability 1. But if these operators represent measurements, then we can calculate the probabilities by simply finding modulus-squared of the bra-ket of the output state and the input state. This works for parts (b) and (c), but not for part (a). Therefore, the operator in part (a) must not represent a measurement, which makes sense since it is the only one of the three that is not Hermitian. Below are three more operators.

$$\hat{O}_{45} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \hat{O}_{\lambda/4} = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} \qquad \hat{O}_{CP} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

Which ones are Hermitian? What is the output state for incoming photons in the $|y\rangle$ state and what is the probability that one of these photons will get through?

EP-51. Photons and polarizers

A photon polarization state for a photon propagating in the z-direction is given by

$$|\psi\rangle = \sqrt{\frac{2}{3}}|x\rangle + \frac{i}{\sqrt{3}}|y\rangle$$

(a) What is the probability that a photon in this state will pass through a polaroid with its transmission axis oriented in the y-direction?

(b) What is the probability that a photon in this state will pass through a polaroid with its transmission axis y' making an angle ϕ with the y-axis ?

(c) A beam carrying N photons per second, each in the state $|\psi\rangle$, is totally absorbed by a black disk with its normal to the surface in the z-direction. How large is the torque exerted on the disk? In which direction does the disk rotate? REMINDER: The photon states $|R\rangle$ and $|L\rangle$ each carry a unit $\hbar$ of angular momentum parallel and antiparallel, respectively, to the direction of propagation of the photon.

EP-52. Quarter-wave plate

A beam of linearly polarized light is incident on a quarter-wave plate (changes relative phase by $90°$) with its direction of polarization oriented at $30°$ to the optic axis. Subsequently, the beam is absorbed by a black disk. Determine the rate angular momentum is transferred to the disk, assuming the beam carries N photons per second.

EP-53. What is happening?

A system of N ideal linear polarizers is arranged in sequence. The transmission axis of the first polarizer makes an angle ϕ/N with the y-axis. The transmission axis of every other polarizer makes an angle ϕ/N with respect to the axis of the preceding polarizer. Thus, the transmission axis of the final polarizer makes an angle ϕ with the y-axis. A beam of y-polarized photons is incident on the first polarizer.

(a) What is the probability that an incident photon is transmitted by the array?
(b) Evaluate the probability of transmission in the limit of large N.
(c) Consider the special case with the angle $\phi = 90°$. Explain why your result is not in conflict with the fact that $\langle x|y\rangle = 0$.

EP-54. Angular Momentum - When we measure the angular momentum of some atoms along a given direction in space (see Stern-Gerlach discussions), say the z-axis, we get three values $(\hbar, 0, -\hbar)$.

(a) Construct an operator for angular momentum in the z-direction (call it $\hat{O}_{L_z}$) using the most simple basis states that are eigenstates of this operator.

When we measure the magnitude squared of the angular momentum of these same atoms, we always get $\hbar^2$. Also, measurements of the z-component and magnitude squared can be done in any order and the results are the same.

(b) Construct an operator for the magnitude squared of the angular momentum (call it $\hat{O}_{L^2}$) using the same basis states you used for $\hat{O}_{L_z}$.

(c) In this basis, the operator for the x-component of angular momentum is

$$\hat{O}_{L_x} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Are the eigenstates of $\hat{O}_{L_z}$ also eigenstates of $\hat{O}_{L_x}$?

(d) Do $\hat{O}_{L_z}$ and $\hat{O}_{L_x}$ commute?

(e) What can you say about successive measurements of the z- and x-components of the angular momentum?

EP-55. Angular Momentum Eigenvectors - Using the $\hat{O}_{L_z}$ operator found in Problem 54, find the eigenvalues and eigenvectors of the $\hat{O}_{L_x}$ operator.

EP-56. Atoms and Angular Momentum - Consider an atom in the following state

$$|\psi\rangle = \frac{1}{\sqrt{14}}\left(|1\rangle + 2|2\rangle + 3i|3\rangle\right)$$

where $|1\rangle$ is the eigenstate of $\hat{O}_{L_z}$ with eigenvalue $\hbar$, $|2\rangle$ is the eigenstate of $\hat{O}_{L_z}$ with eigenvalue 0 and $|3\rangle$ is the eigenstate of $\hat{O}_{L_z}$ with eigenvalue $-\hbar$.

