

Static spherically symmetric solution for the field of a charged particle in a theory of gravity

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The field equations for a new theory of gravity are derived from a variational principle when electromagnetic fields are included in the Lagrangian density \mathcal{L} . A static spherically symmetric solution to the field equations is obtained, which reduces to the Reissner-Nordström solution when a new gravitational coupling constant l equals zero. For $l > m + (m^2 - 4\pi Q^2)^{1/2}$ black-hole event horizons are excluded from physical space-time.

I. INTRODUCTION

A new theory of gravitation¹ has been formulated on the basis of a nonsymmetrical Hermitian $g_{\mu\nu}$ and $\Gamma_{\mu\nu}^\lambda$.

In the following, we shall study the consequences of the new theory when electromagnetic fields are present and derive the static spherically symmetric solution for the field of a charged particle. One important result that emerges from the solution is that if

$$l > m + (m^2 - 4\pi Q^2)^{1/2},$$

where Q is the electric charge on the particle, the charged black-hole event horizons of the type predicted in Einstein-Maxwell theory are excluded from physical space-time. Moreover, test particles are prevented from reaching the essential singularity at $r=0$. These results generalize those obtained in Ref. 1.

II. THE VARIATIONAL PRINCIPLE AND THE FIELD EQUATIONS

The Lagrangian density which includes contributions from the electromagnetic field $F_{\mu\nu}$ is chosen to be¹

$$\mathcal{L} = \sqrt{-g} [g^{\mu\nu} R_{\mu\nu}(W) + 4\pi F^{\mu\nu} F_{\mu\nu}], \quad (2.1)$$

where

$$\begin{aligned} F^{\mu\nu} &= g^{\alpha\mu} g^{\beta\nu} F_{\alpha\beta} = -F^{\nu\mu}, \\ F_{\mu\nu} &= A_{\nu,\mu} - A_{\mu,\nu}. \end{aligned} \quad (2.2)$$

The variational principle requires that

$$\delta \int \mathcal{L} d^4x = 0. \quad (2.3)$$

Consider first the variation of the second term in (2.1) with respect to $g^{\mu\nu}$ with $\delta g^{\mu\nu}$ vanishing at the boundaries of integration and A_μ being kept constant with respect to the variation. We have

$$\begin{aligned} \delta(F^{\mu\nu} F_{\mu\nu} \sqrt{-g}) &= -2E_{\beta\nu} \sqrt{-g} \delta g^{\beta\nu} \\ &= 2\sqrt{-g} \delta g^{\beta\nu} (-\frac{1}{4} g_{\beta\nu} F^{\sigma\tau} F_{\sigma\tau} \\ &\quad + g^{\alpha\mu} F_{\alpha\beta} F_{\mu\nu}), \end{aligned} \quad (2.4)$$

where

$$E_{\mu\nu} = -g^{\alpha\beta} F_{\nu\beta} F_{\mu\alpha} + \frac{1}{4} g_{\mu\nu} F^{\sigma\tau} F_{\sigma\tau} \quad (2.5)$$

is the Hermitian stress-energy tensor of the electromagnetic field.

Next consider the variations δA_μ , the $g_{\mu\nu}$ remaining constant,

$$\delta(F^{\mu\nu} F_{\mu\nu} \sqrt{-g}) = -4(\sqrt{-g} F^{\mu\nu})_{,\mu} \delta A_\nu + 4(\sqrt{-g} F^{\mu\nu} \delta A_\nu)_{,\mu}. \quad (2.6)$$

The variations with respect to $W_{\mu\nu}^\lambda$ and $g_{\mu\nu}$ were calculated in Ref. 1. The resulting field equations are

$$g_{\mu+\nu-;\sigma} = 0, \quad (2.7)$$

$$g^{[\mu\nu]}_{,\nu} = 0, \quad (2.8)$$

$$R_{\mu\nu}(W) - \frac{1}{2} g_{\mu\nu} R(W) = 8\pi E_{\mu\nu}, \quad (2.9)$$

where $\tilde{x} = \sqrt{-g} X$ and

$$\mathfrak{F}^{\mu\nu}_{,\nu} = 0. \quad (2.10)$$

Equation (2.2) yields an additional relation

$$F_{\mu\nu,\sigma} + F_{\nu\sigma,\mu} + F_{\sigma\mu,\nu} = 0. \quad (2.11)$$

The equations

$$R_{\mu\nu}(W) = R_{\mu\nu}(\Gamma) + \frac{2}{3}(W_{\mu,\nu} - W_{\nu,\mu}) \quad (2.12)$$

and

$$E = g^{\mu\nu} E_{\mu\nu} = 0 \quad (2.13)$$

allow (2.9) to be written as

$$R_{(\mu\nu)}(\Gamma) = 8\pi E_{(\mu\nu)}, \quad (2.14)$$

$$R_{[\mu\nu],\sigma}(\Gamma) = 8\pi E_{[\mu\nu],\sigma}. \quad (2.15)$$

The equation of motion of a charged test particle takes the form

$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = \frac{q}{\mu} g^{(\mu\beta)} F_{\nu\beta} \frac{dx^\nu}{ds}, \quad (2.16)$$

where μ and q are the vanishing small mass and charge, respectively, of the particle.

