

Readings: Riley, Hobson and Bence - Chapter 19  
 Boccio - 06\_PDEs, 07\_PDEs, 08\_PDEs

### B. Steady-State Temperature in a Cylinder

3. Separate Laplace's equation in two dimensions in polar coordinates and solve the  $r$  and  $\theta$  equations. Remember that for the  $\theta$  equation, only periodic solutions are of interest. Use your results to solve the problem of the steady state temperature in a circular plate (radius =  $a$ ) if the upper semi-circular boundary is held at  $100^\circ$  and the lower is held at  $0^\circ$ .

Answer is:

$$T = 50 + \frac{200}{\pi} \sum_{n \text{ odd}} \left(\frac{r}{a}\right)^n \frac{\sin n\theta}{n} \quad a = \text{disk radius}$$

4. Find the steady-state temperature distribution in a circular annulus of inner radius  $r = 1$  and outer radius  $r = 2$  if the inner circle is held at  $0^\circ$  and the outer circle has half of its circumference at  $0^\circ$  and half at  $100^\circ$ . (Hint: you cannot neglect  $r$  solutions corresponding to  $k = 0$ ).

Answer is:

$$T = 50 \frac{\log(r)}{\log(2)} + \frac{200}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \left( \frac{r^n - r^{-n}}{2^n - 2^{-n}} \right) \sin n\theta \quad \log = \text{natural logarithm}$$

5. A right circular cylinder is 1 m long and 2 m in diameter. Its left end and lateral surface are maintained at a temperature of  $0^\circ$  and its right end at  $100^\circ$ . Find the temperature at any interior point. Calculate the first three coefficients in the series expansion.

Answer is:

$$T(r, z) = 29.4 J_0(2.4r) \sinh(2.4z) - 0.86 J_0(5.52r) \sinh(5.52z) \\ + 0.03 J_0(8.65r) \sinh(8.65z) - \dots$$

### C. Steady-State Temperature in a Sphere

1. Find the steady-state temperature distribution inside a sphere of radius  $r = 1$  when the surface temperatures are given by:

- (a)  $35(\cos\theta)^4$   
 (b)  $\begin{cases} \cos\theta & 0 < \theta < \pi/2 \\ 0 & \pi/2 < \theta < \pi \end{cases}$   
 (c)  $\sin^2\theta \cos\theta \cos 2\phi - \cos\theta$

Answers are:

- (a)  $T(r, \theta) = 8r^4 P_4(\cos\theta) + 20r^2 P_2(\cos\theta) + 7P_0(\cos\theta)$   
 (b)  $T(r, \theta) = \frac{1}{4} P_0(\cos\theta) + \frac{1}{2} r P_1(\cos\theta) + \frac{5}{16} r^2 P_2(\cos\theta) + \dots$   
 (c)  $T(r, \theta, \phi) = \frac{1}{15} r^3 P_3^2(\cos\theta) \cos 2\phi - r P_1(\cos\theta)$

2. A sphere (radius =  $a$ ) initially at  $0^\circ$  has its surface kept at  $100^\circ$  from  $t = 0$  on (for example, a frozen potato is tossed into in boiling water). Find the time-dependent temperature distribution. (Hint: subtract 100 from all temperatures, solve the problem and then add 100 to the solution; can you justify this procedure?). Show that the Legendre function required for this problem is  $P_0$  and the  $r$  solution is

$$\frac{1}{\sqrt{r}} J_{1/2} \rightarrow j_0$$

Answer is:

$$T = 100 + \frac{200a}{\pi r} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\left(\frac{n\pi r}{a}\right) e^{-(n\pi a/a)^2 t}$$

#### IV. Electric Potentials

1. **Electric Potentials Shells** - Find the potential between two concentric spheres if the outer sphere is maintained at  $V = 100$  and the potential on the inner sphere is maintained at zero. The radii are 2 m and 1 m, respectively.

Answer is:

$$V(r, \theta) = 200 \left[ 1 - \frac{1}{r} \right]$$

2. **Text Example 19.17.** A conducting sphere of radius  $a$  is cut around its equator and the two halves are connected to

voltages of  $+V$  and  $-V$ . Show that an expression for the potential at a point  $(r, \theta, \phi)$  anywhere inside the two hemispheres is

Answer is:

$$u(r, \theta, \phi) = V \sum_{n=0}^{\infty} \frac{(-1)^n (2n)! (4n+3)}{2^{2n+1} n! (n+1)!} \left(\frac{r}{a}\right)^{2n+1} P_{2n+1}(\cos \theta)$$

You only need to modify some of the example in the text.