(a) If the z-component of angular momentum is measured for an atom in the state $|\psi\rangle$, what are the possible results and with what probabilities do they occur?

(b) If the x-component of angular momentum is measured for an atom in the state $|\psi\rangle$, what are the possible results and with what probabilities do they occur?

EP-57. Time development Let the energy operator for a three state system be

$$\hat{H} = \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix}$$

where the basis being used are the energy eigenstates labelled by $|1\rangle$, $|2\rangle$, and $|3\rangle$.

(a) If the state of the system at $t = 0$ is $|\psi(0)\rangle = |2\rangle$, what is the state of the system at time t later $|\psi(t)\rangle$?

(b) If the state of the system at $t = 0$ is $|\psi(0)\rangle = |3\rangle$, what is the state of the system at time t later $|\psi(t)\rangle$?

(c) If the state of the system at $t = 0$ is $|\psi(0)\rangle = \frac{1}{2}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle + \frac{1}{2}|3\rangle$, what is the state of the system at time t later $|\psi(t)\rangle$?

EP-58 - Time development redux

Consider an energy operator $\hat{H}$ with three eigenvectors given by the equations

$$\hat{H}|E = +10\rangle = +|E = +10\rangle$$

$$\hat{H}|E = -10\rangle = -|E = -10\rangle$$

where $\{|E = +10\rangle, |E = -10\rangle\}$ form an orthonormal basis.

Now suppose that we are investigating some physical system and want to predict future value of a "spin" operator $\hat{S}$ with eigenvectors

$$\hat{S}|S = +1\rangle = \hat{S}\left(\frac{1}{\sqrt{2}}\left[|E = +10\rangle + |E = -10\rangle\right]\right) = +\left(\frac{1}{\sqrt{2}}\left[|E = +10\rangle + |E = -10\rangle\right]\right)$$

$$\hat{S}|S = -1\rangle = \hat{S}\left(\frac{1}{\sqrt{2}}\left[|E = +10\rangle - |E = -10\rangle\right]\right) = -\left(\frac{1}{\sqrt{2}}\left[|E = +10\rangle - |E = -10\rangle\right]\right)$$

where the eigenvectors of the "spin" operator have been written in the "energy" basis.

Suppose, in addition, that a physical system is initially prepared to be in the state

$$\left|\psi(t=0)\right\rangle = \frac{1}{\sqrt{2}}\left[\left|E=+10\right\rangle + \left|E=-10\right\rangle\right] = \left|S=1\right\rangle$$

that is, in an eigenvector of the "spin" operator.

(a) Is this state an eigenvector of the energy operator?

What is the probability that we will measure energy values $+10$ *or* -10 in this initial state? What is the probability that we will measure spin values $+1$ *or* -1 in this initial state ?

The time development operator is given by

$$\hat{T} = e^{-i\hat{H}t/\hbar}$$

(b) Determine

$$\hat{T}\left|E=+10\right\rangle = ? \qquad \hat{T}\left|S=+1\right\rangle = ?$$
$$\hat{T}\left|E=-10\right\rangle = ? \qquad \hat{T}\left|S=-1\right\rangle = ?$$

(c) Using (b) determine $\left|\psi(t)\right\rangle = \hat{T}\left|\psi(0)\right\rangle$ = the state of the system at a later time.

What is the probability that we will measure spin values $+1$ *or* -1 at this later time? You must calculate the quantities

$$P(S=1;t) = \left|\left\langle S=1 \middle| \psi(t)\right\rangle\right|^2 = \left|\left\langle S=1\right|\hat{T}\left|\psi(0)\right\rangle\right|^2$$
$$P(S=-1;t) = \left|\left\langle S=-1 \middle| \psi(t)\right\rangle\right|^2 = \left|\left\langle S=-1\right|\hat{T}\left|\psi(0)\right\rangle\right|^2$$

Will the spin value ever "flip" from $S=+1$ to $S=-1$?

EP-59. Time evolution

The matrix representation of the Hamiltonian for a photon propagating along the optic axis (taken to be the z-axis) of a quartz crystal using the linear polarization states $\left|x\right\rangle$ and $\left|y\right\rangle$ as a basis is given by

$$\hat{H} = \begin{pmatrix} 0 & -iE_0 \\ iE_0 & 0 \end{pmatrix}$$

(a) What are the eigenstates and eigenvalues of the Hamiltonian?
(b) A photon enters the crystal linearly polarized in the x direction, that is, $\left|\psi(0)\right\rangle = \left|x\right\rangle$. What is $\left|\psi(t)\right\rangle$, the state of the photon at time t ? Express your answer in the $\{\left|x\right\rangle, \left|y\right\rangle\}$ basis.
(c) What is happening to the polarization of the photon as it travels through the crystal?

