General Relativity

Extra Problems

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Problems

EP #1- The Permutation Symbol I

The alternating symbol is defined by

$$\varepsilon_{abcd} = \begin{cases} 1 & \text{if abcd is an even permutation of 1234} \\ -1 & \text{if abcd is an odd permutation of 1234} \\ 0 & \text{otherwise} \end{cases}$$

Show that if, T, X, Y, and Z are 4−vectors with

$$T = (1, 0),\ X = (0, \vec{x}),\ Y = (0, \vec{y}),\ and\ Z = (0, \vec{z})$$

then

$$\varepsilon_{abcd} T^a X^b Y^c Z^d = \vec{x} \cdot (\vec{y} \times \vec{z})$$

EP #2 - The Permutation Symbol II

Let ε have components ε_{abcd} (defined in EP #1) in every inertial coordinate system

(a) Show that ε is a tensor of rank 4

(b) Write down the values of the components of the contravariant tensor ε^{abcd}

(c) Show that $\varepsilon_{abcd}\varepsilon^{abcd} = -24$ and that $\varepsilon_{abcd}\varepsilon^{abce} = -6\delta^e_d$

EP #3 - Electromagnetic Field Tensor

Let F^{ab} be an electromagnetic field tensor. Write down the components of the dual tensor

$$F^*_{ab} = \frac{1}{2}\varepsilon_{abcd}F^{cd}$$

in terms of the components of the electric and magnetic fields. By constructing the scalars $F_{ab}F^{ab}$ and $F^*_{ab}F^{ab}$, show that $\vec{E} \cdot \vec{B}$ and $\vec{E} \cdot \vec{E} - \vec{B} \cdot \vec{B}$ are invariants.

EP #4 - Vanishing Magnetic Field

An observer has 4−velocity U^a. Show that $U^a U_a = 1$. The observer moves through an electromagnetic field F^{ab}. Show that she sees no magnetic field if $F^{*ab}U_b = 0$, and show that this equation is equivalent to $\vec{B} \cdot \vec{u} = 0$ and $B - \vec{u} \times \vec{E} = 0$. Hence show that there exists a frame in which the magnetic field vanishes at an event if and only if in every frame $\vec{E} \cdot \vec{B} = 0$ and $\vec{E} \cdot \vec{E} - \vec{B} \cdot \vec{B} > 0$ at the event.

EP #5 - Null and Orthogonal

(a) Show that the sum of any two orthogonal (scalar product is zero) spacelike vectors is spacelike.

(b) Show that a timelike vector and a null vector cannot be orthogonal.

EP #6 - Rindler Coordinates

Let $\Lambda_B(\vec{v})$ be a Lorentz boost associated with 3-velocity $\vec{v}$. Consider

$$\Lambda \equiv \Lambda_B(\vec{v}_1) \cdot \Lambda_B(\vec{v}_2) \cdot \Lambda_B(-\vec{v}_1) \cdot \Lambda_B(-\vec{v}_2)$$

where $\vec{v}_1 \cdot \vec{v}_2 = 0$. Assume that $v_1 << 1$, $v_2 << 1$. Show that Λ is a rotation. What is the axis of rotation? What is the angle of rotation?

Let $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$ be inertial coordinates on flat space-time, so the Minkowski metric has components

$$(g_{ab}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Find the metric coefficients $\tilde{g}_{ab}$ in each of the following coordinate systems.

(a) $\tilde{x}^0 = t$, $\tilde{x}^1 = r$, $\tilde{x}^2 = \theta$, $\tilde{x}^3 = z$

(b) $\tilde{x}^0 = t$, $\tilde{x}^1 = r$, $\tilde{x}^2 = \theta$, $\tilde{x}^3 = \varphi$

(c) $\tilde{x}^0 = \tau$, $\tilde{x}^1 = \varphi$, $\tilde{x}^2 = y$, $\tilde{x}^3 = z$

where r, θ in the first case, are plane polar coordinates in the x, y plane, in the second case r, θ, φ are spherical polars coordinates, and in the third, τ, φ are *Rindler coordinates*, defined by $t = \tau \cosh \varphi$, $x = \tau \sinh \varphi$. In each case, say which region of Minkowski space the coordinate system covers. A quick method is to write the metric as $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$ and substitute, for example, $dx = \cos\theta\, dr - r \sin\theta\, d\theta$, and so on. Of course, the penalty is that you must convince yourself that this is legitimate!

EP #7 - $4-$**Velocities**

In some reference frame, the vector fields $\vec{U}$ and $\vec{D}$ have the components

$$U^\alpha \doteq (1 + t^2, t^2, \sqrt{2}t, 0)$$
$$D^\alpha \doteq (x, 5tx, \sqrt{2}t, 0)$$

where t, x, and y are the usual Cartesian coordinates in the specified reference frame. The scalar ρ has the value $\rho = x^2 + t^2 - y^2$. The relationship "$LHS \doteq RHS$" means "the object on the LHS is represented by the object on the RHS in the specified reference frame".

