Ordinary Differential Equations (ODEs)

First-Order Equations

Consider an equation of the form

$$\frac{dy}{dx} + Ay = 0 \qquad (01)$$

where A is a constant.

This equation is called first-order (highest derivative is a first derivative) ordinary linear (all functions and derivatives appear to the first power or less) differential equation with constant coefficients.

We solve this type of ODE using an exponential substitution method which converts the differential equation into a solvable algebraic equation.

Exponential Substitution Method

We make the exponential substitution

$$y = e^{\alpha x} \qquad (02)$$

into the ODE. This will convert the ODE into an **algebraic** equation for α. We have

$$\frac{dy}{dx} = \frac{de^{\alpha x}}{dx} = \alpha e^{\alpha x}$$
$$y = e^{\alpha x}$$

which upon substitution into (01) gives the result

$$(\alpha + A)e^{\alpha x} = 0 \to \alpha + A = 0 \qquad (03)$$

since $e^{\alpha x} \neq 0$.

The solution(s) of this algebraic equation tell us the **allowed** values of α that give **valid** solutions to the ODE. In particular in this case we get only one allowed value, namely,

$$\alpha = -A \qquad (04)$$

as a solution to the linear algebraic equation in (03). This result means that the function

$$y = e^{-Ax} \qquad (04)$$

satisfy the original ODE and therefore a solution.

The general solution to the equation is then written

$$y(x) = Ce^{-Ax} \quad (05)$$

where C is a constant that is determined by the initial conditions as shown below:

Suppose we have the initial condition(s) (number = order of ODE, which is one in this case)

$$y(0) = 7 \quad (06)$$

Then the value of the unknown constant C is determined by making sure that the general solution agrees with (or satisfies) the initial condition(s). We have

$$y(0) = 7 = C \quad (07)$$

Therefore the solution satisfying the initial condition is

$$y(x) = 7e^{-Ax} \quad (08)$$

Check:

$$\frac{dy}{dx} = \frac{d(7e^{-Ax})}{dx} = -7Ae^{-Ax}$$

$$y = 7e^{-Ax}$$

$$\frac{dy}{dx} + Ay = -7Ae^{-Ax} + 7Ae^{-Ax} = 0$$

$$y(0) = 7e^{0} = 7$$

Therefore, for this type of ODE, this exponential substitution method is able to generate a solution.

Second-Order Equations

Consider an equation of the form

$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = 0 \quad (09)$$

where $A, B, and\, C$ are constants.

Many equations derived from Newton's second law in mechanics and from Schrodinger's equation in quantum mechanics take this form.

This equation is called second-order (highest derivative is a second derivative) ordinary linear (all functions and derivatives appear to the first power or less) differential equation with constant coefficients.

We solve this type of ODE using an exponential substitution method which converts the differential equation into a solvable algebraic equation.

Exponential Substitution Method

The Method: consider a typical equation of the form

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0 \tag{10}$$

We make the exponential substitution

$$y = e^{\alpha t} \tag{11}$$

into the ODE. This will convert the ODE into an **algebraic** equation for α. We have

$$\frac{d^2y}{dt^2} = \frac{d^2e^{\alpha t}}{dt^2} = \alpha\frac{de^{\alpha t}}{dt} = \alpha^2 e^{\alpha t}$$

$$\frac{dy}{dt} = \frac{de^{\alpha t}}{dt} = \alpha e^{\alpha t}$$

$$y = e^{\alpha t}$$

which gives the result

$$(\alpha^2 + 3\alpha + 2)e^{\alpha t} = 0 \rightarrow \alpha^2 + 3\alpha + 2 = 0 \tag{12}$$

since $e^{\alpha t} \neq 0$.

The solutions of this algebraic equation tell us the **allowed** values of α that give **valid** solutions to the ODE. In particular, in this case we get

$$\alpha = -1 \; and \; -2 \tag{13}$$

as solutions to the quadratic equation. This result means that the functions $y = e^{-2t}$ and $y = e^{-4t}$ **each** satisfy the original ODE.

