Supplement to Chapter 8: Derivation of the General Geodesic Equation

In this supplement we work through the algebra of showing how Lagrange's equations for timelike geodesics (8.9) with the Lagrangian (8.10) can be cast in the form of the the general geodesic equation (8.14) and derive the relation defining the Christoffel symbols (8.19).

Writing out Lagrange's equations (8.9) using (8.10) for the Lagrangian one finds

$$
\begin{aligned}
& - \frac{d}{d\sigma}\left[\left(-g_{\gamma\delta}\frac{dx^{\gamma}}{d\sigma}\,\frac{dx^{\delta}}{d\sigma}\right)^{-\frac{1}{2}}\left(g_{\alpha\beta}\frac{dx^{\beta}}{d\sigma}\right)\right] \\
& + \frac{1}{2}\left(-g_{\gamma\delta}\frac{dx^{\gamma}}{d\sigma}\,\frac{dx^{\delta}}{d\sigma}\right)^{-\frac{1}{2}}\frac{\partial g_{\varepsilon\beta}}{\partial x^{\alpha}}\,\frac{dx^{\varepsilon}}{d\sigma}\,\frac{dx^{\beta}}{d\sigma} = 0\ . \qquad (1)
\end{aligned}
$$

(Note the care we have taken with the indices in this equation. The only free index is α. The rest are all summation indices so the expression could be written with any indices replacing these provided they don't duplicate ones used elsewhere in the equation.) This expression can be considerably simplified by noting from (8.8) that

$$\left(-\,g_{\alpha\beta}\frac{dx^{\alpha}}{d\sigma}\,\frac{dx^{\beta}}{d\sigma}\right)^{\frac{1}{2}} = \frac{d\tau}{d\sigma}\ . \qquad (2)$$

Multiplying (1) by $d\sigma/d\tau$ and using (2) one finds

$$-\frac{d}{d\tau}\left[g_{\alpha\beta}(x)\frac{dx^{\beta}}{d\tau}\right] + \frac{1}{2}\,\frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\,\frac{dx^{\beta}}{d\tau}\,\frac{dx^{\gamma}}{d\tau} = 0\ . \qquad (3)$$

Now,

$$\frac{dg_{\alpha\beta}}{d\tau} = \frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}\,\frac{dx^{\gamma}}{d\tau}\ , \qquad (4)$$

so that (3) may be written

$$g_{\alpha\beta}\frac{d^2x^{\beta}}{d\tau^2} + \left(-\,\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{1}{2}\,\frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right)\frac{dx^{\beta}}{d\tau}\,\frac{dx^{\gamma}}{d\tau} = 0\ . \qquad (5)$$

This equation can also be written

$$g_{\alpha\delta}\frac{d^2x^\delta}{d\tau^2} = -\frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right)\frac{dx^\beta}{d\tau}\ \frac{dx^\gamma}{d\tau} = 0\ . \tag{6}$$

In the first term the dummy summation index β in (5) was changed to δ. The first two terms on the right hand side contribute identically because $(dx^\beta/d\tau)(dx^\gamma/d\tau)$ is symmetric in β and γ. Just by changing dummy indices we have

$$\frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = \frac{\partial g_{\alpha\gamma}}{\partial x^\beta}\frac{dx^\gamma}{d\tau}\frac{dx^\beta}{d\tau} \tag{7}$$

To identify the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ when the geodesic equation is written in the form (8.14)

$$\frac{d^2x^\delta}{d\tau^2} = -\Gamma^\delta_{\beta\gamma}\ \frac{dx^\beta}{d\tau}\ \frac{dx^\gamma}{d\tau}\ , \tag{8}$$

multiply this equation by $g_{\alpha\delta}$. We arrive at the form (6) provided

$$g_{\alpha\delta}\Gamma^\delta_{\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right)\ . \tag{9}$$

Thus we confirm the form of the geodesic equation (5) and arrive at general relation (8.19) for the Christoffel symbols.

Later in Chapter 20 we will introduce the *inverse metric* $g^{\alpha\beta}$. This is defined as the matrix inverse of $g_{\alpha\beta}$ and so satisfies

$$g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma\ . \tag{10}$$

where δ^α_γ (the Kronecker δ or unit matrix) is 1 when $\alpha = \gamma$ and zero otherwise. Multiplying both sides of (6) by $g^{\epsilon\alpha}$ and using (10) leads directly to the geodesic equation in the form (8). Multiplying both sides of (9) by $g^{\epsilon\alpha}$ and using (10) we find an explicit formula for the Christoffel symbols:

$$\Gamma^\epsilon_{\beta\gamma} = \frac{1}{2}g^{\epsilon\alpha}\left(\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right)\ . \tag{11}$$