(a) Show that $\vec{U}$ is suitable as a 4−velocity. Is $\vec{D}$?

(b) Find the spatial velocity $\vec{v}$ of a particle whose 4−velocity is $\vec{U}$, for arbitrary t. Describe the motion in the limits $t = 0$ and $t \to \infty$.

(c) Find $\partial_\beta U$ for all α, β. Show that $U_\alpha \partial_\beta U^\alpha = 0$. There is a clever way to do this, which you are welcome to point out. Please do it the brute force way as well as practice manipulating quantities like this.

(d) Find $\partial_\alpha D^\alpha$

(e) Find $\partial_\beta (U^\alpha D^\beta)$ for all α

(f) Find $U_\alpha \partial_\beta (U^\alpha D^\beta)$. Why is answer so similar to that for (d)?

(g) Calculate $\partial_\alpha \rho$ for all α. Calculate $\partial^\alpha \rho$

(h) Find $\nabla_{\vec{U}} \rho$ and $\nabla_{\vec{D}} \rho$

EP #8 - Projection Operators

Consider a timelike 4−vector $\vec{U}$ and the tensor $P_{\alpha\beta} = \eta_{\alpha\beta} + U_\alpha U_\beta$. Show that this tensor is a projection operator that projects an arbitrary vector $\vec{V}$ into one orthogonal to $\vec{U}$. In other words, show that the vector $\vec{V}_\perp$ whose components are $V_\perp^\alpha = P_\beta^\alpha V^\beta$ is

(a) orthogonal to $\vec{U}$

(b) unaffected by further projections: $V_{\perp\perp}^\alpha = P_\beta^\alpha V_\perp^\beta = V_\perp^\alpha$

(c) Show that $P_{\alpha\beta}$ is the metric for the space of vectors orthogonal to $\vec{U}$:

$$P_{\alpha\beta} V_\perp^\alpha W_\perp^\beta = \vec{V}_\perp \cdot \vec{W}_\perp$$

(d) Show that for an arbitrary non-null vector $\vec{q}$, the projection tensor is given by

$$P_{\alpha\beta}(q^\alpha) = \eta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^\gamma q_\gamma}$$

Do we need a projection operator for null vectors?

EP #9 - Killing Vectors in Flat Space

Find the Killing vectors for flat space $ds^2 = dx_1^2 + dx_2^2 + dx_3^2$, i.e., write out Killings equation in flat space, differentiate it once and then solve the resulting differential equation.

EP #10 - Oblate Spheroidal Coordinates

Let x, y, z be the usual Cartesian coordinates in flat space. Oblate spheroidal coordinates are defined by the relations

$$x = \sqrt{r^2 + c^2} \sin\theta \cos\varphi \ , \ y = \sqrt{r^2 + c^2} \sin\theta \sin\varphi \ , \ z = r\cos\theta$$

where c is a constant.

(a) What is the shape of surfaces of constant r?

(b) What is the metric in r, θ, φ?

(c) What is the Laplacian operator on a scalar field, $\nabla^2\Phi$, in these coordinates?

(d) Show that $\nabla^2\Phi = 0$ is separable in oblate spheroidal coordinates, $\Phi(r, \theta, \varphi) = R(r)\Theta(\theta)\phi(\varphi)$. Find the θ and φ solutions explicitly, and write an equation for $R(r)$

(e) If $r = r_0$ is the surface of a conducting disk that has net charge Q, what is the electrostatic potential exterior to r_0? What is the surface charge density as a function of $\rho = \sqrt{x^2 + y^2}$ in the limit that the spheroid becomes a flat disk, $r_0 \to 0$?

EP #11 - Uniformly Accelerating Observer A coordinate system for a uniformly accelerating observer

Background definitions: A former physics 008 student is now an astronaut. She moves through space with acceleration g in the $x-$direction. In other words, her 4$-$acceleration $\vec{a} = d\vec{u}/d\tau$ (where τ is time as measured on the astronauts own clock) only has spatial components in the $x-$direction and is normalized such that $\sqrt{\vec{a} \cdot \vec{a}} = g$.

This astronaut assigns coordinates $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$ as follows:

First, she defines spatial coordinates to be $(\tilde{x}, \tilde{y}, \tilde{z})$, and sets the time coordinate $\tilde{t}$ to be her own proper time. She defines her position to be $(\tilde{x} = g^{-1}, \tilde{y} = 0, \tilde{z} = 0)$ (not a unique choice, but a convenient one). Note that she remains *fixed* with respect to these coordinates - thats the point of coordinates for an accelerated observer!