III. THE COVARIANT FORM OF MAXWELL'S EQUATIONS

When current density sources are included, Maxwell's divergence equation (2.10) becomes

$$\mathfrak{F}^{\mu\nu}{}_{;\nu} = \mathfrak{J}^\mu, \quad (3.1)$$

where \mathfrak{J}^μ is the electric current-density four-vector. If we define

$$F^{[\mu+\nu-]}{}_{;\sigma} = \frac{1}{2}(F^{\mu+\nu-}{}_{;\sigma} - F^{\nu+\mu-}{}_{;\sigma}), \quad (3.2)$$

then

$$F^{[\mu+\sigma-]}{}_{;\sigma} = F^{\mu\sigma}{}_{;\sigma} + F^{\mu\sigma} \frac{(\sqrt{-g})_{,\sigma}}{\sqrt{-g}}, \quad (3.3)$$

so (3.1) is equivalent to

$$F^{[\mu+\nu-]}{}_{;\nu} = J^\mu. \quad (3.4)$$

The covariant form of the other set of Maxwell's equations (2.11) is

$$F_{\mu+\nu-;\sigma} + F_{\nu-\sigma-;\mu} + F_{\sigma+\mu+;\nu} = 0. \quad (3.5)$$

IV. STATIC SPHERICALLY SYMMETRIC SOLUTION

In spherical polar coordinates $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, $x^4 = t$, the form of $g_{\mu\nu}$ is²

$$g_{\mu\nu} = \begin{pmatrix} -\alpha & 0 & 0 & w \\ 0 & -r^2 & f \sin \theta & 0 \\ 0 & -f \sin \theta & -r^2 \sin^2 \theta & 0 \\ -w & 0 & 0 & \gamma \end{pmatrix}. \quad (4.1)$$

Since we choose the boundary conditions

$$g_{(\mu\nu)} - \eta_{\mu\nu}, \quad g_{[\mu\nu]} \rightarrow 0 \quad \text{as } r \rightarrow \infty, \quad (4.2)$$

where

$$\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$$

is the Minkowski metric tensor, we have $g_{[23]} \equiv 0$ everywhere.¹ The contravariant $g^{\mu\nu}$ takes the form

$$g^{\mu\nu} = \begin{pmatrix} -\frac{\gamma}{\alpha\gamma - w^2} & 0 & 0 & \frac{-w}{\alpha\gamma - w^2} \\ 0 & -\frac{1}{r^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2 \sin^2 \theta} & 0 \\ \frac{w}{\alpha\gamma - w^2} & 0 & 0 & \frac{\alpha}{\alpha\gamma - w^2} \end{pmatrix}. \quad (4.3)$$

Equation (2.8) yields the solution

$$w^2 = \alpha\gamma K, \quad (4.4)$$

where

$$K = \frac{k^2}{k^2 + r^4} \quad (4.5)$$

and $k = i l^2$ is a purely imaginary constant of integration. A calculation of the $R_{\mu\nu}(\Gamma)$ has been presented before.² Since the tensor $R_{[\mu\nu]}$ has only one nonvanishing component $R_{[14]}$ and $g_{[23]} = 0$, Eq. (2.15) is identically fulfilled.

The electric field is static so that

$$\begin{aligned} A_1 = A_2 = A_3 = 0, \\ F_{14} = -F_{41} = A'_4. \end{aligned} \quad (4.6)$$

With $\alpha = e^\lambda$ and $\gamma = e^\nu$ (a prime denotes differentiation with respect to r),

$$\begin{aligned} F^{41} = (e^{\lambda+\nu} - w^2)^{-1} A'_4, \\ \mathfrak{F}^{41} = \sqrt{-g} F^{41} = (e^{\lambda+\nu} - w^2)^{-1/2} r^2 \sin \theta A'_4. \end{aligned} \quad (4.7)$$

Equation (2.10) then yields the result

$$\mathfrak{F}^{41}{}_{;1} = \sin \theta [(e^{\lambda+\nu} - w^2)^{-1/2} r^2 A'_4]' = 0, \quad (4.8)$$

which gives upon integration

$$A'_4 = \frac{Q}{r^2} (e^{\lambda+\nu} - w^2)^{1/2}, \quad (4.9)$$

where Q is a constant of integration.

We must now solve the field equation (2.14).

