Physics 130 General Relativity Seminar

Assignment 7 March 04, 2013

General topic: **Curved Spacetimes in General Relativity**

Part 1: Readings

Hartle: Ch 12 - Gravitational Collapse and Black Holes

Part 2: Problems Hartle Problems

1. Hartle 12.04 Study the line element
2. Hartle 12.05 Time to singularity
3. Hartle 12.07 Break the meter stick
4. Hartle 12.08 See outside?
5. Hartle 12.09 Darth Vader and Jedi
6. Hartle 12.13 Head and feet near the horizon
7. Hartle 12.14 How much time left?
8. Hartle 12.15 Escape to infinity
9. Hartle 12.16 Radial redshift from collapsing star
10. Hartle 12.18 Horizon inside a collapsing shell
11. Hartle 12.22 Two observers and a black hole

Boccio Extra Problems

1. **Kruskal Coordinates**

   Consider the Schwarzschild metric, which in (t, r, θ, φ) coordinates is

   $$ds^2 = -\left(1 - \frac{2M}{r}\right) c^2 dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

   (a) Show that if we define

   $$u' = \left(\frac{r}{2M} - 1\right)^{1/2} e^{(r+t)/4M} \quad , \quad v' = -\left(\frac{r}{2M} - 1\right)^{1/2} e^{(r-t)/4M}$$

the metric in u', v', θ, φ) coordinates (Kruskal coordinates) is

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} du'dv' + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

(b) Find the locations in the (u,v) plane where this metric has singularities.

(c) What are the possible u,v values for events that can send signals to an event at $(u = u_0, v = v_0)$?

(d) What are the possible u,v values for events that can receive signals to an event at $(u = u_0, v = v_0)$?

(e) Consider a timelike observer in a cicular orbit at $r = 6M$. How is this described in Kruskal coordinates?

(f) What part of the spacetime cannot send signals to this observer? What part of the spacetime cannot receive signals from this observer?

2. **Null Geodesics in Strange Metric**

Consider the metric

$$ds^2 = -dt^2 + (1 - \lambda r^2)dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

where λ is a positive constant. Consider the null geodesics, and choosing coordinates so that the geodesics lie in the plane $\theta = \pi/2$, show that they satisfy

$$\left(\frac{dr}{d\varphi}\right)^2 = r^2(1 - \lambda r^2)\left(\mu r^2 - 1\right)$$

where μ is a constant. Integrate this and show that the paths of light rays are ellipses.

3. **Do Not Touch Anything!**

An astronaut in command of a spaceship equipped with a powerful rocket motor enters the horizon $r = r_s$ of a Schwarzschild black hole.

(a) Prove that in proper time no larger than $r_s\pi/2$, the astronaut reaches the singularity at $r = 0$.

(b) Prove that in order to avoid the singularity for as long as possible, the astronaut ought to move in a purely radial direction. HINT: For purely radial motion, with $dr < 0$ and $dt = d\varphi = d\theta = 0$, show that the increment in proper time is

$$d\tau = -\frac{dr}{\sqrt{\frac{r_s}{r} - 1}} \quad , \quad for\, r \leq r_s$$

and then integrate this between $r = r_s$ and $r = 0$ to obtain

$$\Delta\tau = \frac{\pi r_s}{2}$$

Finally, check that if dt, $d\varphi$, $d\theta$ are different from zero, then the increment $d\tau$, for a given value of $-dr$, is necessarily smaller than the value given above.

(c) Show that in order to achieve the longest proper time the astronaut must use her rocket motor in the following way: outside the horizon, she must brake her fall so as to arrive at $r = r_s$ with nearly zero radial velocity; inside the horizon she must shut off her motor and fall freely. HINT: show that $\Delta\tau = \pi r_s/2$ corresponds to free fall from $r = r_s$ (do not do anything!).

4. **Escape from Black Hole by Ejecting Mass**

A spaceship whose mission is to study the environment around black holes is hovering at the Schwarzschild radius coordinate R outside a spherical black hole of mass M. To escape back to infinity, the crew must eject part of the rest mass of the ship to propel the remaining fraction to escape velocity. What is the largest fraction f of the rest mass that can escape to infinity? What happens to this fraction as R approaches the Schwarzschild radius of the black hole?