Second, at $\tilde{t} = 0$, the astronaut chooses $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$ to coincide with the Euclidean coordinates (t, x, y, z) of the inertial reference frame that momentarily coincides with her motion. In other words, though the astronaut is not inertial, there is an inertial frame that, at $\tilde{t} = 0$, is momentarily at rest with respect to her. This is the frame used to assign $(\tilde{x}, \tilde{y}, \tilde{z})$ at $\tilde{t} = 0$. The clocks of that frame are set such that they are synchronized with her clocks at that moment.

Observers who remain at fixed values of spatial coordinates are called coordinate-stationary observers(CSOs). Note that CSOs are also accelerated observers, though not necessarily accelerating at the same rate as the astronaut. The astronaut requires the CSO worldlines to be orthogonal to hypersurfaces $\tilde{t} = constant$. She also requires that for each $\tilde{t}$ there exists some inertial frame, momentarily at rest with respect to the astronaut, in which all events with $\tilde{t} = constant$ are simultaneous. The accelerated motion of the astronaut can thus be described as movement through a sequence of inertial frames which momentarily coincide with her motion.

It is easy to see that $\tilde{y} = y$ and $\tilde{z} = z$; henceforth we drop these coordinates from the problem.

(a) What is the 4−velocity $\vec{u}$ of the astronaut, as a function of $\tilde{t}$, as measured by CSOs in the initial inertial frame[the frame the uses coordinates (t, x, y, z)]? HINT: by considering the conditions on $\vec{u} \cdot \vec{u}$, $\vec{u} \cdot \vec{a}$, and $\vec{a} \cdot \vec{a}$, you should be able to find simple forms for u^t and u^x. After you have worked out $\vec{u}$, compute $\vec{a}$.

(b) Integrate this 4−velocity to find the position $[T(\tilde{t}), X(\tilde{t})]$ of the astronaut in the coordinates (t, x). Recall that at $t = \tilde{t} = 0$, $X = \tilde{x} = 1/g$. Sketch the astronauts worldline on a spacetime diagram in the coordinates (t, x). You will return to and augment this sketch over the course of this problem, so you may want to to do this on a separate piece of paper.

(c) Find the orthogonal basis vectors $\vec{e}_{\tilde{t}}$ and $\vec{e}_{\tilde{x}}$ describing the momentarily inertial coordinate system at some time $\tilde{t}$. Add these vectors to the sketch of your worldline.

We now *promote* $\tilde{t}$ to a coordinate, i.e., give it meaning not just on the astronauts worldline, but everywhere in spacetime, by requiring that $\tilde{t} = constant$ be a surface of constant time in the Lorentz frame in which the astronaut is instantaneously at rest.

(d) By noting that this *surface* must be parallel to $\vec{e}_{\tilde{x}}$ and that it must pass through the point $[T(\tilde{t}), X(\tilde{t})]$, show that it is defined by the line

$$x = t \cosh g\tilde{t}$$

In other words, it is just a straight line going through the origin with slope $\cosh g\tilde{t}$.

We have now defined the time coordinate $\tilde{t}$ that the astronaut uses to label spacetime. Next, we need to come up with a way to set her spatial coordinates $\tilde{x}$.

(e) Recalling that CSOs must themselves be accelerated observers, argue that their worldlines are hyperbolae, and thus that a CSOs position in (t, x)

must take the form

$$t = \frac{A}{g} \sinh g\tilde{t} \quad , \quad x = \frac{A}{g} \cosh g\tilde{t}$$

From the initial conditions, find A.

(f) Show that the line element $ds^2 = d\vec{x} \cdot d\vec{x}$ in the new coordinates takes the form

$$ds^2 = -dt^2 + dx^2 = -(g\tilde{x})^2 d\tilde{t}^2 + d\tilde{x}^2$$

This is known as the Rindler metric. As the problem illustrates, it is just the flat spacetime of special relativity; but, expressed in coordinates that introduce some features that will be very important in general relativity.

EP #12 - Jumping Seagull

A Newton-Galilean Problem - A seagull sits on the ground. The wind velocity is $\vec{v}$. How high can the gull rise without doing any work? The trick here is

(a) to identify the most convenient reference frame

(b) transform the problem to that frame

(c) solve the problem

(d) transform the result back again so that it is expressed in the original frame

EP #13 - Lagrange Equations for Kepler Orbits

Use Lagrange equations to solve the problem of Kepler planetary orbits in a gravitational field. Work in 3 dimensions in spherical coordinates. Determine the orbital equation $r(\theta)$.

EP #14 - Lagrange Equations for Double Pendulum

Use Lagrange equations to solve the problem of the double pendulum in a gravitational field. The double pendulum is a fixed pivot O, a light rigid rod OA of unit length at an angle α to the vertical, mass m at A, light rigid rod AB of unit length at an angle β to the vertical, mass m at B, constant gravitational field g downwards. Solve the equations in the small angle approximation.