EP-60. Time Evolution redux - Imagine a situation in which there are three energy states $|1\rangle$ with energy $E_1 = 1\,eV$, $|2\rangle$ with energy $E_2 = 2\,eV$ and $|3\rangle$ with energy $E_3 = 4\,eV$. Let the state of the system at $t = 0$ be $|\psi(0)\rangle = \frac{1}{2}|1\rangle + \frac{1}{2}|2\rangle + \frac{1}{\sqrt{2}}|3\rangle$.

(a) What are the possible results of an energy measurement on this system at $t = 0$ and with what probabilities will each of them occur?

(b) What are the possible results of an energy measurement on this system at $t = 5\times10^{-14}\,\text{sec}$ and with what probabilities will each of them occur?

(c) Are you surprised by your answer? Can you make sense of it?

EP-61. Time Evolution again - A state $|X\rangle$ is given in the hardness basis as

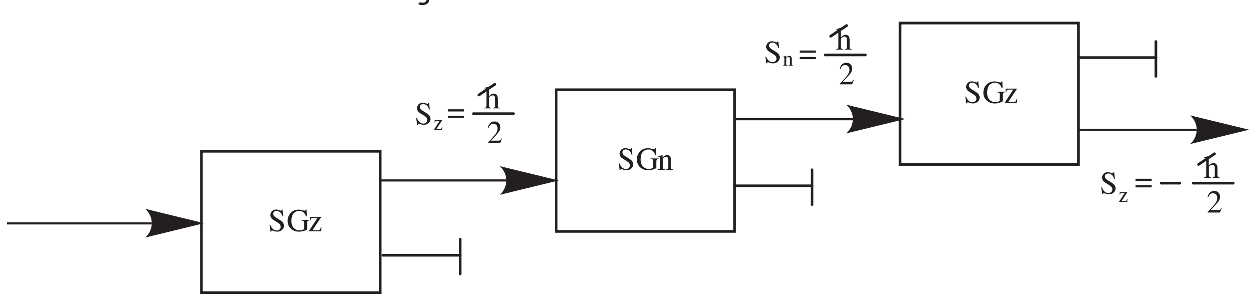
$$|X\rangle = \frac{1}{3}|h\rangle + \frac{i2\sqrt{2}}{3}|s\rangle$$

(a) What are the possible outcomes and probabilities for measurement of hardness and color on this state at $t = 0$?

(b) Imagine that due to an external field, the two color states $|g\rangle$ and $|m\rangle$, are eigenstates of the energy operator with the energy of the green state being zero and the energy of the magenta state being $1.0\times10^{-34}\,J$. What are the possible outcomes and probabilities for measurement of hardness and color on this state after 1 second?

EP-62. What comes out?

A beam of spin 1/2 particles is sent through series of three Stern-Gerlach measuring devices as shown below:



The first SGz device transmits particles with $\hat{S}_z = \hbar/2$ and filters outs particles with $\hat{S}_z = -\hbar/2$. The second device, an SGn device transmits particles with $\hat{S}_n = \hbar/2$ and filters outs particles with $\hat{S}_n = -\hbar/2$, where the axis $\hat{n}$ makes an angle θ in the x-z plane with respect to the z-axis. Thus the particles passing through this SGn device are in the state $|+\hat{n}\rangle = \cos\frac{\theta}{2}|+\hat{z}\rangle + e^{i\phi}\sin\frac{\theta}{2}|-\hat{z}\rangle$ with the angle $\phi = 0$. A last SGz device transmits particles with $\hat{S}_z = -\hbar/2$ and filters outs particles with

$\hat{S}_z = \hbar/2$.

