# PROBLEMS

*Mathematica Lab Number 0*

**Problem 1.** Ask *Mathematica* about **?Binomial** and use that information to evaluate

$$\text{the binomial coefficient } \binom{81}{14}$$

$$\sum_{k=0}^{n} \binom{n}{k}$$

**Problem 2.** The infinite sum

$$G \equiv 1 - \tfrac{1}{9} + \tfrac{1}{25} - \tfrac{1}{49} + \cdots = \sum_{k=0}^{\infty} (-1)^k (2k+1)^{-2}$$

turns up frequently in combinatorial contexts. What does *Mathematica* have to say about that sum? Use **?** to make sense of its answer, and **N[ , ]** to obtain an evaluation to 50 decimal places.

**Problem 3.** Ask *Mathematica* about the closely related sum

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \cdots = \sum_{k=0}^{\infty} (-1)^k (2k+1)^{-1}$$

and obtain an evaluation accurate to 50 decimal places.

**Problem 4.** Obtain 20-place evaluations of $\pi^\pi$, $e^\pi$ , $\pi^e$ and $e^e$ and see what *Mathematica* has to say about the assertion that

$$\pi^\pi > e^\pi > \pi^e > e^e$$

**Problem 5.** Evaluate the following:

$$\sum_{k=0}^{\infty} x^k$$

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^4}$$

$$\sum_{k=1}^{\infty} \frac{1}{2^k k^2}$$

$$\prod_{k=1}^{\infty} \left\{1 + \frac{(-1)^{k+1}}{2k-1}\right\}$$

$$x \prod_{k=1}^{\infty} \left\{1 - \frac{x^2}{k^2\pi^2}\right\}$$

To the final result, bring the command **Simplify[%, $x > 0$]**.

**Problem 6.** Evaluate the indefinite integral

$$\int \frac{1}{1-x^3}\, dx$$

and observe how *Mathematica* responds to the following commands:
**TraditionalForm[%]**
**TeXForm[%]**
Also evaluate

$$\int \frac{1}{1-x^5}\, dx$$

$$\int \frac{1}{1-x^7}\, dx$$

and render the last result in **TraditionalForm**.

**Problem 7.** Ask what *Mathematica* has to say in response to the query **?Table**. Use that information to construct a table of the values assumed by

$$\sum_{k=1}^{n} k \quad : \quad n = 1, 2, \ldots, 10$$

and give that table (list) the name **triangularnumbers**. Do the same for

$$\sum_{k=1}^{n} k^3 \quad : \quad n = 1, 2, \ldots, 10$$

Command **triangularnumbers$^{\mathbf{2}}$**. What do you conclude? Does *Mathematica* support your conjecture? Ask for a respond to the assertion

$$\sum_{k=1}^{n} k^3 = \left[\sum_{k=1}^{n} k\right]^2$$

# PROBLEMS

*Mathematica Lab Number 1*

**Problem 1.** Evaluate

$$\int_0^\pi \cos\big(x\sin\theta\big)\,d\theta$$

and use **Plot[%, {x,0,20}];** to plot the famous result. Demonstrate that **Plot[Evaluate[$\int_0^\pi \cos\big(x\sin\theta\big)\,d\theta$], {x,0,20}];** does the same job without the distraction of intermediate output.

**Problem 2.** The Fibonacci numbers are defined recursively

$$F_1 = F_2 = 1 \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \;:\; n = 3, 4, 5, \ldots$$

and grow very rapidly: ask *Mathematica* about **?Fibonacci** and then evaluate $F_{50}$. Next construct the generating function

$$\sum_{n=1}^{\infty} \frac{1}{n!} F_n x^n$$

and process the output with the successive commands
**Series[%,{x,0,5}]**
**Simplify[%]**
Finally, use **logfib=Table[Log[Fibonacci[k]],{k,1,100}]//N** to construct a list of the logs of the first 100 Fibonacci numbers, and then use **ListPlot** to display that data.

**Problem 3.** A curve is described parametrically by the equations

$$\begin{aligned} x(t) &= \sin(5t)\cos(2t) \\ y(t) &= \sin(3t)\sin(2t) \end{aligned}$$

Assuming $0 \le t \le 2\pi$, use **ParametricPlot** to display that curve. Install these options:
**Axes->None, Frame->True, AspectRatio->Automatic**

**Problem 4.** Let $g_1$, $g_2$, $g_7$ and $g_8$ be the names assigned to plots (assume $0 \le x \le 2\pi$ and adopt the option **`PlotRange->All`**) of the functions

$$\begin{aligned} f_1(x) &= \sin^1 x \\ f_2(x) &= \sin^2 x \\ f_7(x) &= \sin^7 x \\ f_8(x) &= \sin^8 x \end{aligned}$$

Construct a $2 \times 2$ composite figure in which $g_1$, $g_2$, $g_7$ and $g_8$ occupy the NW, NE, SW and SE positions, respectively.

