

**Readings:** Riley, Hobson and Bence - Chapter 16  
Boccio - 011\_ODEs, 012\_ODEs and 013\_ODEs

**Problems:**

16.01 Series Solution

**EP-5** For the ODE

$$36x^2 \frac{d^2y}{dx^2} + (5 - 9x^2)y = 0$$

use the method of **Frobenius** to find both homogeneous solutions.

**EP-6** Solve the ODE for a general solution using the power-series method (non-singular equations).

(b)  $(x^2 - 1)u'' - 4u = 0$

(c)  $(4 - x^2)u'' + 2u = 0$  ,  $u(0) = 0, u'(0) = 1$

**EP-7** Solve the ODE for a general solution using the power-series method (singular equations).

(a)  $2x(1 - x)u'' + u' - u = 0$

(c)  $2x^2u'' + x^2u' + (4/9)u = 0$

**EP-8 Roots Differing by an Integer - Series + Wronskian Method**  
Solve

$$xu'' + (1 - x)u' - u = 0$$

by finding the first solution using the series method and then finding the second solution using the Wronskian method.

**EP-9 Bessel's Equation** (use solutions in text)  
Determine the general solution for

$$xu'' + u' + (x - 1/9x)u = 0$$

**EP-10 Properties of Bessel Functions**

Find an expression in terms of  $J_1(x)$  and  $J_0(x)$  for each integral below:

(a)  $\int \frac{J_4(x)}{x} dx$       (b)  $\int x^3 J_1(x) dx$

**Answers(not in textbook) for Assignment 9**

EP-5  $y(x) = Ax^{1/6} \left[ 1 + \frac{3}{32}x^2 + \frac{9}{5120}x^4 + \dots \right] + Bx^{5/6} \left[ 1 + \frac{3}{64}x^2 + \frac{9}{14336}x^4 + \dots \right]$

EP-6 (b)  $u(x) = b_0 \left[ 1 - 2x^2 + \frac{1}{3}x^4 - \dots \right] + b_1 \left[ x - \frac{2}{3}x^3 - \frac{1}{15}x^5 + \dots \right]$

(c)  $u(x) = x - \frac{1}{12}x^3 - \frac{1}{240}x^5 - \frac{1}{2240}x^7 + \dots$

EP-7 (a)

$$u_1(x) = 1 + x + \frac{1}{6}x^2 + \frac{1}{18}x^3 + \dots$$

$$u_2(x) = x^{1/2} \left( 1 + \frac{1}{6}x + \frac{1}{24}x^2 + \frac{17}{1008}x^3 + \dots \right)$$

(c)

$$u_1(x) = x^{2/3} \left( 1 - \frac{1}{4}x + \frac{5}{112}x^2 + \dots \right)$$

$$u_2(x) = x^{1/3} \left( 1 - \frac{1}{4}x + \frac{1}{20}x^2 + \dots \right)$$

EP-8

$$u_1(x) = e^x$$

$$u_2(x) = e^x \left( \ln x - x + \frac{1}{4}x^2 - \frac{1}{18}x^3 + \dots \right)$$

EP-9

Bessel's equation:  $u(x) = c_1 J_{1/3}(x) + c_2 J_{-1/3}(x)$

EP-10 (a)  $\left( \frac{1}{x} - \frac{12}{x^3} \right) J_1(x) + \frac{6}{x^2} J_0(x) + C$

(b)  $3x^2 J_1(x) - (x^3 - 3x) J_0(x) - 3 \int J_0(x) dx$

## Computational Work:

### Non-linear Oscillator using Runge-Kutta Method

Write a MATLAB code to do solve this ODE system numerically. See MATLAB Lab\_3 notes.

Put the MATLAB code in your homework solutions and include graphs for cases  $a=1.07$  and  $a=1.50$  (see below).

**Problem #20:** Finally, let's do a nonlinear oscillator; one that exhibits chaotic dynamics. There is no analytical solution - numerical solution is the only way to go here. The equation of motion is:

$$\ddot{\theta} = a \cos(\omega_0 t) - \sin(\theta) - q\dot{\theta}$$
$$\theta(0) = \dot{\theta}(0) = 0$$

Some explanation of the terms on the right is necessary:

- (1) The second term is what we'd write down for any pendulum - it's the general case rather than the small-angle limit
- (2) The third is a damping term, with "damping strength"  $q$ .
- (3) The first is a driving term, with driving amplitude  $a$  and frequency  $\omega_0$ .

Some suggested parameter sets are below. Take  $h = 0.05$  throughout.

```
q = 1/2 ,  $\omega_0 = 2/3$ 
a=0.9      ; periodic
a=1.07     ; period doubling
a=1.15     ; chaos
a=1.35     ; periodic
a=1.45     ; period doubling
a=1.47     ; period doubling
a=1.50     ; chaos
```

Use Runge-Kutta to calculate the motion of the pendulum for several cycles. Create a phase-space plot, where you plot  $\dot{\theta}$  as a function of  $\theta$ . Restrict the  $\theta$  motion to  $-\pi \leq \theta \leq \pi$ .