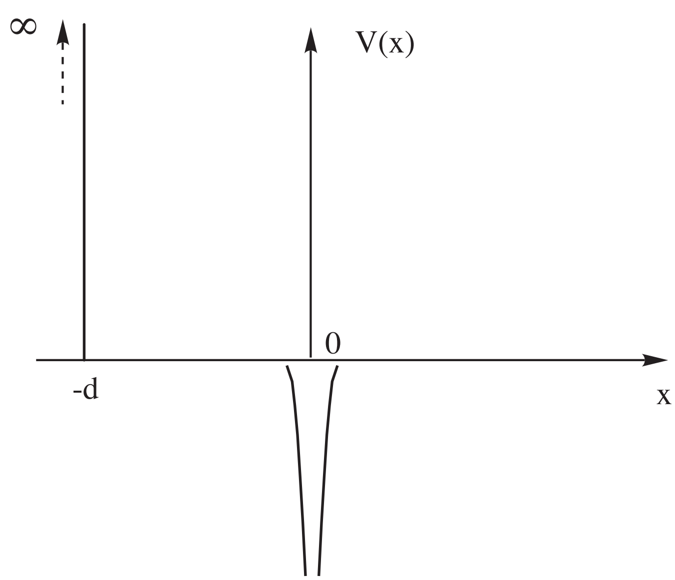
(a) What fraction of the particles transmitted through the first SGz device will survive the third measurement?
(b) How must the angle θ of the SGn device be oriented so as to maximize the number of particles the at are transmitted by the final SGz device? What fraction of the particles survive the third measurement for this value of θ ?
(c) What fraction of the particles survive the last measurement if the SGn device is simply removed from the experiment?

EP-63. Atom near a wall

An approximate model for an atom near a wall is to consider a particle moving under the influence of the one-dimensional potential given by

$$V(x) = \begin{cases} -V_0\delta(x) & x > -d \\ \infty & x < -d \end{cases}$$

as shown below:

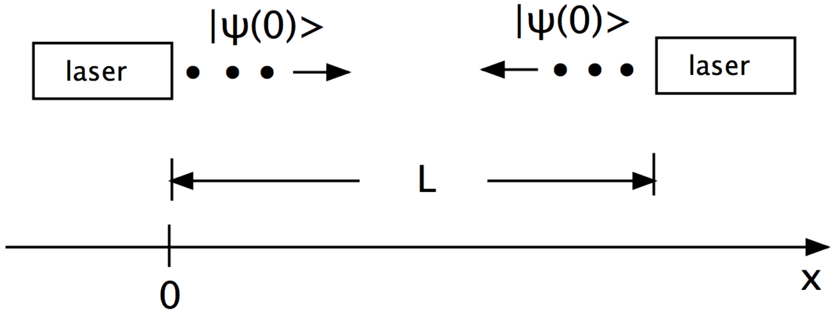


(a) Find the modification of the bound-state energy caused by the wall when it is "far away". Define what you mean by "far away".
(b) What is the exact condition on V_0 and d for the existence of at least one bound state ?

EP-64. Interference - Photons freely propagating through a vacuum have one value for their energy $E = h\nu$. This is therefore a 1-dimensional quantum mechanical system, and since the energy of a freely propagating photon does not change, it must be an eigenstate of the energy operator. So, if the state of the photon at $t = 0$ is denoted as $|\psi(0)\rangle$, then the eigenstate equation can be written $\hat{H}|\psi(0)\rangle = E|\psi(0)\rangle$. To see what happens to the state of the photon with time, we simply have to apply the time evolution operator

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = e^{-ih\nu t/\hbar}|\psi(0)\rangle = e^{-i2\pi\nu t}|\psi(0)\rangle = e^{-i2\pi x/\lambda}|\psi(0)\rangle$$

where the last expression uses the fact that $\nu = c/\lambda$ and that the distance it travels is $x = ct$. Notice that the relative probability of finding the photon at various points along the x-axis (the absolute probability depends on the number of photons emerging per unit time) does not change since the modulus-square of the factor in front of $|\psi(0)\rangle$ is 1. Consider the following situation. Two sources of identical photons face each other an emit photons at the same time. Let the distance between the two sources be L.