**Problem 5.** Define

$$f(x) = \sum_{k=1}^{30} \frac{1}{k + k^{\frac{1}{3}}} \sin 2\pi k x$$

and—using **`//Timing`** to record how long it takes *Mathematica* to do the work—plot that function on the interval $0 < x < 2$. Drag to size and notice certain artifacts (graphical imprecisions). Install the option
**`MaxBend->1`**
and notice what happens both to the rendered detail and to the time required. Install the alternative option
**`PlotPoints->120`**
and do the same. Look up **`MaxBend`** in **Help>Master Index**; you will be referred to material that also provides a description of **`PlotPoints`**. Such trickery is often needed to override the defaults and improve the quality of plotted figures.

# PROBLEMS

*Mathematica Lab Number 2*

**Problem 1.** Use **Plot3D** to plot the function

$$f(x,y) = \sin\left\{\pi\sin(x) + y\right\}$$

on the region $-3 < x, y < 3$. Install the options
**Boxed->False**
**Axes->None**
**PlotPoints->40**

**Problem 2.** The equations

$$\begin{aligned} x^2 \;\, + y^2 \;\, &= 1 \\ x^{10} + y^{10} &= 1 \end{aligned}$$

provide implicit descriptions of a couple of curves, of which the first is familiar, the second isn't. Install the standard package **Graphics`ImplicitPlot`**, then use **ImplicitPlot** to construct a figure in which presents superimposed images of those two curves. Stipulate $-1.5 < x < 1.5$.

**Problem 3.** We have seen that graphs of $x^{2p} + y^{2p} = 1$ become "more and more nearly square" as $p$ advances through the integers. The following figure illustrates the same phenomenon: use **Plot3D** to display the function

$$g(x,y) = e^{-(x^8+y^8)}$$

on the region $-2 < x, y < 2$. Set **PlotPoints->30**.

**Problem 4.** Use **ParametricPlot3D** to display the function

$$x(r,\theta) = r\cos\theta$$
$$y(r,\theta) = r\sin\theta$$
$$z(r,\theta) = \cos\theta\sin\theta$$

with $0 < r < 1$ and $0 < \theta < 2\pi$. Set
**PlotPoints->{10,30}**
**Boxed->False**
**Axes->False**
and (for later convenience) call the figure **g**. Command **Options[%]** to discover the values which *Mathematica* has, by default, assigned to the option
**ViewPoint->{•,•,•}**.
Experiment with that option to display the surface to—in your view—better advantage.

In a separate notebook—not for submission (which would impose excessive demand on the memory resources of the server)—open the standard package **Graphics`Animation`** and use **SpinShow[g]** to accomplish the same objective.

**Problem 5.** Command **?MovieParametricPlot**, and use the information you are thus supplied to construct an animated display of the $t$-indexed set of $s$-parameterized curves

$$x(s,t) = s\cos(2\pi s + t)$$
$$y(s,t) = s\sin(2\pi s + t)$$

where $0 < s < 4$. Command
**{** $t$, **0,** $\frac{14}{15}2\pi$ **}, Frames->15**
to create a 15-frame movie in which the $t$-values are equi-spaced. Set
**Axes->False**
**AspectRatio->Automatic**
**PlotRange->{{−4,4},{−4,4}}**

**Problem 6.** The physical question: *What is the spin angular momentum of the sun, and how does it compare to what would be the angular momentum if an equivalent number of stationary protons each contributed* $\frac{1}{2}\hbar$ (where, by universal convention, $\hbar \equiv h/2\pi$)? To approach the problem you need to install the standard packages
**Miscellaneous`PhysicalConstants`**
**Miscellaneous`Units`**
Ask *Mathematica* about
**?SolarMass**
**?SolarRadius**
**?ProtonMass**
**?PlanckConstant**
and accept as given that the solar rotational period is 25.36 days. Use that information to compute ($i$) the angular momentum of the sun, assumed to rotate as a solid sphere; ($ii$) the equivalent number of protons; ($iii$) the spin angular

momentum (at $\frac{1}{2}\hbar$/each) of such a population; (*iv*) the ratio (latter/former). Be sure to display your final answer as a dimensionless number.

# PROBLEMS

*Mathematica Lab Number 3*

**Problem 1.** **`TrigExpand`** the functions $\tan 2\theta$ and $\cos 6\theta$.

**Problem 2.** **`TrigReduce`** the functions $\sin^2\theta$ and $\tan^2 \frac{1}{2}\theta$.

**Problem 3.** "Magic squares" have fascinated mathematicians for many centuries. The following example

$$\mathbb{M} = \begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix}$$

is taken from an engraving by Albrecht Dürer. It displays all the integers from 1 through 16, and has the "magical" property that

$$\sum_{\text{any row}} = \sum_{\text{any column}} = \sum_{\text{counter diagonal}} = \sum_{\text{principal diagonal}} \equiv \text{trace}$$

Ask *Mathematica* to evaluate

1) $\operatorname{tr}\mathbb{M}$
2) $\det \mathbb{M}$
3) the eigenvalues of $\mathbb{M}$
4) the eigenvectors of $\mathbb{M}$

