PHYSICAL REVIEW D VOLUME 19, NUMBER 12 15 JUNE 1979

Cosmological solution of Bianchi type I in a new theory of gravitation

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(Received 7 July 1978)

We present a homogeneous, plane-symmetric, matter-free solution to a new theory of gravitation. In the limit of large t, the solution goes over into the plane-symmetric Kasner metric of general relativity.

I. INTRODUCTION

Evidence for the present large-scale homogeneity and isotropy of the universe comes from several sources,[1] yet these observations do not rule out cosmologies anisotropic in the early stages of the universe.[2] Such anisotropy as is introduced at the beginning of time tends to disappear[3] as matter terms become important, and the metric reduces to an isotropic one at sufficiently large t.

Isotropic and anisotropic models based on Einstein's equations share one common feature: at $t=0$, the curvature of space-time becomes infinite and the density of matter also goes to infinity. Arguments seeking to avoid this singularity usually invoke quantum effects, but no clear picture of how such effects are supposed to work has yet been given. Thus it would be of considerable interest to find a model with different analytic properties.

We present a homogeneous anisotropic nonstatic solution of the Einstein-Straus[4] field equations, interpreted as a theory of pure gravitation,[5] and discuss the motion of test particles in the metric and certain analytic properties of the solution.

II. THE SOLUTION

The theory for this cosmological model and the notation used herein are described in Ref. 5.

The field equations are

$$g_{\mu\nu,\alpha} - g_{\mu\sigma}\Gamma^{\sigma}_{\alpha\nu} - g_{\sigma\nu}\Gamma^{\sigma}_{\mu\alpha} = 0\,, \tag{2.1}$$

$$\Gamma^{\sigma}_{[\mu\sigma]} = 0\,, \tag{2.2}$$

$$R_{\mu\nu}(W) - \tfrac{1}{2}g_{\mu\nu}R(W) = 8\pi T_{\mu\nu}\,. \tag{2.3}$$

The $g_{\mu\nu}$ corresponding to a plane-symmetric homogeneous anisotropic nonstatic space-time in comoving Cartesian coordinates is

$$g_{\mu\nu} = \begin{bmatrix} -\alpha(t) & 0 & 0 & w(t) \\ 0 & -\beta(t) & f(t) & 0 \\ 0 & -f(t) & -\beta(t) & 0 \\ -w(t) & 0 & 0 & 1 \end{bmatrix}, \tag{2.4}$$

with line element

$$ds^2 = dt^2 - \alpha(dx^1)^2 - \beta[(dx^2)^2 + (dx^3)^2]\,. \tag{2.5}$$

Equation (2.1) can be solved to yield the affine connections

$$\begin{aligned}
&\Gamma^1_{(14)} = \tfrac{1}{2}[(w^2/\alpha)\dot\psi + \dot\alpha/\alpha], \quad \Gamma^2_{[34]} = -\Gamma^3_{[24]} = -\tfrac{1}{2}D\,,\\
&\Gamma^2_{[12]} = \Gamma^3_{[13]} = \tfrac{1}{2}w\dot A, \quad \Gamma^4_{11} = w^2\dot\psi + \tfrac{1}{2}\dot\alpha\,,\\
&\Gamma^2_{(13)} = -\Gamma^3_{(12)} = \tfrac{1}{2}wD, \quad \Gamma^4_{[14]} = -\tfrac{1}{2}w\dot\psi\,,\\
&\Gamma^2_{(24)} = \Gamma^3_{(34)} = \tfrac{1}{2}\dot A\,, \quad \Gamma^4_{22} = \Gamma^4_{33} = -\tfrac{1}{2}(fD - \beta\dot A), \quad \Gamma^4_{[23]} = -\tfrac{1}{2}(f\dot A + \beta D)\,,
\end{aligned} \tag{2.6}$$

where $A = \frac{1}{2}\ln(\beta^2 + f^2)$, $\psi = \ln(1 - \alpha/w^2)$, and $D = (\dot\beta f - \beta\dot f)/(\beta^2 + f^2)$, with $\dot\gamma = d\gamma/dt$. All the other $\Gamma^{\lambda}_{\mu\nu}$'s are zero.

