Derivation of the Uncertainty Relations in General

Given two Hermitian operators $\hat{A}$ and $\hat{B}$ we define the two new operators

$$\hat{D}_A = \hat{A} - \left\langle \hat{A} \right\rangle \quad \text{and} \quad \hat{D}_B = \hat{B} - \left\langle \hat{B} \right\rangle$$

where $\left\langle \hat{O} \right\rangle = \langle \psi | \hat{O} | \psi \rangle$ equals the average or expectation value in the state $|\psi\rangle$.

In the statistical analysis of data, we use a quantity called the standard or mean-square deviation as a measure of the uncertainty of an observed quantity. It is defined, for a set of N measurements of the quantity q by

$$\begin{aligned}
(\Delta q)^2 &= (\text{standard deviation})^2 = \frac{1}{N}\sum_{i=1}^{N}(q_i - q_{average})^2 \\
&= \frac{1}{N}\sum_{i=1}^{N}(q_i)^2 - \frac{1}{N}\sum_{i=1}^{N}(q_i q_{average}) - \frac{1}{N}\sum_{i=1}^{N}(q_{average} q_i) + \frac{1}{N}\sum_{i=1}^{N}(q_{average})^2 \\
&= (q^2)_{average} - (q_{average})^2
\end{aligned}$$

where we have used $q_{average} = \frac{1}{N}\sum_{i=1}^{N} q_i$. In analogy, we define the mean-square deviations for $\hat{A}$ and $\hat{B}$ as

$$\left(\Delta\hat{A}\right)^2 = \left\langle \hat{A}^2 \right\rangle - \left\langle \hat{A} \right\rangle^2 = \left\langle \left(\hat{A} - \left\langle \hat{A} \right\rangle\right)^2 \right\rangle = \left\langle \hat{D}_A^2 \right\rangle$$

$$\left(\Delta\hat{B}\right)^2 = \left\langle \hat{B}^2 \right\rangle - \left\langle \hat{B} \right\rangle^2 = \left\langle \left(\hat{B} - \left\langle \hat{B} \right\rangle\right)^2 \right\rangle = \left\langle \hat{D}_B^2 \right\rangle$$

We then have

$$\left(\Delta\hat{A}\right)^2 \left(\Delta\hat{B}\right)^2 = \left\langle \hat{D}_A^2 \right\rangle \left\langle \hat{D}_B^2 \right\rangle$$

Now we assume that

$$\left[\hat{D}_A, \hat{B}\right] = \left[\hat{A}, \hat{B}\right] = \left[\hat{D}_A, \hat{D}_B\right] = i\hat{C}$$

where $\hat{C}$ is also a Hermitian operator and we let

$$|\alpha\rangle = \hat{D}_A |\psi\rangle = \left(\hat{A} - \left\langle \hat{A} \right\rangle\right)|\psi\rangle \quad \text{and} \quad |\beta\rangle = \hat{D}_B |\psi\rangle = \left(\hat{B} - \left\langle \hat{B} \right\rangle\right)|\psi\rangle$$

we have

$$\left(\Delta\hat{A}\right)^2 = \left\langle \hat{D}_A^2 \right\rangle = \langle \psi | \hat{D}_A^2 | \psi \rangle = \left(\langle \psi | \hat{D}_A\right)\left(\hat{D}_A | \psi \rangle\right) = \langle \alpha | \alpha \rangle$$

$$\left(\Delta\hat{B}\right)^2 = \left\langle \hat{D}_B^2 \right\rangle = \langle \psi | \hat{D}_B^2 | \psi \rangle = \left(\langle \psi | \hat{D}_B\right)\left(\hat{D}_B | \psi \rangle\right) = \langle \beta | \beta \rangle$$

The Schwarz inequality says that for any two vectors we must have the relation

$$\langle \alpha | \alpha \rangle \langle \beta | \beta \rangle \geq \left|\langle \alpha | \beta \rangle\right|^2$$

We therefore have

$$\left(\Delta\hat{A}\right)^2\left(\Delta\hat{B}\right)^2 = \left\langle\hat{D}_A^2\right\rangle\left\langle\hat{D}_B^2\right\rangle = \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq \left|\langle\alpha|\beta\rangle\right|^2$$
$$= \left|\left\langle\psi\left|\hat{D}_A\hat{D}_B\right|\psi\right\rangle\right|^2 = \left|\left\langle\hat{D}_A\hat{D}_B\right\rangle\right|^2$$