Notice that you had no reason to expect the eigenvalues to real, but that they turned out "magically" to be so. Let the eigenvectors (which *Mathematica* has presented as lists) be called $a$, $b$, $c$ and $d$. Evaluate each of the following ten dot products:

$$\begin{matrix} a\cdot a & a\cdot b & a\cdot c & a\cdot d \\ & b\cdot b & b\cdot c & b\cdot d \\ & & c\cdot c & c\cdot d \\ & & & d\cdot d \end{matrix}$$

What do you conclude about the relation of $d$ to $a$, $b$ and $c$? Look finally to the validity of each of the four claims that

$$(\text{matrix})(\text{eigenvector}) = (\text{eigenvalue})(\text{eigenvector})$$

**Problem 4.** Here is a "Latin square" (each row and each column presents a permutation of $\{1, 2, 3, 4\}$):

$$\mathbb{L}_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 4 & 3 & 1 & 2 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

Evaluate $\det \mathbb{L}_1$ and list the eigenvalues of $\mathbb{L}_1$.

**Problem 5.** The following Latin square

$$\mathbb{L}_2 = \begin{pmatrix} 1 & 4 & 2 & 5 & 3 \\ 4 & 2 & 5 & 3 & 1 \\ 2 & 5 & 3 & 1 & 4 \\ 5 & 3 & 1 & 4 & 2 \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix}$$

is distinguished by (among other properties) its symmetry about the principal diagonal. . . on which grounds we are assured that the eigenvalues must be real. Evaluate $\det \mathbb{L}_2$ and list the eigenvalues of $\mathbb{L}_2$. Use **N[%]** to list approximate *numerical values* of the eigenvalues (which *Mathematica* prefers to give exactly, when it can).

**Problem 6.** Now introduce one small symmetry-preserving typo into the description of the preceding matrix, writing

$$\mathbb{L}_3 = \begin{pmatrix} 1 & 4 & 2 & 5 & 3 \\ 4 & 3 & 5 & 3 & 1 \\ 2 & 5 & 3 & 1 & 4 \\ 5 & 3 & 1 & 4 & 2 \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix}$$

Evaluate $\det \mathbb{L}_3$ and list the eigenvalues of $\mathbb{L}_3$; you find that *Mathematica*, confronted with the solution of a quintic, responds unhelpfully to the latter command. So use

```
CharacteristicPolynomial[L3,x]
spectrum=x/.NSolve[% == 0, x]
```

to obtain a numerical description of the spectrum of $\mathbb{L}_3$. How does that compare to the spectrum of $\mathbb{L}_2$? Notice that this little exercise—mere child's play for *Mathematica*—involves labor you would not want to undertake by hand!

**Problem 7.** In the theory of one-dimensional crystals one encounters high-dimensional symmetric matrices of the form

$$\mathbb{S} = \begin{pmatrix}
a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
b & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & b & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & b & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & b & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & b & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & b & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & b & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & b & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & b & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & b & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & b & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & b & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & b & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & b\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a
\end{pmatrix}$$

with $a = 2$ and $b = -1$. Proceed as in Problem 6 to construct a list of the eigenvalues; then use
**`ListPlot[spectrum, PlotStyle->PointSize[0.02]]`** to display those data.

It is found to be analytically convenient to impose "periodic boundary conditions" (physically: to tie the last atom to the first, forming a *ring* of atoms, each interacting only with its nearest neighbors). The matrix $\mathbb{S}$ has then to be adjusted

$$0 \rightarrow b \text{ in upper right and lower left corners}$$

Proceed as before to plot of the spectrum of the modified matrix, and compare it to the previous spectrum.

To model the introduction of an impurity we might study (for example)

$$\mathbb{S} = \begin{pmatrix}
a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
b & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & b & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & b & a & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & B & A & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & B & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & b & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & b & a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & b & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & b & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & b & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & b & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & b & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & b & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a & b\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & a
\end{pmatrix}$$

with (say) $A = 3$ and $B = -2$. Proceed as before: plot the spectrum, and compare it with the original spectrum.

The results just obtained would have been much more revealing—more valuable as "toy solid state physics"—if we had assumed the crystal to contain not just 16 "atoms" but (say) 16,000. But to describe a $16000 \times 16000$ matrix to *Mathematica* requires a higher level of technique than the one to which we presently aspire.

# PROBLEMS

*Mathematica Lab Number 4*

**Problem 1.** Think for a moment about how you would proceed to solve *by hand* the linear first-order ODE

$$y'(x) = -\cos x \cdot y(x)$$

Use **DSolve** to ask *Mathematica* to do the same thing. The answer will contain a single constant of integration, called `C[1]`. The following command
**somesolutions=Table[** place general solution here **/.C[1]->i,{*i*, −5, 5}]**
produces a list of particular solutions. Plot **Evaluate[somesolutions]** on the interval $0 < x < 4\pi$. Place ticks at the points $\{0, \pi, 2\pi, 3\pi, 4\pi\}$.