We now restrict ourselves to empty space, so that

$$T_{\mu\nu} = 0\,, \tag{2.7}$$

and the field equations involving $R_{\mu\nu}(\Gamma)$ read

$$\begin{aligned}
&R_{11} = \frac{\partial}{\partial t}(w^2\dot\psi + \tfrac{1}{2}\dot\alpha) + (w^2\dot\psi + \tfrac{1}{2}\dot\alpha)[\dot A - (w^2/2\alpha)\dot\psi - \dot\alpha/2\alpha] + \tfrac{1}{2}w^2(\dot A^2 + D^2) + \tfrac{1}{4}w^2\dot\psi^2 = 0\,,\\
&R_{22} = R_{33} = -\frac{1}{2}\frac{\partial}{\partial t}(fD - \beta\dot A) - \tfrac{1}{4}(fD - \beta\dot A)\frac{\partial}{\partial t}\ln(w^2 - \alpha) - \tfrac{1}{2}D(f\dot A + \beta D) = 0\,,\\
&R_{44} = -\ddot A - \tfrac{1}{2}(\dot A^2 + D^2) - \frac{1}{4}\left(\frac{w^2}{\alpha}\dot\psi + \frac{\dot\alpha}{\alpha}\right)\left(\frac{\dot\alpha}{\alpha} + \frac{w^2}{\alpha}\dot\psi\right) - \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\dot\alpha}{\alpha} + \frac{w^2}{\alpha}\dot\psi\right) = 0\,,\\
&R_{[23]} = -\tfrac{1}{4}(f\dot A + \beta D)\frac{\partial}{\partial t}\ln(w^2 - \alpha) - \frac{1}{2}\frac{\partial}{\partial t}(f\dot A + \beta D) + \tfrac{1}{2}D(fD - \beta\dot A) = c\,,
\end{aligned} \tag{2.8}$$

where c is a constant of integration. Equation (2.2) implies that either $w=0$ or

$$\dot{\psi}=2\dot{A}\ , \tag{2.9}$$

which leads immediately to

$$1-\frac{w^2}{\alpha}=\frac{\beta^2+f^2}{k^2+\beta^2+f^2}\ , \tag{2.10}$$

where k is a constant of integration.

As a further simplification, we take $f=0$ and consider only the effect of $w\neq 0$. Then the equation $R_{[23]}=c$ reduces trivially to zero and we are left with three equations to satisfy.

The linear combination $R_{11}+\alpha R_{44}=0$ yields with

$$y=\frac{\alpha\beta^2}{\beta^2+k^2} \tag{2.11}$$

the equation

$$2\frac{\ddot{\beta}}{\beta}-\left(\frac{\dot{\beta}}{\beta}\right)^2-\frac{\dot{\beta}}{\beta}\frac{\partial}{\partial t}\ln y=0\ . \tag{2.12}$$

Subtracting from R_{44} and integrating, we obtain

$$(\dot{y})^2=\lambda y/\beta^2\quad(\lambda=\text{constant})\ . \tag{2.13}$$

The R_{22} equation becomes

$$2\frac{\ddot{\beta}}{\beta}+\frac{\dot{\beta}}{\beta}\frac{\partial}{\partial t}\ln y=0\ . \tag{2.14}$$

Equations (2.12) and (2.14) give

$$\beta=Ct^{4/3}\ , \tag{2.15}$$

where C is a constant and one constant of integration has been set arbitrarily equal to 1.

If $\lambda=0$, y is a constant, which implies that $\ddot{\beta}=0$ from either Eq. (2.12) or (2.14). Compatibility with Eq. (2.15) then requires $C=0$, leading to a trivial solution.

