Supplment to Chapter 24:
Energy Levels of a Free Particle in a Box

Section 24.1's derivation of the equation of state of a gas of free, spin-1/2 fermions assumed some elementary and standard facts about the energy levels of single quantum mechanical particle confined to a box. For completeness, we review those facts here, although they can be found in any standard quantum mechanics text.

We consider a single particle of mass m moving freely in one-dimension (x) and confined to a box which extends from $x = 0$ to $x = \mathcal{L}$. The quantum state of such a particle is described by a wave function $\Psi(x)$. The Schrödinger equation for the allowed values of the energy E is

$$\hat{H}\Psi = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} = E\Psi \tag{1}$$

where a hat denotes an operator. As a convenience we define p by

$$E \equiv \frac{p^2}{2m} \ . \tag{2}$$

The quantity p can be thought of as the magnitude of the momentum. Then (2) can be written in the form

$$\hat{p}^2\Psi(x) = -\hbar^2\frac{d^2\Psi(x)}{dx^2} = p^2\Psi(x) \tag{3}$$

The most general solution of (3) is

$$\Psi(x) = A\sin(px/\hbar) + B\cos(px/\hbar) \tag{4}$$

where A and B are constants. If the particle is confined to the box then the wave function must vanish outside it. Continuity of Ψ at the walls at $x = 0$ and $x = \mathcal{L}$ implies the boundary conditions:

$$\Psi(0) = \Psi(\mathcal{L}) = 0 \ . \tag{5}$$

These require $B = 0$ in (4) and the discrete values of p

$$p_k \equiv \frac{k\pi\hbar}{\mathcal{L}} \ , \qquad k = 1, 2, \cdots \ . \tag{6}$$

(The value $k = 0$ corresponds to $\Psi = 0$ everywhere.) This is (24.2). The corresponding discrete energy levels are given by (2) as

$$E_k = \frac{1}{2m}\left(\frac{k\pi\hbar}{\mathcal{L}}\right)^2 \equiv \frac{p_k^2}{2m} , \qquad k = 1, 2, \cdots \tag{7}$$

This is (24.1).

The energy levels of a *free* relativistic fermion can be understood in much the same way. *Interacting* relativistic particles can be created and destroyed — a process which is most efficiently described in terms of quantum field theory. But it is possible to think of a *free* relativistic particle like a non-relativistic one with the Hamiltonian

$$\hat{H} = [(mc^2)^2 + (\hat{p}c)^2]^{1/2} . \tag{8}$$

Here, $\hat{p} = -i\hbar(d/dx)$ is the usual momentum operator. (Momentum is the infinitesimal generator of displacements in x and a displacement is a displacement no matter what the kinematics.) If you are worried about what the square root of an operator means, think of specifying $\hat{H}$ by its matrix elements in a basis of definite momentum states. The content of (8) is that the diagonal elements are $[(mc^2)^2 + (pc)^2]^{1/2}$ Once specified in one basis the operator is defined in all.

The Schrodinger equation $\hat{H}\Psi = E\Psi$ for the energy eigenvalues E leads to

$$\hat{H}^2\Psi = [(mc^2)^2 + (\hat{p}c)^2]\Psi = E^2\Psi . \tag{9}$$

But if we define a quantity p by

$$E \equiv [(mc^2)^2 + (pc)^2]^{1/2} , \tag{10}$$

then (9) becomes

$$\hat{p}^2\Psi(x) = -\hbar^2\frac{d^2\Psi(x)}{dx^2} = p^2\Psi(x) . \tag{11}$$

This is just the same as (3) and all of its consequences follow in particular (6). The allowed energy levels for a relativistic particle in a box are given by

$$E_k = [(mc^2)^2 + (p_kc)^2]^{1/2} \quad k = 1, 2\cdots . \tag{12}$$