**Problem 2.** Lord Rayleigh was led from the physics of violin strings to a nonlinear ODE of the form

$$x''(t) + \left\{\tfrac{1}{3}\big[x'(t)\big]^2 - 1\right\}x'(t) + x(t) = 0$$

The command **DSolve** informs us that no analytical solution exists; we must use **NDSolve**. Assign names

**violin1** to the solution that proceeds from $x(0) = 0.1, x'(0) = 0$;

**violin2** to the solution that proceeds from $x(0) = 1.0, x'(0) = 0$;

**violin3** to the solution that proceeds from $x(0) = 1.9, x'(0) = 0$.

Taking the procedure described at the beginning of Lab 4 Part C as your model, and working on the interval $0 < t < 15$, construct a figure that superimposes the numerical solutions **violin1**, **violin2**, and **violin3**, using the option
**PlotStyle->{RGBColor[1,0,0],RGBColor[0,0,1],{ }}**
to assign to those curves the colors just indicated.

**Problem 3.** The simple oscillator equation reads $mx'' + kx = 0$ if the spring is "perfect," but if the spring is what engineers call "hard" then

$$k \text{ constant} \longrightarrow k + ax^2$$

in better approximation ($a$ is a phenomenological constant, and refers to a second property of the spring). The equation of motion has become

$$mx''(t) + kx(t) + ax^3(t) = 0$$

Set $m = 1$, $k = 0.3$, $a = 0.04$, $x(0) = 0$ and

$$x'(0) = 1, 2, 3, 4, 5$$

Working again on the interval $0 < t < 15$, and on the pattern rehearsed in the preceding problem, give names **hardspring1** through **hardspring5** to the five numerical solutions supplied by **NDSolve**, then plot those solutions. To distinguish one from another, create a 5-place list of gray values

```
grays={GrayLevel[0.2], GrayLevel[0.35], GrayLevel[0.5],
GrayLevel[0.65], GrayLevel[0.8]}
```

and then adopt the option **PlotStyle->grays**. Notice that increasing the initial velocity (energy) causes the particle to venture further into the stiff spring domain, and decreases the period.

**Problem 4.** To model an oscillator in which the spring "gets tired" we might write

$$mx''(t) + ke^{-t/\tau}x(t) = 0$$

We will look to the case $m = 1$, $k = \tau = 4$. Again, no analytical solution is available, and we must proceed numerically. Set $x'(0) = 0$, $x(0) = 1, 2, 3, 4, 5$. Call the solutions **tired1** through **tired5**. Proceed as before to plot the solutions reported by **NDSolve**, using differentiated gray levels to distinguish one curve from another. Notice that the zero-crossings are energy independent, but are spaced farther and farther apart as the spring loses its poop.

**Problem 5.** To "pump" a swing we use our internal energy reserves to manipulate a system parameter (that that instance, the *effective length* of the rope), thus creating an instance of a "parametrically stimulated oscillator." We are careful to pump in synchrony with the natural period of the swing, since our objective is to pump energy into the system. The equation

$$x''(t) + \big\{1 + 0.2\sin 2t\big\}x(t) = 0$$

serves to illustrate the process. Construct

```
stimulated=x[t]/.NDSolve[{x''[t]+(1+0.2Sin[2t])x[t]==0, x'[0]==0,
x[0]==1},x[t],{t,0,50}]
```

and

**unstimulated=** same thing with **0.2** → **0.0**

and then plot those functions. You will want to install the option

**PlotRange->All**

To demonstrate the importance of synchrony, construct and plot

**outofsync=** same as **stimulated** but with **2** → **2.2**

# PROBLEMS

*Mathematica Lab Number 5*

NOTE: Your ability to make effective use of *Mathematica* in the course of your own future work will hinge critically on your ability to look things up, as needed; there is no way to store The *Mathematica* Book, the *Guide to the Standard Add-on Packages* plus other supplementary material between your ears. . . but what you can acquire and store is the technique for discovering the techniques of which you have momentary need. To encourage you in that effort I will, in this last pair of problem sets, allow myself to move farther and farther from material specifically covered in the lab manual. But not to worry: I will provide lots of hints.

**Problem 1: Invaded Planetary System.** First, use techniques sketched in the Graphics Primitives section of Lab 2 Part A to draw a figure consisting of

- a black point of **`AbsolutePointSize[10]`** at $\{0, 0\}$
- a red point of **`AbsolutePointSize[4]`** at $\{1, 0\}$
- a blue point of **`AbsolutePointSize[8]`** at $\{10, 4\}$
- a blue circle of unit radius, centered at $\{0, 0\}$
- a blue line from $\{-10, 4\}$ to $\{+10, 4\}$.

Drag to size.