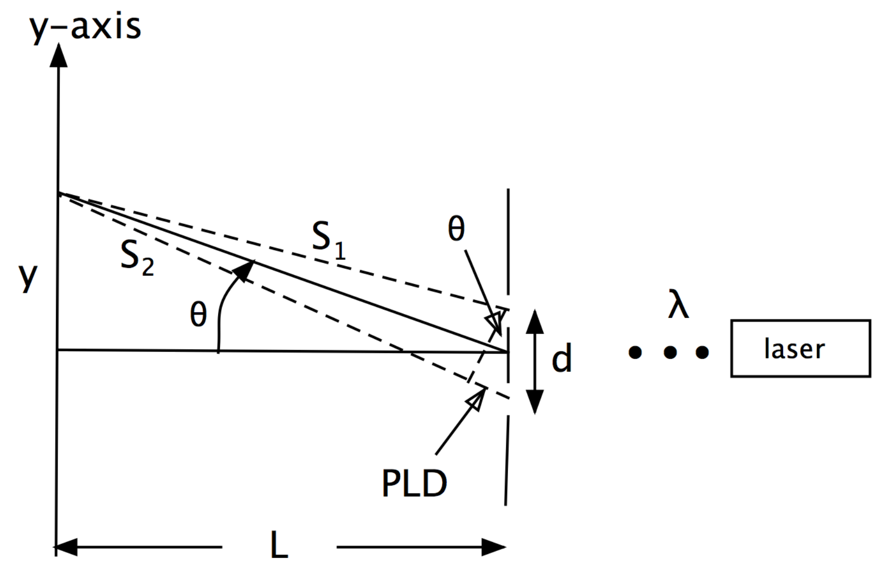
Notice that we are assuming the photons emerge from each source in state $|\psi(0)\rangle$. In between the two light sources we can detect photons but we do not know from which source they originated. Therefore, we have to treat the photons at a point along the x-axis as a superposition of the time-evolved state from the left source and the time-evolved state from the right source.

(a) What is this superposition state $|\psi(t)\rangle$ at a point x between the sources? Assume the photons have wavelength λ.

(b) Find the relative probability of detecting a photon at point x by evaluating $|\langle\psi(t)|\psi(t)\rangle|^2$ at the point x.

(c) Describe in words what your result is telling you. Does this correspond to anything you have seen when light is described as a wave?

EP-65. More Interference - Now let us tackle the two slit experiment with photons being shot at the slits one at a time. The situation looks something like the figure below. The distance between the slits, d is quite small (less than a mm) and the distance up the y-axis(screen) where the photons arrive is much,much less than L (the distance between the slits and the screen). In the figure, S_1 and S_2 are the lengths of the photon paths from the two slits to a point a distance y up the y-axis from the midpoint of the slits. The most important quantity is the difference in length between the two paths. The path length difference or PLD is shown in the figure.



We calculate PLD as follows:

$$PLD = d\sin\theta = d\left[\frac{y}{\left[L^2+y^2\right]^{1/2}}\right] \approx \frac{yd}{L} \quad , \quad y << L$$

Show that the relative probability of detecting a photon at various points along the screen is approximately equal to $4\cos^2\left(\frac{\pi yd}{\lambda L}\right)$.

EP-66. Spin in Magnetic Field - Suppose that we have a spin = 1/2 particle interacting with a magnetic field via the Hamiltonian

$$\hat{H} = \begin{cases} -\vec{\mu}\cdot\vec{B} \quad , \quad \vec{B} = B\hat{e}_z \qquad 0 \le t < T \\ -\vec{\mu}\cdot\vec{B} \quad , \quad \vec{B} = B\hat{e}_y \qquad T \le t < 2T \end{cases}$$

where

$$\vec{\mu} = \mu_B\vec{\sigma}$$

and the system is in the initial($t=0$) state

$$|\psi(0)\rangle = |x+\rangle = \frac{1}{\sqrt{2}}\left(|z+\rangle + |z-\rangle\right)$$

Determine the probability that the state of the system at $t=2T$ is

$$|\psi(2T)\rangle = |x+\rangle$$

in two ways:

(1) Using the Schrodinger equation (solving differential equations)
(2) Using the time development operator (using operator algebra)

EP-67. Compatibility - When the angular momentum along a direction in space of some atoms is measured, only two values result. Again, keeping things simple for measurements along the z-direction, we can define $\hat{O}_{L_z}$ and the eigenstates of $\hat{O}_{L_z}$ as usual.

$$\hat{O}_{L_z} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ with } |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In this basis, the operators for angular momentum along the x- and y-axes are as follows:

$$\hat{O}_{L_x} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \hat{O}_{L_y} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

(a) What are the eigenvalues and corresponding eigenstates of $\hat{O}_{L_x}$ and $\hat{O}_{L_y}$?
(b) What are the commutators of pairs of these operators?
(c) Are measurements of angular momentum along different direction in space compatible for these atoms?