If there is more than one allowed value of α(as in this case), then the most general solution will be a linear combination of all possible solutions(because this is a linear ODE, that is, all derivative and functions enter in the equation to the first-power). Since, in this case, the allowed values of α are

$$\alpha = -1 \; and \; -2 \tag{14}$$

the most general solution of the ODE is

$$y(t) = ae^{-t} + be^{-2t} \tag{15}$$

where a and b are constants to be determined by the **initial conditions.**

Again, the number of arbitrary constants that need to be determined by the initial conditions is equal to the order(highest derivative → 2 in this case) of the ODE.

Suppose the initial conditions are $y(0)=0$ and $\left.\frac{dy}{dt}\right|_{t=0}=1$ (at $t=0$). Then we have

$$\begin{aligned} y(t) &= ae^{-t}+be^{-2t} \\ y(0) &= 0 = a+b \\ \frac{dy}{dt} &= -ae^{-t}-2be^{-2t} \\ \frac{dy}{dt}(0) &= -a-2b=1 \end{aligned} \tag{16}$$

which gives $a=-b=1$ and

$$y(t)=e^{-t}-e^{-2t} \tag{17}$$

If we substitute this solution into the original equation we have

$$\begin{aligned} y(t) &= e^{-t}-e^{-2t} \Rightarrow y(0)=0 \\ \frac{dy}{dt} &= -e^{-t}+2e^{-2t} \Rightarrow \frac{dy}{dt}(0)=1 \end{aligned}$$

as required and

$$\begin{aligned} y(t) &= e^{-t}-e^{-2t} \\ \frac{dy}{dt} &= -e^{-t}+2e^{-2t} \\ \frac{d^2y}{dt^2} &= e^{-t}-4e^{-2t} \\ \frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y &= (e^{-t}-4e^{-2t})+3(-e^{-t}+2e^{-2t})+2(e^{-t}-e^{-2t})=0 \end{aligned}$$

as required. Therefore, we see that the method works and generates a solution with the correct initial conditions!!

Although this method is very powerful as described, we can make it even more powerful by using the new mathematical quantity called the **complex exponential** as defined in the notes. This will allow us to use the method for the Schrodinger equation case.

Complex Exponentials - Alternative Very Powerful Method

Remember our discussion earlier in class about power series expansions of a function around some point

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \tag{18}$$

the point is $x = 0$ in this case.

If we apply this to the exponential function we get

$$f(x) = e^{\alpha x}$$
$$f^{(0)}(0) = f(0) = 1$$
$$f^{(1)}(0) = \left.\frac{df}{dx}\right|_{x=0} = \alpha e^{\alpha x}\Big|_{x=0} = \alpha$$
$$f^{(2)}(0) = \left.\frac{d^2 f}{dx^2}\right|_{x=0} = \alpha^2 e^{\alpha x}\Big|_{x=0} = \alpha^2$$

and so on ....

or

$$e^{\alpha x} = 1 + \alpha x + \frac{\alpha^2}{2!} x^2 + \frac{\alpha^3}{3!} x^3 + \frac{\alpha^4}{4!} x^4 + ... = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} x^n \tag{19}$$

If we apply this to the sine and cosine functions in the same manner we get (you should do this for at least one of these functions)

$$\sin \alpha x = \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^{2n+1}}{(2n+1)!} x^{2n+1} = \alpha x - \frac{\alpha^3}{3!} x^3 + \frac{\alpha^5}{5!} x^5 + ... \tag{20}$$

$$\cos \alpha x = \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^{2n}}{(2n)!} x^{2n} = 1 - \frac{\alpha^2}{2!} x^{24} + \frac{\alpha^4}{4!} x^4 + ... \tag{21}$$

We can then show the neat result that

$$e^{\pm i\alpha t} = \cos \alpha t \pm i \sin \alpha t$$

which will be very useful throughout the course. It is called Euler's formula.