Neglect the blue body, imagine the black body to be pinned at the origin, and allow the red body to move dynamically; if we work in units where $G = 1$ and assume the black body to have mass 9, then the motion of the red body is described

$$\ddot{x}(t) = -\frac{9x(t)}{[x^2(t) + y^2(t)]^{\frac{3}{2}}}$$

$$\ddot{y}(t) = -\frac{9y(t)}{[x^2(t) + y^2(t)]^{\frac{3}{2}}}$$

The prescribed initial data will be

$$x(0) = 1, \quad y(0) = 0, \quad \dot{x}(0) = 0, \quad \dot{y}(0) = 3$$

Give the name **`unperturbedorbit`** to the result of using **`NDSolve`** to solve that system on $0 < t < 3$, then plot **`unperturbedorbit`** and color it red; this exercise serves to demonstrate that we have successfully rigged the initial data to yield a *circular* orbit of unit radius. Be sure to install the options

**AspectRatio->Automatic**
**PlotRange->All**

Now install the blue body (assumed to have mass 6, and to be pinned at $\{10,4\}$); the equations of motion become

$$\ddot{x} = -\frac{9x}{[x^2+y^2]^{\frac{3}{2}}} + \frac{6(10-x)}{[(10-x)^2+(4-y)^2]^{\frac{3}{2}}}$$
$$\ddot{y} = -\frac{9y}{[x^2+y^2]^{\frac{3}{2}}} + \frac{6(4-y)}{[(10-x)^2+(4-y)^2]^{\frac{3}{2}}}$$

Preserving the former initial conditions, and working now on $0<t<20$, again solve the equations (call the solution **perturbedorbit**) and plot the orbit.

Now assume the blue body to move uniformly along the blue line, from right to left, and to arrive at the left end at $t=40$; make, in other words, the substitution

$$10 \longrightarrow 10 - \tfrac{1}{20}t$$

Again solve and plot the solution (called now **impactedorbit**), working this time on $0<t<40$, which will require that you set **MaxSteps->5000**.

Repeat that exercise on the assumption that the blue body starts at the left end of the blue line and moves to the right:

$$10 \longrightarrow -10 + \tfrac{1}{20}t$$

When trying to make carry-home intuitive sense of your result, don't forget to take into account the fact that in both cases the red particle is moving counterclockwise.

**Problem 2: Nonlinear Oscillations.** In the following "scrubbed" instance of the unforced damped oscillator equation

$$\ddot{x} + b\dot{x} + x = 0$$

we will allow $b$ to become $x$-dependent (which will destroy the linearity of the equation: solution + solution will no longer be a solution!). In particular, we set

$$b \longrightarrow b(x) \equiv \mu(x^2-1) \quad : \quad \mu \geqslant 0$$

and obtain the "unforced Van der Pol equation," which is fundamental to the theory of nonlinear oscillations.[1] **Plot** $b(x)$ and notice that the $\dot{x}$-term describes damping (energy loss) if $x<-1$ or $1<x$, but "antidamping" (energy injection)

[1] Balthasar van der Pol (1889–1959) was an engineer working for the N. V. Philips' Glowlamp Works in Eindhover, Holland when his study of the operating characteristics of triodes (early radio tubes with cathode, anode & grid) led him to publish "Forced oscillations in a system with non-linear resistance," Phil. Mag. **3**, 65 (1927), of which the Dutch version had appeared already in 1924.

if $x$ is small: $-1 < x < +1$. The Van der Pol oscillator is "self-exciting," with consequences that will soon become evident.

Set $\mu = \frac{1}{8}$, $x(0) = 1$ and $\dot{x}(0) = 0$, then on $0 < t < 30$ **NDSolve** and **Plot** the solution of

$$\ddot{x}(t) + \mu[x^2(t) - 1]\dot{x}(t) + x(t) = 0$$

Notice that the oscillation grows to an apparently stable value.

Do the same with $x(0) = 4$. The oscillation damps to an apparently stable value.

Now set $\mu = 8$, $x(0) = 1$ and $\dot{x}(0) = 0$ and proceed as before. You have entered the regime most characteristic of Van der Pol oscillators.

The idea now is to promote $\dot{x}$ to the status of an independent variable, display the Van der Pol equation as a coupled pair of first order equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} + \mu(x^2 - 1)y + x &= 0 \end{aligned}$$

and to look at the phase plot. To that end, give names **phase1**, **phase2** and **phase3** to the results of **NDSolve**'ing the preceding system—work as before on $0 < t < 30$—with

$$\begin{aligned} &\mu = 8, \quad x(0) = 0.0, \quad y(0) = 01.0 \\ &\mu = 8, \quad x(0) = 0.0, \quad y(0) = 12.0 \\ &\mu = 8, \quad x(0) = 2.1, \quad y(0) = 17.2 \end{aligned}$$

Plot those results, calling the results **phasemap1**, **phasemap2** and **phasemap3**. Adopt the options

```
PlotRange->All
AspectRatio->Automatic
Ticks->None
PlotPoints->200
```

and make the figures respectively red, black and blue.