Since

$$\begin{aligned} E_{11} = -\frac{1}{2} e^\lambda \frac{Q^2}{r^4}, \quad E_{22} = \frac{1}{2} r^2 \frac{Q^2}{r^4}, \\ E_{33} = \frac{1}{2} r^2 \sin^2 \theta \frac{Q^2}{r^4}, \quad E_{44} = \frac{1}{2} e^\nu \frac{Q^2}{r^4}, \end{aligned} \quad (4.10)$$

we get

$$e^{-\lambda} R_{11}(\Gamma) + e^{-\nu} R_{44}(\Gamma) = \frac{\lambda' + \nu'}{r} + \frac{4K}{r^2} = 0, \quad (4.11)$$

where K is given by (4.5). The $R_{22}(\Gamma)$ equation is

$$-\frac{\nu'}{r} e^{-\lambda} + \frac{\lambda'}{r} e^{-\lambda} - \frac{2}{r^2} (e^{-\lambda} - 1) - \frac{4K}{r^2} e^{-\lambda} = 8\pi \frac{Q^2}{r^4}. \quad (4.12)$$

Multiplying (4.11) by $e^{-\lambda}$ and substituting the resulting expression for $-(\nu'/r)e^{-\lambda}$ into (4.12) we find

$$e^{-\lambda} \left(\frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} = 4\pi \frac{Q^2}{r^4}. \quad (4.13)$$

This can be integrated to give

$$\alpha = \left(1 - \frac{2m}{r} + \frac{4\pi Q^2}{r^2} \right)^{-1}. \quad (4.14)$$

By virtue of (4.5), Eq. (4.11) can be written in the form

$$\left[\lambda + \nu + \ln \left(\frac{r^4}{r^4 + k^2} \right) \right]' = 0. \quad (4.15)$$

Hence

$$\lambda + \nu + \ln \left(\frac{r^4}{r^4 + k^2} \right) = \text{const} \quad (4.16)$$

This gives, since $k^2 = -l^4$, the result

$$\alpha \gamma \frac{r^4}{r^4 - l^4} = C^2, \quad (4.17)$$

where C^2 is a constant of integration. Using Eq. (4.4) we get

$$w^2 = \frac{C^2 k^2}{r^4}. \quad (4.18)$$

For $r \rightarrow \infty$ we have $w \rightarrow 0$ and $g_{(uv)} = \eta_{uv}$. Thus from (4.17) we must have $C^2 = 1$ and (4.17) and (4.18) give

$$\gamma = \left(1 - \frac{l^4}{r^4} \right) \left(1 - \frac{2m}{r} + \frac{4\pi Q^2}{r^2} \right), \quad (4.19)$$

$$w = \pm \frac{il^2}{r^2}.$$

V. PHYSICAL INTERPRETATION OF THE SOLUTION

From Eqs. (4.14) and (4.19) we obtain the metric exterior to a charged particle

$$ds^2 = \left(1 - \frac{l^4}{r^4} \right) \left(1 - \frac{2m}{r} + \frac{4\pi Q^2}{r^2} \right) dt^2 - \left(1 - \frac{2m}{r} + \frac{4\pi Q^2}{r^2} \right)^{-1} dr^2 - r^2 d\Omega^2. \quad (5.1)$$

Moreover, from (4.9), (4.14), and (4.19) we find that

$$F_{14} = A'_4 = \frac{Q}{r^2}. \quad (5.2)$$

This justifies our identifying Q with the electric charge on the particle.

The metric (5.1) has three null surfaces

$$r_1 = l, \quad r_2 = m + (m^2 - 4\pi Q^2)^{1/2},$$

$$r_3 = m - (m^2 - 4\pi Q^2)^{1/2},$$

where the latter two null surfaces occur only when $m^2 > 4\pi Q^2$. When $r < l$, we see that $ds^2 < 0$ and the radius $r_1 = l$ defines a sphere inside which space is Euclidean four-dimensional. This sphere excludes the essential singularity at $r = 0$ from physical space-time, as in the case of the vacuum solution for an uncharged particle with $Q = 0$.¹

When $l < r_2$, the line element (5.1) can form black-hole event horizons at $r = r_2$ and $r = r_3$, similar to those that occur in the Reissner-Nordström solution,^{3,4} when $m^2 > 4\pi Q^2$. However, when $l > r_2$, the standard black-hole event horizons are *excluded* from physical space-time, that space-time for which the metric is locally Minkowskian and the proper time along the path of a test particle remains real. Test particles moving along paths determined by the equation of motion (2.16) will be deflected for $r > l$, so no particle can reach $r = 0$, as in the uncharged case.¹

In the present work there has been no attempt made to unify the gravitational and electromagnetic fields. In contrast to previous work,⁵ the total gravitational field is described by the non-symmetric metric g_{uv} , while the electromagnetic field F_{uv} is treated as a separate field in the same way as in the Einstein-Maxwell theory.

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¹J. W. Moffat, this issue, Phys. Rev. D **19**, 3554 (1979).

²A. Papapetrou, Proc. R. Ir. Acad., Sect. A **52**, 69 (1948).

³H. Reissner, Ann. Phys. (Leipzig) **50**, 106 (1916); G. Nordström, K. Ned. Akad. Wet. Versl. Gewone

Vergad. Afd. Natuurkd. **20**, 1238 (1918).

⁴C.f. S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge Univ. Press, London, 1973).

⁵J. W. Moffat, Phys. Rev. D **15**, 3520 (1977).