

Readings: Riley, Hobson and Bence - Chapter 21
 Boccio - 14_PDEs, 15_PDEs, 16_PDEs

II. Diffusion or Heat Flow Equation

6. Solve the **Diffusion Equation** for $T(x,t)$ under the following conditions. We have a laterally insulated 2-m long rod with conductivity $10^{-4} \text{ m}^2 / \text{s}$ and $T(x,0)=100(2x-x^2)$, $T(0,t)=0$, $T(2,t)=0$

Answer is:

$$T(x,t) = \sum_{n=1}^{\infty} A_n e^{-\pi^2 D n^2 t / 4} \sin \frac{n\pi x}{2}$$

$$A_n = \int_0^2 100(2x-x^2) \sin \frac{n\pi x}{2} dx$$

III. Laplace Equation

A. Steady-state temperature in a rectangular plate (5 problems)

1. Find the steady-state temperature distribution for a semi-infinite plate if the temperature of the bottom edge is $T = f(x) = x$, the temperature of the sides is 0° and the width of the plate is 10 cm.

Answer is:

$$T = \frac{20}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-\frac{n\pi y}{10}} \sin\left(\frac{n\pi x}{10}\right)$$

2. Solve the semi-infinite plate problem if the bottom edge of width 20 cm is held at

$$T = \begin{cases} 0^\circ & 0 < x < 10 \\ 100^\circ & 10 < x < 20 \end{cases}$$

and the sides are at 0° .

Answer is:

$$T = \sum_n b_n e^{-\left(\frac{n\pi}{20}\right)y} \sin\left(\frac{n\pi x}{20}\right)$$

$$b_n = \begin{cases} 0 & n = 4, 8, 12, \dots \\ -\frac{400}{\pi n} & n = 2, 6, 10, \dots \\ \frac{200}{\pi n} & n = \text{odd} \end{cases}$$

3. Solve problem 2 if the plate is cut off at height 10 cm and the temperature of the top edge is 0° .

Answer is:

$$T = \sum_n b_n \sinh\left(\frac{n\pi}{20}(10-y)\right) \sin\left(\frac{n\pi x}{20}\right)$$

$$b_n \sinh\frac{n\pi}{2} = \begin{cases} 0 & n = 4, 8, 12, \dots \\ -\frac{400}{\pi} & n = 2, 6, 10, \dots \\ \frac{200}{\pi} & n \text{ odd} \end{cases}$$

4. Find the temperature distribution in a rectangular plate 10 cm by 30 cm if two adjacent sides are held at 100° and the other two sides are at 0° .

Answer is:

$$T = \frac{400}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \left[\frac{1}{\sinh(3n\pi)} \sinh\left(\frac{n\pi}{10}(30-y)\right) \sin\left(\frac{n\pi x}{10}\right) + \frac{1}{\sinh(n\pi/3)} \sinh\left(\frac{n\pi}{30}(10-x)\right) \sin\left(\frac{n\pi y}{30}\right) \right]$$

5. Using the diffusion equation find the **steady-state temperature distribution** in a 1 m x 1 m slab if the flat surfaces are insulated and the edge conditions are as follows:

$$T(0,y) = 0 \quad , \quad \left(\frac{\partial T(x,y)}{\partial y}\right)_{y=0} = 0 \quad , \quad \left(\frac{\partial T(x,y)}{\partial x}\right)_{x=1} = 0 \quad , \quad T(x,1) = 100$$

Answer is:

$$T(x,y) = 2 \sum_{n=1}^{\infty} A_n \sin\left(\frac{2n-1}{2}\pi x\right) \cosh\left(\frac{2n-1}{2}\pi y\right)$$

$$A_n = \frac{200}{(2n-1)\pi \cosh\left(\frac{2n-1}{2}\pi\right)}$$

B. Steady-State Temperature in a Cylinder

1. Find the steady-state temperature distribution in a solid semi-infinite cylinder if the boundary temperatures are $u=0$ at $r=1$ and $u=y=r\sin\theta$ at $z=0$.

Answer is:

$$T = \sum_{m=1}^{\infty} \frac{2}{k_m J_2(k_m)} J_1(k_m r) e^{-k_m z} \sin\theta \quad k_m = \text{zeroes of } J_1$$

2. Water at 100° is flowing through a long pipe of radius $r=1$ rapidly enough so that we may assume that the temperature is 100° at all points. At $t=0$, the water is turned off and the surface of the pipe is maintained at 40° from then on (neglect wall thickness). Find the temperature distribution of the water as a function of r and t . (Hint: you only need to consider a cross-section of the pipe).

Answer is:

$$T = 40 + \sum_{m=1}^{\infty} \frac{120}{k_m J_1(k_m)} J_0(k_m r) e^{-(\alpha k_m)^2 t} \quad k_m = \text{zeroes of } J_0$$