Finally, use **Show** to superimpose those figures. Drag to size, and notice that after initial transcients have died down they *precisely coincide*! This is but one of many wonderful properties—properties of high practical importance—exhibited by nonlinear oscillators.[2]

**Problem 3: Larmor's Formula by Dimensional Analysis.** Charged particles are known to radiate when accelerated. One might therefore expect to have a formula of the form

$$\text{instantaneous radiated power} = a^x (e^2)^y c^z$$

[2] For a nice account of this subject—a famously "difficult" subject which *Mathematica* makes much more accessible—see A. H. Nayfeh & D. T. Mook, *Nonlinear Oscillations* (1979).

Use the technique described in the lab manual to deduce the enforced values of $x$, $y$ and $z$. Joseph Larmor's analysis (1897) yielded precisely this result, but with $\frac{2}{3}$ as a numerical prefactor.

You might suppose that the radiated power depends also on the mass of the charged particle. Work from

$$\text{power} = a^x (e^2)^y c^z m^u$$

and get a $u$-parameterized *family* of possibilities. By assigning appropriate values to $u$ you could kill the $a$-dependence, or the $e^2$-dependence, or the $c$-dependence. Can you argue that $u = 0$ is the most plausible state of affairs?

# PROBLEMS

*Mathematica Lab Number 6*

**Problem 1: River Meanders.** It was apparently Einstein (1926) who first discussed why rivers, in their progress across gently sloped alluvial plains, do not pursue straight courses (such motion is evidently unstable), but meander. In recent reading I was alerted[1] to the existence of a recent contribution to that pastoral subject: Hans-Henrik Støllem, in "River meandering as a self-organization process," Science **271**, 1710 (1996), brings modern computer resources, the theory of fractals and field data to bear on the problem, and is led to the striking conclusion that typically—on average—one can expect to find

$$\frac{\text{wet distance from source to mouth}}{\text{distance as a crow flys}} \sim \pi$$

The question arise: what would such a river look like in an idealized case? We begin by drawing a picture of idealized "River Alpha." Enter these commands:

```
α = π/5

a=Sin[α]
b=Cos[α]

x1[θ_]:=a+Cos[θ]
x2[θ_]:=3a+Cos[θ]
x3[θ_]:=5a+Cos[θ]
x4[θ_]:=7a+Cos[θ]
x5[θ_]:=9a+Cos[θ]
y[θ_]:=+b+Sin[θ]
z[θ_]:=-b+Sin[θ]

upperloops
=ParametricPlot[{{x1[θ],y[θ]}, {x3[θ],y[θ]}, {x5[θ],y[θ]}},
{θ,-π/2+α,3π/2-α},
AspectRatio->Automatic, PlotStyle->{
```

[1] See Simon Singh, *Fermat's Enigma* (1997), page 17. See also page 329 in A. Pais, *Subtle is the Lord...:The Science and the Life of Albert Einstein* (1982).

```
{RGBColor[0,0,1], Thickness[0.02]},
{RGBColor[0,0,1], Thickness[0.02]},
{RGBColor[0,0,1], Thickness[0.02]}}, Axes->None];

lowerloops
=ParametricPlot[{{x2[θ],z[θ]}, {x4[θ],z[θ]}},
{θ,-3π/2+α,π/2-α},
AspectRatio->Automatic, PlotStyle->{
{RGBColor[0,0,1], Thickness[0.02]},
{RGBColor[0,0,1], Thickness[0.02]}}, Axes->None];

Show[upperloops, lowerloops, Graphics[
{Thickness[0.01], RGBColor[1,0,0], Line[{{0,0}, {10a,0}}]}]];
```

You have with these commands constructed an illustration of the meandering river (five semi-circular loops) and the flight of a crow. The figure can be considered to be have been assembled from "primitive cells":

```
bluearc=ParametricPlot[{Cos[θ], Sin[θ]}, {θ,α,π},
AspectRatio->Automatic, Ticks->None, Axes->None,
PlotStyle->{RGBColor[0,0,1], Thickness[0.02]}];

lines=Graphics[{
Line[{{-1,0}, {Cos[α],0}}],
Line[{{0,0}, {Cos[α],Sin[α]}}],
{RGBColor[1,0,0], Thickness[0.01],
Line[{{Cos[α],0}, {Cos[α],Sin[α]}}]},
Text[Styleform["α", FontSize->14], {0.15, 0.04}]}];

Show[bluearc, lines]
```

Working from that figure, use **NSolve** to show that Støllem's relation requires

$$\alpha = 47.4328^\circ$$

Now back up to the beginning, assign the computed value (in radian measure) to $\alpha$, and reconstruct the picture of the river... which is now a "Støllem meander."

One can think of ways to make the model more realistic, to make it dynamical. Note the characteristic events that result in the creation of oxbow lakes and temporary straightening of the channel.

**Problem 2: Solitons.** An equation central to the theory of nonlinear waves (continuous analog of the theory of nonlinear oscillators) is the "KdV equation"[2]

$$u_t + 6uu_x + u_{xxx} = 0$$

[2] Subscripts are used here to denote partial derivatives.

which D. J. Korteweg & G. de Vries (1882) devised to account for certain solitary hydraulic waves first noted by John Scott Russell in 1834. Notice that it is again the middle term that contributes the nonlinearity, and causes it to be the case that solutions added to solutions are not again solutions.