EP-68. Angular Momentum - When measuring the angular momentum along the z-direction of some atoms, there are some states for which the average value of the measurements is zero.

(a) Find the probabilities of measuring each eigenvalue of $\hat{O}_{L_z}$ for the state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right)$.
(b) Use these probabilities to show that the average measurement is zero.
(c) Is there a state for which the average value of the measurements of the x-, y-, and z-components of the angular momentum are all zero? Show that the answer is no by choosing a general state $|\psi\rangle = \alpha|1\rangle + \beta|2\rangle$ and forcing the probabilities to be such that the average measurement of all three components is zero, yielding conditions on α and β that can only be satisfied if $\alpha = \beta = 0$.

EP-69. Angular Momentum Measurement - A measurement of a component of the intrinsic angular momentum (or spin) of an electron always yields one of two values, namely, $\pm\hbar/2$. If we call the component we are measuring the z-component, we can easily form an operator representing this measurement (call it $\hat{O}_{S_z}$) and the eigenstates representing the two states resulting from the measurement.

$$\hat{O}_{S_z} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad |+z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In the $\left(|+z\rangle, |-z\rangle\right)$ basis, the operator representing measurements of spin along the y-axis is

$$\hat{O}_{S_y} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

(a) Find the eigenvalues and associated eigenvectors of $\hat{O}_{S_y}$.
(b) What state (call it $|X\rangle$) results when the $|+z\rangle$ state is operated on

by the operator

$$\hat{O}_X = e^{-i\theta \hat{O}_{S_y}}$$

where θ is a constant?

(c) What are the possible outcomes of a measurement of the z-component of the spin of an electron in the state $|X\rangle$ and with what probability will each occur?

(d) What are the possible outcomes of a measurement of the y-component of the spin of an electron in the state $|X\rangle$ and with what probability will each occur?

EP-70. Angular Momentum - For a physical situation in which there are only 2 possible values for the z-component of the angular momentum, it is often convenient to use the eigenstates of the $\hat{L}_z$ operator as the basis for calculations. In this basis, the $\hat{L}_z$ operator and its eigenstates take the following form:

$$\hat{L}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{with eigenstates} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(a) Let us define 2 new states using the $\hat{L}_z$ eigenstates as a basis.

$$|A\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \quad , \quad |B\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} -i \\ \sqrt{2} \end{pmatrix}$$

Are these two states normalized? Are these two states orthogonal?

(b) Using the eigenstates of $\hat{L}_z$ as a basis, the operator representing measurements of the y-component of the angular momentum is given as follows:

$$\hat{L}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Find the eigenvalues and eigenstates of the $\hat{L}_y$ operator in the $\hat{L}_z$ basis.

(c) Express the states $|A\rangle$ and $|B\rangle$ using the eigenstates of $\hat{L}_y$ as a basis.

(d) Do $\hat{L}_z$ and $\hat{L}_y$ commute? What is the significance of whether they commute or not?

EP-71. Ammonia Molecule

In the ammonia molecule, NH_3,the three hydrogen atoms lie in a plane at the vertices of an equilateral triangle. The single nitrogen atom can lie either above or below the plane containing the hydrogen atoms, but in either case the nitrogen atom is equidistant from each of the hydrogen atoms (they form an equilateral tetrahedron). Let us call the state of the ammonia molecule when the nitrogen atom is above the plane of the hydrogen atoms $|1\rangle$. Let us call the state of the ammonia molecule when the nitrogen atom is below the plane of the hydrogen atoms $|2\rangle$.

How do we determine the energy operator for the ammonia molecule?

If these were the energy eigenstates, they would clearly have the same energy (since we cannot distinguish them in any way). So diagonal elements of the energy operator must be equal if we are using the ($|1\rangle$,$|2\rangle$) basis. But there is a small probability that a nitrogen atom above the plane will be found below the plane and vice versa (called tunnelling). So the off-diagonal element of the energy operator must not be zero, which also reflects the fact that the "above" and "below" states are not energy eigenstates. We therefore arrive with the following matrix as representing the most general possible energy operator for the ammonia molecule system:

$$\hat{H} = \begin{pmatrix} E_0 & A \\ A & E_0 \end{pmatrix}$$

where E_0 and A are constants.

