Supplement to Chapter 18: Derivation of the Robertson-Walker Line Element

This supplement shows how coordinates can be chosen so that the line element of a homogeneous, isotropic spacetime has the general Robertson-Walker form (18.4).

As discussed in Section 18.1, a spacetime is homogeneous and isotropic when each of its points lies on some member of a family of spacelike three-surfaces whose *three-dimensional* geometry is homogeneous and isotropic. Robertson-Walker coordinates employ one coordinate t to label the surfaces and three coordinates x^i to label points in them. They thus constitute a particular way of dividing spacetime up into space and time. (See Section 7.9.) Begin their construction by picking one of these surfaces and using choosing a coordinate system x^i to label points it. Consider the particle of the cosmological fluid at x^i on the surface. Use the same three coordinates to label the location of that particle on all other surfaces, ie for all time. The coordinates x^i are thus *comoving coordinates* in which the four-velocity of a particle of the cosmological fluid satisfies

$$u^i \equiv \frac{dx^i}{d\tau} = 0\ . \tag{1}$$

Consider another surface of homogeneity to the future of the first. Suppose one observer riding on a particle of the cosmological fluid reaches the second surface after a proper time t. All other observers starting from other spatial positions must reach the second surface in the same proper time. Different times for observers at different locations would violate the assumption of homogeneity. The proper time t can be used as the coordinate that labels the spacelike surfaces because it is constant on each of them. In the coordinates (t, x^i) the line element takes the form

$$ds^2 = -dt^2 + 2g_{it}(t,x^k)dx^i dt + g_{ij}(t,x^k)dx^i dx^j\ . \tag{2}$$

The only departure from full generality is that $g_{tt} = -1$. It has to have this value so that t coincides with proper time for comoving ($x^i = \text{const.}$) observers.

Isotropy implies that $g_{it} = 0$. That is because the four velocity of the cosmological fluid $\mathbf{u}$ must be orthogonal (normal) to the surfaces of homogeneity. Were is not, its component in the surface would single out some direction violating the assumption of isotropy. Explicitly [cf. (7.68)] $\mathbf{u}\cdot\mathbf{t} = 0$ for any tangent vector to the surface, that is vectors of the form $t^\alpha = (0,\ t^1,\ t^2,\ t^3)$. But u^α has only a time component because the coordinates x^i are comoving [cf. (1)]. Thus

$$\mathbf{u}\cdot\mathbf{t} = g_{ti}u^t t^i = 0 \tag{3}$$

which implies that $g_{it} = 0$ since (3) holds for all t^i.

Choosing coordinates in this way the line element of a homogeneous, isotropic spacetime takes the form

$$ds^2 = -dt^2 + dS^2(t) \tag{4}$$

where $dS^2(t)$ defines the geometry of the surface of homogeneity labeled by t

$$dS^2(t) = g_{ij}(t, x^k)dx^i dx^j \ . \tag{5}$$

To study how these geometries can change in time and preserve homogeneity and isotopy, fix one time t_r to define a reference spatial geometry and denote the metric of that spatial geomtry by $\gamma_{ij}(x^k) \equiv g_{ij}(t_r, x^k)$. The squared distance between a particle of the cosmological fluid at x^k and a nearby particle at $x^k + \Delta x^k$ in the reference geometry is

$$\Delta S_r^2 = \gamma_{ij}(x^k)\Delta x^i \Delta x^j \ . \tag{6}$$

The ratio of this squared distance to the squared distance between the same pair of particles at different time t must be independent of x^k. Different ratios at different locations would violate the assumption of homogeneity. Thus,

$$\Delta S^2(t) = a^2(t)\Delta S_r^2 \tag{7}$$

for some function of time alone which we have denoted by $a^2(t)$. This is the statement that the evolution in time is uniform in space. Further, this relation must hold for any nearby pair of particles we may choose, that is for any small values of Δx^i. It must therefore hold term by term in the sums over Δx^i on both sides of (7). That is, it must hold for each metric component separately. Using (5) and (6) in (7) we find

$$g_{ij}(t, x^k) = a^2(t)\gamma_{ij}(x^k) \tag{8}$$

Thus we obtain for the line element of a homogeneous, isotropic spacetime

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}(x^k)dx^i dx^j \tag{9}$$

where $\gamma_{ij}(x^k)$ defines a time independent homogeneous and isotropic spatial geometry. This is the Robertson-Walker line element (18.4) with

$$d\mathcal{L}^2 = \gamma_{ij}(x^k)dx^i dx^j \ . \tag{10}$$