Physics 130 General Relativity Seminar

Assignment 14 April 29, 2013

General topic: **The Einstein Equation**

Part 1: Readings

Hartle: Chap 18 - Sections 3-7 Cosmological Models

Hartle: Chap19 Which Universe and Why?

Hartle: Chap 18 Supplement - Derivation of Robertson-Walker Line Element

Part 2: Problems Hartle Problems

1. Hartle 18.10 One new galaxy
2. Hartle 18.13 Duration of universe
3. Hartle 18.15 Homogeneous, isotropic cosmological model
4. Hartle 18.18 De Sitter space
5. Hartle 18.19 Friedman equation and the big bang
6. Hartle 18.24 Einstein static universe
7. Hartle 18.28 FRW singularity theorem

Boccio Extra Problems

1. **Red Shift in Model Galaxy**

   Assume that the universe is isotropic and spatially flat. The metric then takes the form

   $$ds^2 = -dt^2 + a^2(t)\left(dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right)$$

   where r, θ, and φ are co-moving coordinates. By this is meant any galaxy will have constant values of r, θ, φ(peculiar motions of galaxies are neglected). The universe is assumed to be matter-dominated with matter density $\rho(t)$ at time t.

   (a) Under this circumstance show that the Einstein equations are

   $$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 \quad and \quad \ddot{a} = -\frac{4\pi G}{3}\rho a$$

(b) From the fact that light propagates along null geodesics, show that the cosmological red shift of spectral lines emitted at time t_e and received at time t_0, defined as

$$Z = \frac{wavelength\ of\ received\ line\ -\ wavelength\ of\ emitted\ line}{wavelength\ of\ emitted\ line}$$

is

$$Z = \frac{a_0}{a_e} - 1$$

where $a_0 = a(t_0)$, $a_e = a(t_e)$.

(c) In the cosmological model under discussion a given galaxy will decrease in angular size with increasing distance from the observer - up to a critical distance. Beyond this the angular size will increase with distance. What is the red shift Z_{crit} corresponding to the minimum in angular size?

2. **Expanding Universe**

The metric of the expanding universe has the form

$$ds^2 = dt^2 - R^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

where the possible curvature of space has been neglected. The detailed form of $R(t)$ depends on the matter content of the universe.

(a) A particle of mass m has energy E_0 and momentum p_0 at time t_0; assume $R(t_0) = R_0$. The particle thereafter propagates freely except for the effects of the above metric. Calculate the energy and momentum as a function of time.

(b) Suppose that the early universe contained a gas of non-interacting massless particle (perhaps photons) subject to gravitational effects only. Show that if at time t_0 they were in a thermal distribution at temperature T_0, they remained in a thermal distribution later, but with a temperature that depends on time in a fashion you should determine. HINT: EP60 shows that

$$photon\ frequencies\ change\ like: \tfrac{\nu'}{\nu} = \tfrac{R(t)}{R(t')}$$
$$volumes\ change\ like: \tfrac{V(t')}{V(t)} = \tfrac{R^3(t')}{R^3(t)}$$

(c) Show that, instead, a gas of non-interacting massive particles initially in a thermal distribution would not remain in a thermal distribution under the influence of the expansion of the universe.

(d) Suppose that the early universe contained a non-interacting gas of massless photons and also a non-interacting gas of massive particles of mass m (massive neutrinos to be definite). Suppose that at some early time the photons and neutrinos were both in a thermal distribution with a temperature $kT = mc^2$ (m being the neutrino mass) for both photons and neutrinos. It has been observed that in todays universe the photons are in a thermal distribution with kT about $3 \times 10^{-4}\, eV$. In terms of the neutrino mass, what (roughly) would be the typical velocity and kinetic energy of a neutrino today? Assume $m >> 3 \times 10^{-4}\, eV$.

3. **Flat Universe with Period of Inflation**

Consider a simplified model of the history of a flat universe involving a period of inflation. The history is split into four periods

(a) $0 < t < t_3$ radiation only

(b) $t_3 < t < t_2$ vacuum energy dominates with an effective cosmological

constant $\Lambda = 3/(4t_3^2)$

(c) $t_2 < t < t_1$ a period of radiation dominance

(d) $t_1 < t < t_0$ matter domination

(a) Show that in (3) $\rho(t) = \rho_r(t) = 3/(32\pi t^2)$ and in (4) $\rho(t) = \rho_m(t) = 1/(6\pi t^2)$. The functions ρ_r and ρ_m are introduced for later convenience.

(b) Give simple analytic formulas for $a(t)$ which are approximately true in the four epochs.

(c) Show that during the inflationary epoch the universe expands by a factor

$$\frac{a(t_2)}{a(t_3)} = \exp\left(\frac{t_2 - t_3}{2t_3}\right)$$

(d) Show that

$$\frac{\rho_r(t_0)}{\rho_m(t_0)} = \frac{9}{16}\left(\frac{t_1}{t_0}\right)^{2/3}$$

(e) If $t_3 = 10^{-35}$ seconds, $t_2 = 10^{-32}$ seconds, $t_1 = 10^4$ years and $t_0 = 10^{10}$ years, give a sketch of $\log(a)$ versus $\log(t)$ marking any important epochs.

(f) Define what is meant by the particle horizon and calculate how it behaves for this model. Indicate this behavior on the sketch you made. How does inflation solve the horizon problem?