

Readings: Riley, Hobson and Bence - Chapter 19

Boccio - 14_PDEs, 15_PDEs, 16_PDEs

I. Wave Equation

1. **Vibrating string** - A string of length L with fixed ends has a zero initial velocity and a displacement

$$y_o = \begin{cases} 8hx/L & 0 < x < L/8 \\ 8h(L/4 - x)/L & L/8 < x < L/4 \\ 0 & L/4 < x < L \end{cases}$$

(this initial displacement might be caused by holding the string at the center and plucking half of it).

Find the displacement as a function of x and t .

Answer is:

$$y = \frac{16h}{\pi^2} \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi vt}{L}\right)$$

where

$$b_n = \frac{2 \sin\left(\frac{n\pi}{8}\right) - \sin\left(\frac{n\pi}{4}\right)}{n^2}$$

2. **Vibration of a Membrane** - Separate the wave equation in 2-dimensional rectangular coordinates (x,y) . Consider a rectangular membrane with corners $(0,0),(a,0),(a,b),(0,b)$ that is rigidly attached to supports along its sides. Show that its characteristic frequencies are

$$f_{nm} = \frac{v}{2} \sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}$$

where (m,n) are positive integers. Sketch the normal modes of vibration corresponding to the first few frequencies, that is, indicate the nodal lines. Now suppose that the membrane is square. Show that in this case there may be two or more normal modes corresponding to a single frequency. This is an example of a phenomenon called **degeneracy**.

3. **Vibrating string** - A string π meters long is started into motion by giving the middle one-half ($\pi/4 \leq x \leq 3\pi/4$) an initial velocity of 20 m/s (while keeping the initial displacement

zero everywhere). The string is stretched until the wave speed is 60 m/s. Determine the resulting displacement of the string as a function of x and t .

Answer is:

$$y(x,t) = \sum_{n=1}^{\infty} \frac{2}{3\pi n^2} \left(\cos \frac{n\pi}{4} - \cos \frac{3n\pi}{4} \right) \sin(60nt) \sin(nx)$$

II. Diffusion or Heat Flow Equation

- Heat flow in a bar or slab** - A bar 10 cm long with insulated sides is initially at 100° . Starting at $t = 0$, the ends are held at 0° . Find the temperature distribution in the bar at time t .

Answer is:

$$T = \frac{400}{\pi} \sum_{n \text{ odd}} \frac{1}{n} e^{-\left(\frac{n\pi\alpha}{10}\right)^2 t} \sin\left(\frac{n\pi x}{10}\right)$$

- Heat flow in a bar or slab** - In the initial steady-state distribution of an infinite slab of thickness d , the face $x = 0$ is at 0° and the face $x = d$ is at 100° . From $t = 0$ on, the $x = 0$ face is held at 100° and the $x = d$ face is held at 0° . Find the temperature distribution in the bar at time t .

Answer is:

$$T = 100 - \frac{100x}{d} - \frac{400}{\pi} \sum_{n \text{ even}} \frac{1}{n} e^{-\left(\frac{n\pi\alpha}{d}\right)^2 t} \sin\left(\frac{n\pi x}{d}\right)$$

You need to consider the $k=0$ ($k_n = k_0 = 0$) solution.

- Heat flow in a bar or slab** - Two slabs, each 1 inch thick, each have one surface at 0° and the other at 100° . At $t = 0$, they are stacked with their 100° faces together and then the outside surfaces are held at 100° . Find the temperature distribution for $t > 0$.

Answer is:

$$T = 100 + \frac{1}{\pi} \sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi\alpha}{2}\right)^2 t} \sin\left(\frac{n\pi x}{2}\right)$$

where

$$b_n = 400 \begin{cases} 0 & \text{even } n \\ \frac{2}{n^2\pi^2} - \frac{1}{n\pi} & n=1+4k \\ -\frac{2}{n^2\pi^2} - \frac{1}{n\pi} & n=3+4k \end{cases}$$

You need to consider the $k=0$ ($k_n = k_0 = 0$) solution.