This gives

$$\left(\Delta\hat{A}\right)^2\left(\Delta\hat{B}\right)^2 \geq \left|\left\langle\hat{D}_A\hat{D}_B\right\rangle\right|^2$$

$$\left|\left\langle\hat{D}_A\hat{D}_B\right\rangle\right|^2 = \left|\left\langle\Delta\hat{A}\Delta\hat{B}\right\rangle\right|^2 = \left|\left\langle\frac{1}{2}\left[\Delta\hat{A},\Delta\hat{B}\right]+\frac{1}{2}\left\{\Delta\hat{A},\Delta\hat{B}\right\}\right\rangle\right|^2 = \left|\left\langle\frac{1}{2}\left[\hat{A},\hat{B}\right]+\frac{1}{2}\left\{\hat{A},\hat{B}\right\}\right\rangle\right|^2$$

$$\left[\hat{A},\hat{B}\right]^+ = -\left[\hat{A},\hat{B}\right] \rightarrow \text{anti - Hermitian} \rightarrow \text{expectation value is imaginary}$$

$$\left\{\hat{A},\hat{B}\right\}^+ = \left\{\hat{A},\hat{B}\right\} \rightarrow \text{Hermitian} \rightarrow \text{expectation value is real}$$

Therefore,

$$\left(\Delta\hat{A}\right)^2\left(\Delta\hat{B}\right)^2 \geq \left|\left\langle\frac{1}{2}\left[\hat{A},\hat{B}\right]+\frac{1}{2}\left\{\hat{A},\hat{B}\right\}\right\rangle\right|^2$$

$$\left(\Delta\hat{A}\right)^2\left(\Delta\hat{B}\right)^2 \geq \frac{1}{4}\left|\left\langle i\hat{C}\right\rangle + a\right|^2 \quad \text{where } a = \text{ real number}$$

and

$$\left(\Delta\hat{A}\right)^2\left(\Delta\hat{B}\right)^2 \geq \frac{1}{4}\left|a + i\left\langle\hat{C}\right\rangle\right|^2 = \frac{1}{4}|a|^2 + \frac{1}{4}\left|\left\langle\hat{C}\right\rangle\right|^2$$

$$\left(\Delta\hat{A}\right)^2\left(\Delta\hat{B}\right)^2 \geq \frac{1}{4}\left|\left\langle\hat{C}\right\rangle\right|^2 \tag{7.43}$$

since $\frac{1}{4}|a|^2 \geq 0$.

If $\left[\hat{A},\hat{B}\right] = i\hat{C} = 0$, we then get

$$\left(\Delta\hat{A}\right)^2\left(\Delta\hat{B}\right)^2 \geq 0$$

and we have no uncertainty relation between the observables.

On the other hand, if $\hat{A} = \hat{x}$ and $\hat{B} = \hat{p}_x$ we have $\left[\hat{x},\hat{p}_x\right] = i\hbar = \hat{C}$

$$\left(\Delta\hat{A}\right)^2\left(\Delta\hat{B}\right)^2 \geq \frac{\hbar^2}{4}$$

or

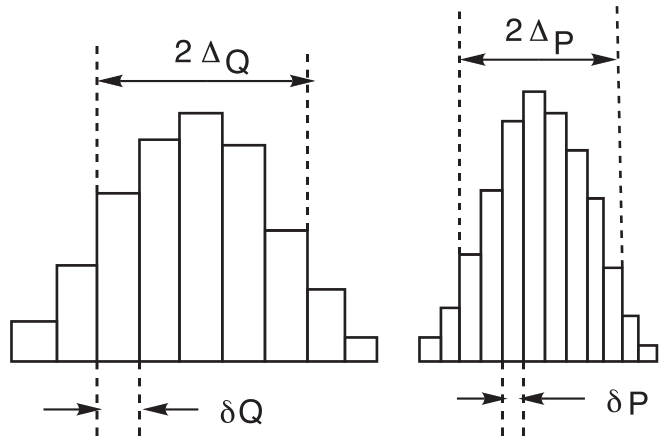
$$\Delta\hat{x}\Delta\hat{p}_x \geq \frac{\hbar}{2} \tag{7.44}$$

which is the Heisenberg uncertainty principle. **It is simply the Schwarz inequality!!**

The Meaning of the Indeterminacy Relations

What is the significance of indeterminacy relations in the world of experimental physics?

Consider the experimental results shown below:



These are frequency distributions for the results of independent measurements of Q and P on an ensemble if similarly prepared systems, i.e., on each of a large number of similarly prepared systems one performs a single measurement (either Q or P) . The histograms are the statistical distribution of the results.

The standard deviations (variances) as shown must satisfy (according to the theory) the relation

$$\Delta_Q \Delta_P \geq \frac{\hbar}{2}$$

They must be distinguished from the resolution of the individual measurements, δQ and δP.

Let me emphasize these points:

(1) The quantities Δ_Q and Δ_P are **not** errors of measurement. The "**errors**" or preferably the **resolutions** of the Q and P measuring instruments are δQ and δP. They are logically unrelated to Δ_Q and Δ_P and to the uncertainty relations except for the practical requirement that if

$$\delta Q > \Delta_Q \ \text{(or } \delta P > \Delta_P\text{)}$$

then it will not be possible to determine Δ_Q (or Δ_P) in the experiment and the experiment cannot test the uncertainty relation.

(2) The experimental test of the indeterminacy relation **does not** involve simultaneous measurements of Q and P, but rather it involves the measurement of one or the other of these dynamical variables on each independently prepared representative of the particular state being studied.

