Physics 50 **Assignments #10** **Spring 2008**

Readings: Riley, Hobson and Bence - Chapter 19
Boccio - 06_PDEs, 07_PDEs, 08_PDEs

I. Wave Equation

1. **Vibrating string -** A string of length L with fixed ends has a zero initial velocity and a displacement

$$y_o = \begin{cases} 8hx/L & 0<x<L/8 \\ 8h(L/4 - x)/L & L/8<x<L/4 \\ 0 & L/4<x<L \end{cases}$$

(this initial displacement might be caused by holding the string at the center and plucking half of it).

Find the displacement as a function of x and t.

Answer is:

$$y = \frac{16h}{\pi^2}\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi v t}{L}\right)$$

where

$$b_n = \frac{2\sin\left(\frac{n\pi}{8}\right) - \sin\left(\frac{n\pi}{4}\right)}{n^2}$$

2. **Vibration of a Membrane -** Separate the wave equation in 2-dimensional rectangular coordinates (x,y). Consider a rectangular membrane with corners $(0,0),(a,0),(a,b),(0,b)$ that is rigidly attached to supports along its sides. Show that its characteristic frequencies are

$$f_{nm} = \frac{v}{2}\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}$$

where (m,n) are positive integers. Sketch the normal modes of vibration corresponding to the first few frequencies, that is, indicate the nodal lines. Now suppose that the membrane is square. Show that in this case there may be two or more normal modes corresponding to a single frequency. This is an example of a phenomenon called **degeneracy.**

3. **Vibrating string -** A string π meters long is started into motion by giving the middle one-half($\pi/4 \le x \le 3\pi/4$) an initial velocity of 20 m/s (while keeping the initial displacement

zero everywhere). The string is stretched until the wave speed is 60 m/s. Determine the resulting displacement of the string as a function of x and t.

Answer is:

$$y(x,t)=\sum_{n=1}^{\infty}\frac{2}{3\pi n^2}\left(\cos\frac{n\pi}{4}-\cos\frac{3n\pi}{4}\right)\sin(60nt)\sin(nx)$$

II. Diffusion or Heat Flow Equation

1. **Heat flow in a bar or slab -** A bar 10 cm long with insulated sides is initially at $100°$. Starting at t = 0, the ends are held at $0°$. Find the temperature distribution in the bar at time t.

   Answer is:

$$T=\frac{400}{\pi}\sum_{n\ odd}\frac{1}{n}e^{-\left(\frac{n\pi\alpha}{10}\right)^2 t}\sin\left(\frac{n\pi x}{10}\right)$$

2. **Heat flow in a bar or slab -** In the initial steady-state distribution of an infinite slab of thickness d, the face x = 0 is at $0°$ and the face x = d is at $100°$. From t = 0 on, the x = 0 face is held at $100°$ and the x = d face is held at $0°$. Find the temperature distribution in the bar at time t.

   Answer is:

$$T=100-\frac{100x}{d}-\frac{400}{\pi}\sum_{n\ even}\frac{1}{n}e^{-\left(\frac{n\pi\alpha}{d}\right)^2 t}\sin\left(\frac{n\pi x}{d}\right)$$

   You need to consider the $k=0 \quad (k_n=k_0=0)$ solution.

3. **Heat flow in a bar or slab -** Two slabs, each 1 inch thick, each have one surface at $0°$ and the other at $100°$. At t = 0, they are stacked with their $100°$ faces together and then the outside surfaces are held at $100°$. Find the temperature distribution for t > 0.

   Answer is:

$$T = 100 + \frac{1}{\pi}\sum_{n=1}^{\infty} b_n e^{-\left(\frac{n\pi\alpha}{2}\right)^2 t} \sin\left(\frac{n\pi x}{2}\right)$$

where

$$b_n = 400\begin{cases} 0 & \text{even n} \\ \frac{2}{n^2\pi^2} - \frac{1}{n\pi} & \text{n=1+4k} \\ -\frac{2}{n^2\pi^2} - \frac{1}{n\pi} & \text{n=3+4k} \end{cases}$$

You need to consider the $k = 0$ $(k_n = k_0 = 0)$ solution.

4. **Heat flow in a bar or slab -** A bar of length L with insulated sides has its ends also insulated from time t = 0 on. Initially the temperature distribution is T = x, where x = distance from one end. Determine the temperature distribution inside the bar at time t. (Hint: you cannot neglect the k = 0 solutions in this case).

   Answer is:

$$T = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n\,odd}\frac{1}{n^2} e^{-\left(\frac{n\pi\alpha}{L}\right)^2 t} \cos\left(\frac{n\pi x}{L}\right)$$

5. **Text problem 19.19 - Heat flow in a bar or slab -** Two identical copper bars are each of length a. Initially, one is at $0°C$ and the other is at $100°C$. They are then joined together end to end and thermally isolated. Obtain in the form of a Fourier series and expression T(x,t) for the temperature at any point a distance x from the join at a later time t.

   Taking a = 0.5 m, estimate the time it takes for one of the free ends to reach a temperature of $55°C$. The thermal conductivity of copper is $3.8\times10^2\,Jm^{-1}K^{-1}s^{-1}$ and its specific heat capacity is $3.4\times10^6\,Jm^{-1}K^{-1}$.

   Answer is:

$$T(x,t) = 50 + \frac{200}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\frac{(2n+1)\pi x}{2a} e^{-\frac{D(2n+1)^2\pi^2 t}{4a^2}}$$