- 4. Heat flow in a bar or slab** - A bar of length L with insulated sides has its ends also insulated from time $t = 0$ on. Initially the temperature distribution is $T = x$, where $x =$ distance from one end. Determine the temperature distribution inside the bar at time t . (Hint: you cannot neglect the $k = 0$ solutions in this case).

Answer is:

$$T = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right)$$

- 5. Text problem 19.19 - Heat flow in a bar or slab** - Two identical copper bars are each of length a . Initially, one is at 0°C and the other is at 100°C . They are then joined together end to end and thermally isolated. Obtain in the form of a Fourier series and expression $T(x,t)$ for the temperature at any point a distance x from the join at a later time t .

Taking $a = 0.5$ m, estimate the time it takes for one of the free ends to reach a temperature of 55°C . The thermal conductivity of copper is $3.8 \times 10^2 \text{ Jm}^{-1}\text{K}^{-1}\text{s}^{-1}$ and its specific heat capacity is $3.4 \times 10^6 \text{ Jm}^{-1}\text{K}^{-1}$.

Answer is:

$$T(x,t) = 50 + \frac{200}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin \frac{(2n+1)\pi x}{2a} e^{-\frac{D(2n+1)^2 \pi^2 t}{4a^2}}$$

Optics Laboratory #4 Analysis

After doing the lab last week should have a datafile with two columns as shown below. My sample file on the website (slits2.dat) has 1528 data values. Column 1 is the coordinate along the interference pattern and column #2 is the intensity at that coordinate value.

```
-6.7680 0.33300
-6.7590 0.50000
-6.7500 0.50000
-6.7410 0.50000
-6.7320 0.66700
-6.7230 0.66700
-6.7140 1.00000
.....
.....
```

There is MATLAB analysis program on the website also. It is called FTbrute.m.

```
clear all
% all length units are mm
lambda = 0.000635;
L = 110;
for i = 1:801
    yp(i) = (i-401)*0.001;
    A(i) = 0;
end
% a = slit width, d = slit separation, N = number of slits
a = 0.04;
d = 0.125;
N = 2;
% generate aperature function (must fit into 0.8 mm)
Na = round(1000*a/2);
Nd = round(1000*d/2);
for i = 1:N
    NN = -(N-1) + 2*(i-1);
    for j = 1:2*Na
        jj = 401+NN*Nd-Na+(j-1);
        A(jj) = 1;
    end
end
% compute diffraction pattern (Fourier Transform)
for j = 1:833
    y(j) = (j-417)*0.009;
    for i = 1:801
        Acos(i) = (1/(lambda*L))*A(i)*cos(2*pi*yp(i)*y(j)/(lambda*L));
```

