Physics 130 General Relativity Seminar

Assignment 8 March 18, 2013

General topic: **Curved Spacetimes in General Relativity**

Part 1: Readings

Hartle: Ch 13 - Astrophysical Black Holes
Hartle: Ch 14 - A Little Rotation
Hartle: Ch 14 Supplement - Construction of Freely Falling Frames

Part 2: Problems Hartle Problems

1. Hartle 13.02 The Roche Lobe
2. Hartle 13.05 Mass of black hole at beginning
3. Hartle 13.07 Exploding primordial black hole
4. Hartle 14.02 Orientation of spin
5. Hartle 14.04 How does it precess?
6. Hartle 14.07 Precession and PPN
7. Hartle 14.08 Stays in the plane
8. Hartle 14.09 Rotation deflects light

Boccio Extra Problems

1. **Geodesic Effect**

If in flat spacetime a spacelike vector λ^μ is transported along a timelike geodesic without changing its spatial orientation, then, in Cartesian coordinates, it satisfies $d\lambda^\mu/dt\tau = 0$ where τ is the proper time along the geodesic. That is, λ^μ is parallel transported through spacetime along the geodesic. Moreover, if at some point λ^μ is orthogonal to the tangent vector $\dot{x}^\mu = dx^\mu/d\tau$ to the geodesic, then $\eta_{\mu\nu}\lambda^\mu\dot{x}^\nu = 0$, and this relationship is preserved under parallel transport. This orthogonality condition simply means that λ^μ has no temporal component in an instantaneous rest frame of an observer traveling along the geodesic. The corresponding criteria for transporting a spacelike vector λ^μ in this fashion in the curved spacetime of general relativity are, therefore,

$$\frac{d\lambda^\mu}{d\tau} + \Gamma^\mu_{\nu\sigma}\lambda^\nu \dot{x}^\sigma = 0 \quad , \quad g_{\mu\nu}\lambda^\mu \dot{x}^\nu = 0$$

(a) Explain why these are the correct equations.

(b) Consider a spinning particle (perhaps a gyroscope) moving in a gravitational field. No non-gravitational forces are present. Write down and explain the equation which governs the behavior in time of the spin(vector) of the particle.

(c) Consider a slowly rotating thin spherical shell of mass M, radius R and rotation frequency ω. The metric of the field due to this shell can be written as

$$ds^2 = -c^2 H(r) dt^2 + \frac{1}{H(r)}\left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta (d\varphi - \Omega dt)^2\right]$$

where $\Omega = 4GM\omega/3Rc^2$ for $r < R$, $\Omega \to 0$ for $r \to \infty$, and

$$H(r) = \begin{cases} 1 - \frac{2GM}{rc^2} & r > R \\ 1 - \frac{2GM}{Rc^2} & r < R \end{cases}$$

This form of the metric is valid if $GM/Rc^2 << 1$. Consider a spinning particle at rest at the center of the sphere($r = 0$). Using the equation from part (b), with what frequency will the spin of the particle precess? What is the precession frequency quantitatively, if ω is the rotational frequency of the Earth and M and R, the mass and radius of the Earth, are $M \approx 6.0 \times 10^{27}\, gm$ and $R \approx 6.4 \times 10^3\, km$? A rough estimate is enough.

2. **A Charged Black Hole**

The metric for the spacetime around a static spherically symmetric source of mass M and charge Q(in appropriate units) is

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

This is called the Reissner-Nordstrom metric.

(a) Show that if $Q > M$, this metric is only singular at $r = 0$.

(b) For $Q < M$, the metric in this coordinate system is also singular at $r = r_\pm (r_+ > r_-)$. Find $r_\pm$ in terms of Q, M.

(c) Define a new coordinate u (analogous to Eddington-Finkelstein coordinates) so that the metric in (u, r, θ, φ) coordinates is regular at r_+.