Physics 130 General Relativity Seminar

Assignment 6 February 25, 2013

General topic: **Curved Spacetimes in General Relativity**

Part 1: Readings

Hartle: Ch 10 - Solar System Tests

Hartle: Ch 11 - Relativistic Gravity

Part 2: Problems Hartle Problems

1. Hartle 10.07 Solar oblateness and precession of the perihelion
2. Hartle 10.09 Solar corona and bending of light
3. Hartle 10.10 Real data analysis
4. Hartle 11.02 Odd number of gravitational lens images
5. Hartle 11.04 Path length difference
6. Hartle 11.06 A microlensing event
7. Hartle 11.07 A lensing event

Boccio Extra Problems

1. **Weak Gravity**

In weak gravity, the metric of a mass M at rest at the origin is

$$ds^2 = -(1+2\varphi)dt^2 + (1-2\alpha\varphi)\delta_{ij}dx^i dx^j$$

where α is a constant and $\varphi = -GM/r$.

(a) What is the value of α in general relativity?

(b) Instead of sitting at rest at the origin, the mass M moves in the $+x-$direction with speed v, passing through the origin at time $t = 0$, so that its position as a function of time is $x = vt$. What is the metric in this case?

(c) A photon moves along a trajectory originally in the $+y-$direction with offset b behind the $y-$axis, so that its undeflected trajectory is $x_0 = -b\hat{x} + t\hat{y}$. By what angle is the path of this test particle deflected?

(d) What is change in energy of deflected photon in part (c).

2. **Star with Constant Density**

   The metric of a star with constant density is

$$ds^2 = -\left(1 - \frac{2\mathrm{M}(r)}{r}\right) c^2 dt^2 + \left(1 - \frac{2\mathrm{M}(r)}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

   where

$$M(r) = \begin{cases} M(r/R)^3 & 0 < r < R \\ M & R < r \end{cases}$$

   is the mass interior to radius r, M is the total mass of the star, and R is the coordinate radius of the surface of the star. Assume $R > 2M$. We consider the orbits of photons where $g_{\mu\nu} u^\mu u^\nu = 0$.

   (a) Are there any singularities (coordinate or otherwise) of the metric?

   (b) Write the timelike and spacelike Killing vectors for this spacetime. There are actually two spacelike Killing vectors, but we will only need one since the photon orbits are planar. You may set $\theta = \pi/2$. Write out the associated conserved quantities.

   (c) Derive an expression for $dr/d\lambda$ where λ is the affine parameter. Put your expression in the form

$$\frac{1}{b^2} = \frac{1}{l^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r)$$

   and define b in terms of the constants of motion and W_{eff}.

   (d) Sketch W_{eff} and describe the photon orbits. How do these differ from the photon orbits in the standard Schwarzschild geometry?

   (e) Calculate the coordinate time t for a photon to travel from the center of the star at $r = 0$ to the surface at $r = R$.

   (f) Assume $R >> M$ and find the approximate delay, i.e., the extra time relative to the result from special relativity ($t = R$) to leading order. What is the value for the Sun where $M = 1.5\,km$ and $R = 7.0 \times 10^3\,km$.

3. **In the Schwarzschild Geometry**

Consider a spacetime described by the Schwarzschild line element:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

(a) A clock at fixed (r, θ, φ) measures an (infinitesimal) proper time interval, which we denote by dT. Express dT (as a function of r) in terms of the coordinate time interval dt.

(b) A stationary observer at fixed (t, θ, φ) measures an (infinitesimal) radial distance, which we denote by dR. Express dR (as a function of r) in terms of the coordinate radial distance dr.

(c) Consider the geodesic equations for free particle motion in the Schwarzschild geometry. Write out explicitly the equation corresponding to the time component. The equations corresponding to the space components will not be required. The resulting equation can be used to determine $dt/d\tau$ (where τ is the proper time and $t = x^0/c$ is the coordinate time). In particular, show that the quantity

$$k = \left(1 - \frac{2GM}{c^2 r}\right) \frac{dt}{d\tau}$$

is a constant independent of τ. Using the time component of the geodesic equation obtained earlier, compute the values of $\Gamma^0_{\alpha\beta}$ for this geometry. Consider all possible choices of α and β.

(d) Consider a particle falling radially into the center of the Schwarzschild metric, i.e., falling in radially towards $r = 0$. Assume that the particle initially starts from rest infinitely far away from $r = 0$. Since this is force-free motion, the particle follows a geodesic. Using the results of part (c), evaluate the constant k and thereby obtain a unique expression for $dt/d\tau$ that is valid at all points along the radial geodesic path. HINT: What is the value of $dt/d\tau$ at $r \to \infty$ (where the initial velocity of the particle is zero)?

(e) Since $ds^2 = -c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$ it follows that

$$g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -c^2$$

In this problem, $g_{\mu\nu}$ is determined from the Schwarzschild line element. Using these results and the result obtained in part (d) for $dt/d\tau$, compute the particles inward coordinate velocity, $v = dr/dt$, as a function of the coordinate radial distance r. Invert the equation, and integrate from $r = r_0$ to $r = r_s$, where r_0 is some finite coordinate distance such that $r_0 > r_s$ and $r_s = 2GM/c^2$ is the Schwarzschild radius. Show that the elapsed coordinate time is infinite, independent of the choice of the starting radial coordinate r_0, i.e., it takes an infinite coordinate time to reach the Schwarzschild radius. HINT: For radial motion, θ and φ are constant independent of τ. Note that for inward radial motion $dt/d\tau$ is negative.

(f) Compute the velocity dR/dT as measured by a stationary observer at a coordinate radial distance r. Verify that $|dR/dT| \to c$ as $r \to r_s$. HINT: Use the result for dR and dT obtained in parts (a) and (b).

4. **Time delay to Jupiter**

The Solar System is accurately described by the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

where M is the mass of the Sun, t the time coordinate, r the radial coordinate, and θ and φ are polar angles.

A radio pulse is sent from the Earth, reflected off a satellite of Jupiter (the satellite is a point), and received on Earth. Jupiter is a distance r_2 from the Sun, the Earth is a distance r_1. Assume that Jupiter is on the other side of the Sun relative to the Earth. Let r_0 be the distance of closest approach of the radio pulse to the Sun. Calculate the gravitational delay in the round-trip time of the radio pulse as a function of r_0, to lowest order in G. Estimate very roughly the magnitude of the effect, given that

$$\begin{array}{l} \text{mass of Sun} \approx 2 \times 10^{33} gm \\ \text{radius of Sun} \approx 7 \times 10^{10} cm \\ \text{Sun} - \text{Earth distance} \approx 1.5 \times 10^{13} cm \\ \text{Sun} - \text{Jupiter distance} \approx 8 \times 10^{13} cm \\ G \approx 6.67 \times 10^{-8} cm^3/gm - \sec^2 \end{array}$$