(a) Find the eigenvalues and eigenvectors of the energy operator. Label them as ($|I\rangle$,$|II\rangle$).

(b) Let the initial state of the ammonia molecule be $|I\rangle$, that is $|\psi(0)\rangle = |I\rangle$. What is $|\psi(t)\rangle$, the state of the ammonia molecule after some time t? What is the probability of finding the ammonia molecule in each of its energy eigenstates? What is the probability of finding the nitrogen atom above or below the plane of the hydrogen atoms?

(c) Let the initial state of the ammonia molecule be $|1\rangle$, that is $|\psi(0)\rangle = |1\rangle$. What is $|\psi(t)\rangle$, the state of the ammonia molecule after some time t? What is the probability of finding the ammonia molecule in each of its energy eigenstates? What is the probability of finding the nitrogen atom above or below the plane of the hydrogen atoms?

EP-72. Neutrino Oscillations

It is generally recognized that there are at least three different kinds of neutrinos. They can be distinguished by the reactions in which the neutrinos are created or absorbed. Let us call these three types of neutrino ν_e, ν_μ and ν_τ. It has been speculated that each of these neutrinos has a small but finite rest mass, possibly different for each type. Let us suppose, for this exam question, that there is a small perturbing interaction between these neutrino types, in the absence of which all three types of neutrinos have the same nonzero rest mass M_0. The Hamiltonian of the system can be written as

$$\hat{H} = \hat{H}_0 + \hat{H}_1$$

where

$$\hat{H}_0 = \begin{pmatrix} M_0 & 0 & 0 \\ 0 & M_0 & 0 \\ 0 & 0 & M_0 \end{pmatrix} \rightarrow \text{ no interactions present}$$

and

$$\hat{H}_1 = \begin{pmatrix} 0 & \hbar\omega_1 & \hbar\omega_1 \\ \hbar\omega_1 & 0 & \hbar\omega_1 \\ \hbar\omega_1 & \hbar\omega_1 & 0 \end{pmatrix} \rightarrow \text{ effect of interactions}$$

where we have used the basis

$$|\nu_e\rangle = |1\rangle \quad , \quad |\nu_\mu\rangle = |2\rangle \quad , \quad |\nu_\tau\rangle = |3\rangle$$

(a) First assume that $\omega_1 = 0$, i.e., no interactions. What is the time development operator? Discuss what happens if the neutrino initially was in the state

$$|\psi(0)\rangle = |\nu_e\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ or } \quad |\psi(0)\rangle = |\nu_\mu\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ or } \quad |\psi(0)\rangle = |\nu_\tau\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

What is happening physically in this case ?

(b) Now assume that $\omega_1 \neq 0$, i.e., interactions are present. Also assume that at $t = 0$ the neutrino is in the state

$$|\psi(0)\rangle = |\nu_e\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

What is the probability as a function of time, that the neutrino will be in each of the other two states ?

(c) An experiment to detect the "neutrino oscillations" is being performed. The flight path of the neutrinos is 2000 meters. Their energy is 100 GeV. The sensitivity of the experiment is such that the presence of 1% of neutrinos different from those present at the start of the flight can be measured with confidence. Let $M_0 = 20\,eV$. What is the smallest value of $\hbar\omega_1$ that can be detected ? How does this depend on M_0? Don't ignore special relativity.

EP-73. Particle in a box with a membrane
Do matrix solution and pure Dirac algebra solution.

A box, containing a particle, is divided into a right and a left compartment by a thin partition. Suppose that the amplitude for the particle being on the left side of the box is ψ_1 and the amplitude for the particle being on the right side of the box is ψ_2. Neglect spatial variations of these amplitudes within the halves of the box. Suppose that the particle can tunnel through the partition and that the rate of change of the amplitude on the right is given by

$$i\hbar \frac{\partial \psi_2}{\partial t} = K\psi_1$$

where K is real. Assume that in the absence of tunneling, i.e., an impermeable membrane, that $\frac{\partial \psi_1}{\partial t} = 0$.

(a) What is the equation that determines the rate of change of the amplitude on the left?
(b) Find the normalized energy eigenstates (2-component vectors) of the particle in the box. Have these states definite parities?
(c) Suppose that at time t = 0, the amplitude on the right equals $e^{i\delta}$ time the amplitude on the left. Calculate, as a function of time, the time rate of change of the probability of observing the particle on the left.