Proof:

$$\begin{aligned} e^{i\alpha t} &= 1 + i\alpha t + \frac{(i\alpha t)^2}{2!} + \frac{(i\alpha t)^3}{3!} + \frac{(i\alpha t)^4}{4!} + ... \\ &= \left(1 - \frac{(\alpha t)^2}{2!} + \frac{(\alpha t)^4}{4!} - ....\right) + i\left(\alpha t - \frac{(\alpha t)^3}{3!} + \frac{(\alpha t)^5}{5!} - ....\right) \\ &= \cos \alpha t + i \sin \alpha t \end{aligned} \tag{22}$$

and similarly

$$e^{-i\alpha t} = \cos \alpha t - i \sin \alpha t \tag{23}$$

From these results we can also derive the relations

$$\frac{e^{i\alpha t} - e^{-i\alpha t}}{2i} = \frac{\cos\alpha t + i\sin\alpha t - \cos\alpha t + i\sin\alpha t}{2i} = \sin\alpha t$$
$$\frac{e^{i\alpha t} + e^{-i\alpha t}}{2} = \frac{\cos\alpha t + i\sin\alpha t + \cos\alpha t - i\sin\alpha t}{2} = \cos\alpha t \tag{24}$$

Finally, we use these results to solve the equation which results from applying Newton's second law to a simple harmonic oscillator (a spring for example). We have

$$F = -ky \rightarrow M\frac{d^2y}{dt^2} + ky = 0 \rightarrow \frac{d^2y}{dt^2} + \omega^2 y = 0 \quad , \quad \omega^2 = \frac{k}{M}$$

Now we use the exponential substitution method to solve the equation

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \tag{25}$$

Substituting $y = e^{\alpha t}$ we get the algebraic equation

$$\alpha^2 + \omega^2 = 0 \tag{26}$$

which has solutions (allowed values of α) of

$$\alpha = \pm i\omega \tag{27}$$

so that the most general solution takes the form

$$y(t) = Ae^{i\omega t} + Be^{-i\omega t}$$

Suppose now that the initial conditions are $y = y_0$ and $dy/dt = 0$ at $t = 0$. Then we have

$$y(t) = Ae^{i\omega t} + Be^{-i\omega t} \Rightarrow y(0) = y_0 = A + B$$
$$\frac{dy}{dt} = i\omega Ae^{i\omega t} - i\omega Be^{-i\omega t} \Rightarrow \frac{dy}{dt}(0) = i\omega A - i\omega B = 0 \rightarrow A - B = 0$$

or

$$A = B = \frac{y_0}{2}$$

and

$$y(t) = y_0\frac{e^{i\omega t} + e^{-i\omega t}}{2} = y_0\cos\omega t \tag{28}$$

which correctly corresponds to the motion of a mass M that is released from rest while attached to a spring with spring constant k.

Schrodinger Equation

As we will see in seminar, the Schrodinger equation takes the following general form in 1-dimension

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+V(x)\psi(x)=E\psi(x) \tag{29}$$

where E is a constant corresponding to an energy eigenvalue. If the function $V(x)$ (the potential energy function) is constant (say V_0) in some region, then the equation becomes

$$\frac{d^2\psi(x)}{dx^2}+\frac{2m}{\hbar^2}(E-V_0)\psi(x)=0 \tag{30}$$

which is a second-order ordinary linear differential equation with constant coefficients. Therefore the exponential substitution method will generate a solution.

In particular, if we substitute

$$\psi(x)=e^{\alpha x} \tag{31}$$

into eq (30) we have the algebraic equation

$$(\alpha^2+\frac{2m}{\hbar^2}(E-V_0))e^{\alpha x}=0 \tag{32}$$

so that the allowed values are

$$\alpha=\pm i\sqrt{\frac{2m}{\hbar^2}(E-V_0)} \tag{33}$$

Therefore the most general solutions in the case are

Case 1 : $E>V_0$

$$\psi(x)=Ae^{+i\sqrt{\frac{2m}{\hbar^2}(E-V_0)}x}+Be^{-i\sqrt{\frac{2m}{\hbar^2}(E-V_0)}x} \tag{34}$$

(complex exponentials)

Case 2 : $E<V_0$

$$\psi(x)=Ce^{+\sqrt{\frac{2m}{\hbar^2}(V_0-E)}x}+De^{-\sqrt{\frac{2m}{\hbar^2}(V_0-E)}x} \tag{35}$$

(real exponentials)

We will use solutions (34) and (35) later to solve Schrodinger equation problems.