Remind yourself what *Mathematica* has to say about **?D**, then use that information to ask *Mathematica* for a demonstration that every function of the form

$$u(x,t;b) \equiv 2b^2 \text{sech}^2(bx - 4b^3 t) \quad : \quad b \text{ arbitrary} \tag{1}$$

is a particular solution of the KdV equation. Command

```
Table[Plot[{u[x,t,1], u[x,t,1/√2]}, {x,-4,16},
PlotRange->{0,4}];,{t,0,3,0.1}];
```

to animate the motion of two such solutions.

Notice that ($i$) each wave *preserves its shape* as it moves (that's why such waves are called "solitons"); ($ii$) amplitude is coupled to wavespeed (tall/narrow waves move faster than shorter/broader ones).

The occurence/preservation of stable forms ("fixed points" in some broad sense) is reminiscent of our experience with the with the Van der Pol equation, and also with iterated maps. For helpful discussion of nonlinear wave systems (and a valuable bibliography) see the recent thesis of Adam Halverson.[3]

**Problem 3: Fourier Analysis of a KdV Soliton.** It is our habit to resolve complex waves into their elementary (Fourier) components. In nonlinear wave theory that exercise is still possible but less useful: the Fourier components will not themselves be primitive *solutions* of the wave equation! But let's pursue the idea anyway, to see where it leads.

Ask what *Mathematica* has to say in response to
**?FourierTransform**
**?InverseFourierTransform**
Let's see how this works in some simple cases. Assign the name **normedgauss** to the function

$$g(x;a) \equiv \sqrt{a/2\pi}\, e^{-\frac{1}{2}ax^2}$$

and demonstrate that

$$\int_{-\infty}^{+\infty} g(x;a)\,dx = 1 \quad : \quad \text{all positive real } a$$

Plot $\big\{g(x;1), g(x;2), g(x;3)\big\}$ on $-5 < x < 5$, with installed options
**PlotStyle->{Thickness[0.005],Thickness[0.008],Thickness[0.011]}, PlotRange->All**.

---

[3] "The inverse scattering transform and the Korteweg-de Vries equation," (Reed College 2000).

Now use **FourierTransform[*expr*,x,y]** to construct

$$G(y;a) \equiv \text{Fourier transform of } g(x;a)$$

and plot $\{G(y;1), G(y;2), G(y;3)\}$ on $-5 < y < 5$, subject to this modification of the former options:
**Thickness[0.005]** $\longrightarrow$ **{RGBColor[1,0,0], Thickness[0.005]}**, etc.
Note that ($i$) the transform of a normalized Gaussian is again gaussian (though *not* normalized), and ($ii$) that as the Gaussian gets skinnier its transform gets fatter.

Use **InverseFourierTransform** to recover the functions from which we started.

What, by the way, is the Fourier transform of the unit function $u(x) \equiv 1$ (all $x$)? What is the inverse Fourier transform of the function to which *Mathematica* just led you?

Do for the functions

$$h(x;a) \equiv \frac{a}{\pi}\,\text{sech}(ax) \quad \text{and its transform} \quad H(y;a)$$

all the things just done with $g(x;a)$ and $G(y;a)$. Notice that—here again—Fourier transformation has carried us from a function to a function *of similar design*. This is an exceptional state of affairs, as will immediately emerge:

Look to the KdV soliton $u(x,t;b)$ defined at (1). Set $t = 0$ and obtain

$$v(x;b) \equiv 2b^2\text{sech}^2(bx)$$

Do for the functions $v(x;b)$ and

$$V(y;b) \equiv \text{Fourier transform of } v(x;b)$$

the things we did earlier with $g(x;a)$ and $G(y;a)$.

Show finally that the functions

$$w(x,t;b) \equiv \cos(bx - 4b^3t)$$

do *not* satisfy the KdV equation (though when appropriately superimposed they yield a function which does).

**Problem 4: Square Waves, Gibbs' Phenomenon & Griffiths' "Needle Problem."** As we have just seen, *Mathematica* comes out of the box with a powerful and easy-to-use Fourier analytic capability, but that power is greatly expanded when one installs the "Calculus\`FourierTransform\`" Standard Add-on Package.

Do so. Look into **Help** to find clear discussion and examples. See what *Mathematica* has to say about

**?FourierTrigSeries**

of which we will have need in a moment.

Define

```
squarewave[x_]:=-UnitStep[x+3]+2UnitStep[x+2]-2UnitStep[x+1]
+2UnitStep[x+0]-2UnitStep[x-1]+2UnitStep[x-2]
```

and give the name **exact** to the figure produced when you **Plot** that function on $-3 < x < 3$, subject to the options
**AspectRatio->Automatic,**
**PlotStyle->{RGBColor[1,0,0], Thickness[0.01]}**.
Evidently, *squarewave*$(x)$ is an odd periodic function, with period 2. We expect it to be describable as a Fourier sine series.