Why am I being so picky here?

The quantities Δ_Q and Δ_P as defined here are often misinterpreted as the errors of individual measurements. This probably arises because

Heisenberg's original paper on this subject, published in 1927, was based on an early version of quantum mechanics that predates the systematic formulation and statistical interpretation of quantum mechanics as it exists now. The derivation, as carried out here was not possible in 1927!

Time-Energy Uncertainty Relations

The use of time-energy uncertainty relations in most textbooks is simply incorrect. Let us now derive the most we can say about such relations.

We need to derive the time-dependence of expectation values. We have

$$\langle \hat{Q} \rangle = \langle \psi(t) | \hat{Q} | \psi(t) \rangle$$

so that

$$\begin{aligned} \frac{d\langle \hat{Q} \rangle}{dt} &= \frac{d}{dt}\langle \psi(t) | \hat{Q} | \psi(t) \rangle = \frac{d}{dt}\langle \psi(0) | e^{i\hat{H}t/\hbar}\hat{Q}e^{-i\hat{H}t/\hbar} | \psi(0) \rangle = \langle \psi(0) | \frac{d}{dt}\left(e^{i\hat{H}t/\hbar}\hat{Q}e^{-i\hat{H}t/\hbar}\right) | \psi(0) \rangle \\ &= \langle \psi(0) | \left(\frac{i\hat{H}}{\hbar}e^{i\hat{H}t/\hbar}\hat{Q}e^{-i\hat{H}t/\hbar}\right) | \psi(0) \rangle - \langle \psi(0) | \left(e^{i\hat{H}t/\hbar}\hat{Q}\frac{i\hat{H}}{\hbar}e^{-i\hat{H}t/\hbar}\right) | \psi(0) \rangle \\ &= \langle \psi(0) | e^{i\hat{H}t/\hbar}\frac{i}{\hbar}\left[\hat{H},\hat{Q}\right]e^{-i\hat{H}t/\hbar} | \psi(0) \rangle = \frac{i}{\hbar}\langle \psi(t) | \left[\hat{H},\hat{Q}\right] | \psi(t) \rangle \end{aligned} \qquad (1)$$

$$\frac{d\langle \hat{Q} \rangle}{dt} = \frac{1}{i\hbar}\left\langle \left[\hat{Q},\hat{H}\right] \right\rangle$$

We consider the dynamical state of the system at a given time t. Let $|\psi\rangle$ be the vector representing that state. Call ΔQ, ΔE the root-mean-square deviations of $\hat{Q}$ and of $\hat{H}$, respectively. Applying the Schwarz inequality (as above) to the vectors $(\hat{Q} - \langle Q \rangle)|\psi\rangle$ and $(\hat{H} - \langle H \rangle)|\psi\rangle$ and carrying out the same manipulations as (as above) we find after some calculations

$$\Delta Q \Delta E \geq \frac{1}{2}\left| \left\langle \left[\hat{Q},\hat{H}\right] \right\rangle \right| \qquad (2)$$

the equality being realized when $|\psi\rangle$ satisfies the equation

$$(\hat{Q} - \alpha)|\psi\rangle = i\gamma(\hat{H} - \varepsilon)|\psi\rangle$$

where α, γ and ε are arbitrary real constants (as above). Using (1) in (2) we have

$$\frac{\Delta Q}{\left| \frac{d\langle \hat{Q} \rangle}{dt} \right|}\Delta E \geq \frac{\hbar}{2} \qquad (3)$$

or

$$\tau_Q \Delta E \geq \frac{\hbar}{2} \qquad (4)$$

where we have defined

$$\tau_Q = \frac{\Delta Q}{\left| \frac{d\langle \hat{Q} \rangle}{dt} \right|} \qquad (5)$$

τ_Q appears as a time characteristic of the evolution of the expectation value of $\hat{Q}$. It is the time required for the center of the center $\langle \hat{Q} \rangle$ of the statistical distribution of $\hat{Q}$ to be displaced by an amount equal to its width ΔQ. In other words, the time necessary for this statistical distribution to be appreciably modified. In this way we can define a characterisitc evolution time for each dynamical variable of the system.

Let τ be the shortest of the times thus defined. τ may be considered as the characteristic time of evolution of the system itself, that is, whatever the measurement carried out on the system at an instant of time t', the statistical distribution of the results is essentially the same as would be obtained at the instant t, as long as the difference $|t - t'|$ is less than τ.

According to (4), this time τ and the energy spread ΔE satisfy the time-energy uncertainty relation

$$\tau \Delta E \geq \frac{\hbar}{2} \qquad (6)$$

If, in particular, the system is in a stationary state where

$$\frac{d\langle \hat{Q} \rangle}{dt} = 0$$

no matter what $\hat{Q}$, and consequently τ is infinite, then $\Delta E = 0$ according to (6).

Ordinary time t is just a parameter in non-relativistic QM and not an operator! Eq (6) does not say that

$$\Delta t \Delta E \geq \frac{\hbar}{2}$$

which is an equation that has no meaning!