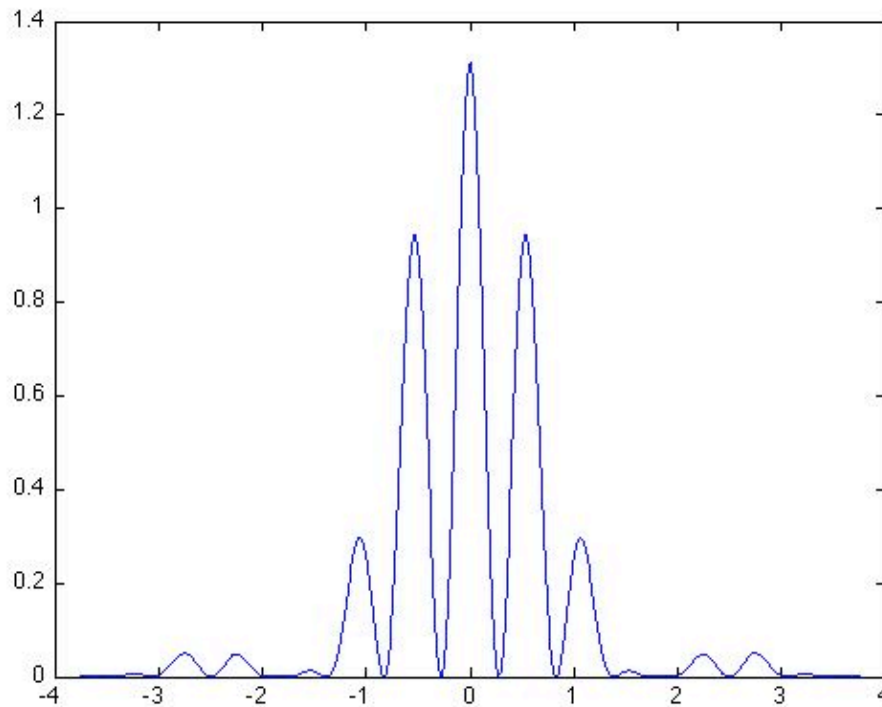
```

        Asin(i) = (1/(lambda*L))*A(i)*sin(2*pi*yp(i)*y(j)/(lambda*L));
    end
    Er(j) = trapz(yp,Acos);
    Ei(j) = trapz(yp,Asin);
    TheorI(j) = Er(j)^2 + Ei(j)^2;
end
plot(y, TheorI)
figure
% slits2.dat is intensity data (a=0.04, d=0.125, N=2)
load slits2.dat
yexp = slits2(:,1);
ExpI = slits2(:,2);
plot(yexp,ExpI,'r',y,230*TheorI,'b')
xlim([-4 +4])

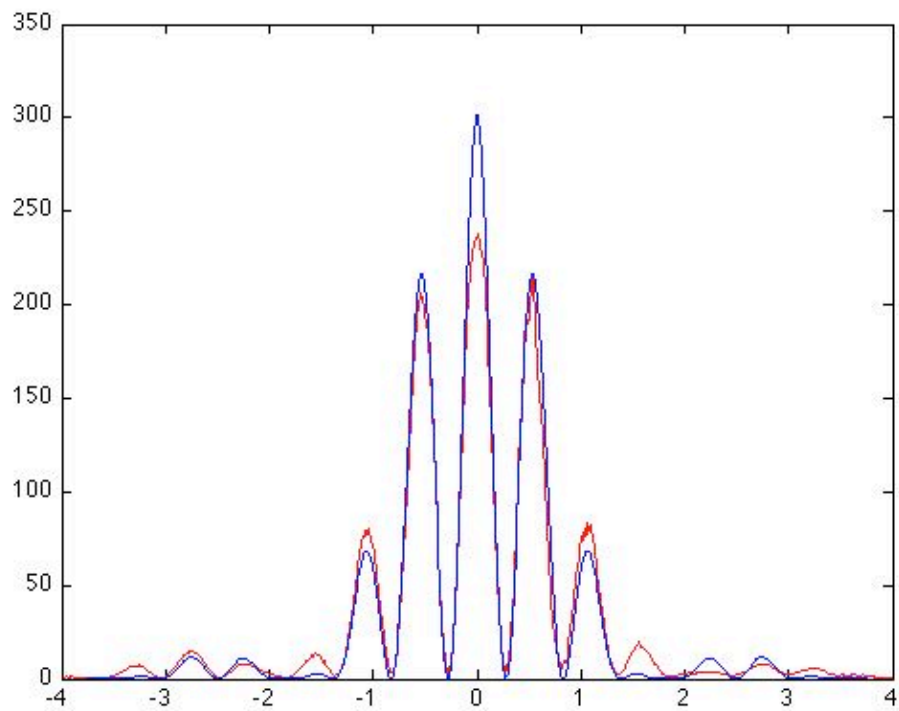
```

Make sure you understand all aspects of this program. Describe to me what it is doing and what all the variables, etc mean.

When I use this program to analyze my data, my results are:



plot of data



data versus theory

Using your experimental data, generate similar plots.

In your homework include:

- a copy of your data file
- an explanation of all aspects of FTbrute.m
- two plots as above.