Command (this will take a minute)

```
FourierTrigSeries[squarewave[x],x,15,FourierParameters->{0,1/2}]
```

execute the command **Expand[%]** and assign the name **f[x_]** to the function thus produced. Plot that function (same interval as before, but turn off the **PlotStyle** option, and in its place install **MaxBend->1**), and give the name **approx** to the resulting figure.

```
Show[exact, approx];
```

You see how the approximating function (which is continuous) struggles to achieve the discontinuity which we built into the design of the exact squarewave, and how in particular it acquires "ears." This is illustrative of a general circumstance, known as "Gibbs' phenomenon." It is to expose the situation more clearly that we

```
Plot[Abs[squarewave[x]-f[x]], {x,-3,3},
PlotRange->All, MaxBend->1];
```

A magnified view is provided by

```
Plot[Abs[squarewave[x]-f[x]], {x,0,1},
PlotRange->{0,1}, MaxBend->1];
```

but on that interval it is cleaner to define

```
gibbs[x_]:=1-f[x]
absolutegibbs[x_]:=Abs[1-f[x]]
```

and create

```
plotabsolutegibbs=Plot[absolutegibbs[x], {x,0,1},
PlotRange->{0,1}, MaxBend->1];
```

We undertake now to locate the maxima, and to plant a red dot on top of each little mountain. There are alternative ways to proceed:

FIRST METHOD: Compute **∂x gibbs[x]**, give the name **gibbsprime[x_]** to the function thus produced, **Plot** that function on $0 < x < 1$ and using the figure to discover the "seeds" in computations like this one

```
FindRoot[gibbsprime[x]==0, {x,0.06}]
```

construct a list of the fifteen zero-crossings; it will have the form

**xlist=**$\{x_1, x_2, \ldots, x_{15}\}$

Construct

```
ylist=absolutegibbs[xlist]
```

Recall the effect of the command **Transpose[{{a,b,c}, {p,q,r}}]**. Use that command to construct

```
xylist=Transpose[{xlist, ylist}]
```

Finally create

```
redpeaks=ListPlot[xylist, PlotRange->{0,1},
PlotStyle->{RGBPlot[1,0,0],PointSize[0.03]}];

Show[plotabsolutegibbs, redpeaks];
```

SECOND METHOD: Command

```
Solve[{gibbsprime[x]==0, x==Abs[x]},x]
```

and proceed in direct imitation of the discussion which begins at In[84] in Lab 3 Part B:

```
N[%]

peaks={x, absolutegibbs[x]}/.%

Plot[absolutegibbs[x], {x,0,1}, PlotRange->{0,1},
Epilog->{PointSize[0.03], Hue[1], Map[Point, peaks]}]
```

You will notice that **peaks** and **xylist** are identical except for order, and that we have arrived at the same decorated figure as before.

It seems to be a principle of scholarship that if you know two things—*any* two things—you are destined to see a relationship between them. As it happens, I have seen such point distributions before... in the thesis of Ye Li.[4] An effort was made there to fit the points to a curve of the form

$$w_p(x) = \frac{1}{[x(1-x)]^p}$$

Griffiths had theoretical reasons for setting $p = \frac{1}{3}$, but Li (motivated only by a desire to achieve goodness of fit) set $p = \frac{1}{2}$. Let us follow Li's lead. Command

```
Fit[peaks, {1, 1/(x(1-x))^(1/2)}, x]
```

and assign the name **li[x_]** to the function thus produced. Finally create

```
Plot[li[x], {x,0,1}, PlotRange->{0,1},
Epilog->{PointSize[0.03], Hue[1], Map[Point, peaks]}]
```

Looks pretty good (and $p = \frac{1}{3}$ looks a bit less good).

As the Fourier series is (prior to truncation) carried to higher and higher, we—on this evidence

```
Plot[{(4x(1-x))^(-1/2), (4x(1-x))^(-1/4), (4x(1-x))^(-1/8)}, {x,0,1},
PlotRange->{0,4},
PlotStyle->{RGBColor[0,0,1], {}, RGBColor[0,0,1]}];
```

—expect to have to make $p$ smaller and smaller. But Gibbs' phenomenon is persistent to very high order; we expect the process $p \downarrow 0$ to be quite slow.

By a Fourier-analytic argument rooted in the factional calculus[5] I have shown—I think to Griffiths' satisfaction—that the charge density on a needle is actually uniform, and that the seeming "horns" are artifacts of his numerical approximation procedure (but artifacts that persist to very high order).

Many questions now arise, among them this one: Is it possible that Gibbs' phenomenon has only incidentally to do with Fourier series, and is really an aspect of the general theory of approximation? I will not pursue the matter; it seems fitting that this introduction to *Mathematica* should end with a question —a question which we would scarcely have been in position even to pose had we been deprived of the assistance of this powerful new tool.

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[4] "Charge density on a needle," (Reed College 1994). An expanded version of that work was published as David Griffiths & Ye Li, "Charge density on a conducting needle," AJP **64**, 706 (1996).

[5] See §10 in "Construction & physical application of the fractional calculus" (1997).