Physics 50 **Assignments #11** **Spring 2009**

Readings: Riley, Hobson and Bence - Chapter 21
Boccio - 14_PDEs, 15_PDEs, 16_PDEs

II. Diffusion or Heat Flow Equation

6. Solve the **Diffusion Equation** for T(x,t) under the following conditions. We have a laterally insulated 2-m long rod with conductivity $10^{-4}\ m^2/s$ and $T(x,0)=100(2x-x^2)$, $T(0,t)=0$, $T(2,t)=0$

Answer is:

$$T(x,t)=\sum_{n=1}^{\infty}A_n e^{-\pi^2 Dn^2 t/4}\sin\frac{n\pi x}{2}$$

$$A_n=\int_0^2 100(2x-x^2)\sin\frac{n\pi x}{2}dx$$

III. Laplace Equation

A. Steady-state temperature in a rectangular plate (5 problems)

1. Find the steady-state temperature distribution for a semi-infinite plate if the temperature of the bottom edge is $T=f(x)=x$, the temperature of the sides is $0°$ and the width of the plate is 10 cm.

Answer is:

$$T=\frac{20}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-\frac{n\pi y}{10}}\sin\left(\frac{n\pi x}{10}\right)$$

2. Solve the semi-infinite plate problem if the bottom edge of width 20 cm is held at

$$T=\begin{cases}0° & 0<x<10\\ 100° & 10<x<20\end{cases}$$

and the sides are at $0°$.

Answer is:

$$T=\sum_n b_n e^{-\left(\frac{n\pi}{20}\right)y}\sin\left(\frac{n\pi x}{20}\right)$$

$$b_n=\begin{cases} 0 & n=4,8,12,\dots\\ -\dfrac{400}{\pi n} & n=2,6,10,\dots\\ \dfrac{200}{\pi n} & n=odd \end{cases}$$

3. Solve problem 2 if the plate is cut off at height 10 cm and the temperature of the top edge is 0°.

Answer is:

$$T=\sum_n b_n \sinh\left(\frac{n\pi}{20}(10-y)\right)\sin\left(\frac{n\pi x}{20}\right)$$

$$b_n\sinh\frac{n\pi}{2}=\begin{cases} 0 & n=4,8,12,\dots\\ -\dfrac{400}{\pi} & n=2,6,10,\dots\\ \dfrac{200}{\pi} & n\ odd \end{cases}$$

4. Find the temperature distribution in a rectangular plate 10 cm by 30 cm if two adjacent sides are held at 100° and the other two sides are at 0°.

Answer is:

$$T=\frac{400}{\pi}\sum_{n\ odd}\frac{1}{n}\left[\frac{1}{\sinh(3n\pi)}\sinh\left(\frac{n\pi}{10}(30-y)\right)\sin\left(\frac{n\pi x}{10}\right)+\frac{1}{\sinh(n\pi/3)}\sinh\left(\frac{n\pi}{30}(10-x)\right)\sin\left(\frac{n\pi y}{30}\right)\right]$$

5. Using the diffusion equation find the **steady-state temperature distribution** in a 1 m x 1 m slab if the flat surfaces are insulated and the edge conditions are as follows:

$$T(0,y)=0\quad,\quad\left(\frac{\partial T(x,y)}{\partial y}\right)_{y=0}=0\quad,\quad\left(\frac{\partial T(x,y)}{\partial x}\right)_{x=1}=0\quad,\quad T(x,1)=100$$

Answer is:

$$T(x,y) = 2\sum_{n=1}^{\infty} A_n \sin\left(\frac{2n-1}{2}\pi x\right)\cosh\left(\frac{2n-1}{2}\pi y\right)$$

$$A_n = \frac{200}{(2n-1)\pi\cosh\left(\frac{2n-1}{2}\pi\right)}$$

B. Steady-State Temperature in a Cylinder

1. Find the steady-state temperature distribution in a solid semi-infinite cylinder if the boundary temperatures are $u=0$ at r = 1 and u = y = $r\sin\theta$ at z = 0.

Answer is:

$$T = \sum_{m=1}^{\infty} \frac{2}{k_m J_2(k_m)} J_1(k_m r) e^{-k_m z} \sin\theta \qquad k_m = zeroes\ of\ J_1$$

2. Water at $100°$ is flowing through a long pipe of radius r = 1 rapidly enough so that we may assume that the temperature is $100°$ at all points. At t = 0, the water is turned off and the surface of the pipe is maintained at $40°$ from then on(neglect wall thickness). Find the temperature distribution of the water as a function of r and t. (Hint: you only need to consider a cross-section of the pipe).

Answer is:

$$T = 40 + \sum_{m=1}^{\infty} \frac{120}{k_m J_1(k_m)} J_0(k_m r) e^{-(\alpha k_m)^2 t} \qquad k_m = zeroes\ of\ J_0$$