If $\lambda\neq 0$, the solution of Eq. (2.13) is

$$y=Bt^{-2/3} \tag{2.16}$$

and we deduce from Eq. (2.11) that

$$\alpha=Bt^{-2/3}\left(1+\frac{k^2}{B}t^{-8/3}\right). \tag{2.17}$$

Finally,

$$w^2=k^2\frac{B}{C}t^{-10/3}\ . \tag{2.18}$$

Setting $B=C=1$ and noticing that $k^2=-\kappa^2<0$ since w is pure imaginary, we obtain the solution

$$\begin{aligned}&\alpha=t^{-2/3}(1-\kappa^2t^{-8/3})\ ,\\&\beta=t^{4/3},\\&w=i\kappa t^{-5/3}\ .\end{aligned} \tag{2.19}$$

III. ANALYSIS OF THE SOLUTION

The solution given in Eq. (2.19) is open in the plane of symmetry, as expected, and closed in the direction of anisotropy. Moreover, the solution has the correct correspondence with the solution of Einstein's equations[6] when $\kappa\to 0$.

The interesting features of the model occur as t decreases toward zero. When $t<\kappa^{3/4}$, α becomes negative and the signature of the metric vanishes. This means that the singularity present at $t=0$ lies in an unphysical region of space-time. To gain more understanding of this phenomenon, we shall study the behavior of world lines.

The test-particle equation of motion is[5]

$$\frac{d^2x^\mu}{ds^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=0\ . \tag{3.1}$$

Using the $\Gamma^\lambda_{\mu\nu}$ given in Eq. (2.6), we find that

$$\frac{d}{ds}\left[(\alpha-w^2)\frac{dx^1}{ds}\right]=\frac{d}{ds}\left(\beta\frac{dx^1}{ds}\right)=0\ , \tag{3.2}$$

$$\frac{d}{ds}\left(\beta\frac{dx^2}{ds}\right)=0\ , \tag{3.3}$$

and

$$\frac{d}{ds}\left(\beta\frac{dx^3}{ds}\right)=0\ . \tag{3.4}$$

Thus, the motion is restricted to a straight line. A first integral of Eq. (3.1) is

$$\left(\frac{dt}{ds}\right)^2-\alpha\left(\frac{dx^1}{ds}\right)^2-\beta\left(\frac{dx^2}{ds}\right)^2-\beta\left(\frac{dx^3}{ds}\right)^2=1\ , \tag{3.5}$$

which yields

$$\begin{aligned}\frac{dt}{ds}&=\left[1+p^2\frac{\alpha}{(\alpha-w^2)^2}+\frac{q^2+u^2}{\beta}\right]^{1/2}\\&=\left[1+p^2t^{2/3}\left(1-\frac{\kappa^2}{t^{8/3}}\right)+\frac{q^2+u^2}{t^{4/3}}\right]^{1/2}\ ,\end{aligned} \tag{3.6}$$

where p, q, and u are real constants of integration. It follows that dt/ds is imaginary at

$$\begin{aligned}t_{\min}&=\left[\frac{\kappa^2}{1+1/p^2t_{\min}^{2/3}+(q^2+u^2)/p^2t_{\min}^2}\right]^{3/8}\\&\leq\kappa^{3/4}\ .\end{aligned} \tag{3.7}$$

Hence the world lines of test particles are well behaved everywhere in physical space-time for $t>\kappa^{3/4}$: The branch point in Eq. (3.6) occurs at $t=\kappa^{3/4}$ only as $p\to\infty$, corresponding to a null ray moving in the direction of anisotropy x^1. The non-Riemannian 4-geometry, however, is finite on the surface $t=\kappa^{3/4}$, even though the Riemannian submanifold is singular. In the new theory, the initial state of this model (empty) universe would have different analytic properties than in Einstein's gravitational theory. A more realistic cosmology, describing a universe containing matter, will be considered in a forthcoming paper.

ACKNOWLEDGMENT

This research was supported by the Natural Sciences and Engineering Research Council of Canada.

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