EP #15 - Uniform Relativistic Circular Motion

A particle (in special relativity) moves in uniform circular motion, that is (with $c = 1$),

$$x^\mu = (t, r\cos\omega t, r\sin\omega t, 0)$$

(a) Write down its worldline according to an observer moving with velocity $\vec{v}$ along the $y-$axis. You will need to use the old time t as a parameter. HINT: this follows directly from the Lorentz transformation.

(b) If the particle at rest decays with half-life $\tau_{1/2}$, what is its observed half-life?

(c) Show that the proper acceleration α is given by

$$\alpha = \frac{r\omega^2}{1 - r^2\omega^2}$$

EP #16 - Accelerated Motion

(a) Consider a particle moving along the $x-$axis with velocity u and acceleration a, as measured in frame S. S moves relative to S with velocity v along the same axis. Show that

$$\frac{du}{dt} = \frac{1}{\gamma^3} \frac{1}{(1 + u'v)^3} a'$$

(b) Suppose that S is chosen to be the instantaneous rest frame of the particle and $a = 9.806\, m/s^2$. That is, $u' = 0$, $a' = g$, and $u = v$. Using the result from part (a), derive expressions for u (velocity as measured in S) and x as a function of time. Write x as a function of τ, the proper time as measured along the particles worldline, and evaluate for $\tau = 20\, years$. Discuss the significance of this result for space travel. You can use $dt/d\tau = \gamma$ to derive an expression for τ as a function of t.

EP #17 - High Energy Kinematics

(a) In a high energy accelerator, the energy available to create new particles is the energy in the center-of-mass(CM) frame. Consider a proton with momentum $1\, TeV/c$ incident on a target proton at rest. What is the available CM energy?

(b) Next consider a $1\, TeV$ proton heading east colliding with a $1\, TeV$ proton headed west. What is the available CM energy? What momentum would be needed in a fixed-target experiment to obtain the same available energy?

(c) A Λ^0 baryon ($m_\Lambda = 1115.7\, MeV$) decays into a proton ($m_p = 938.3\, MeV$) anda negative pion ($m_\pi = 139.6\, MeV$). What is the momentum of the proton or pion in the CM frame?

(d) The decaying Λ^0 has momentum $28.5\, GeV/c$ in the lab frame. What is the maximum angle between the proton and the pion in the lab?

EP #18 - Tensor Properties

Consider a tensor T_{ij} in three-dimensional Euclidean space. Under an arbitrary rotation of the three-dimensional coordinate space, the tensor is transformed. Show that

(a) If T_{ij} is symmetric and traceless, then the transformed tensor is symmetric and traceless.

(b) If T_{ij} is antisymmetric, then the transformed tensor is symmetric.

State the analogous result for a tensor $T^{\mu\nu}$ in four-dimensional Minkowski space. Define carefully what you mean by a traceless tensor in this case.

EP #19 - Transforming Electromagnetic Fields

Under a Lorentz transformation, a tensor transforms as follows:

$$F'^{\mu\nu} = \Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}F^{\alpha\beta}$$

where Λ is the Lorentz transformation matrix. Consider an inertial frame K at rest, and a second inertial frame K' moving with velocity v along the $x-$direction with respect to K. Using the explicit result for Λ corresponding to the transformation between K and K', determine the electric and magnetic fields in frame K' in terms of the corresponding fields in frame K.

EP #20 - Twins In Relativity

Consider a pair of twins that are born somewhere in spacetime. One of the twins decides to explore the universe. She leaves her twin brother behind and begins to travel in the $x-$direction with constant acceleration $a = 10\ m/s^2$ as measured in her rocket frame. After 10 years according to her watch, she reverses the thrusters and begins to accelerate with a constant acceleration $-a$ for a while.

(a) At what time on her watch should she again reverse her thrusters so she ends up home at rest?

(b) According to her twin brother left behind, what was the most distant point on her trip?

(c) When the sister returns, who is older, and by how much?

EP #21 - Multiple Lorentz Transformations

A Lorentz transformation is the product of a boost with rapidity ζ in the direction $\hat{n}_1$, followed by a boost with rapidity ζ in the direction $\hat{n}_2$, followed by a boost with rapidity ζ in the direction $\hat{n}_3$, where ζ is the same in each case, and the three directions $\hat{n}_1$, $\hat{n}_2$, and $\hat{n}_3$ lie in the same plane separated by 120°. What is the resulting transformation? To lowest order for small ζ, is it a boost or a rotation? At what order does the other (boost or rotation) enter?

EP #22 - Tensor Transformations

(a) Observer O at *rest* sees a symmetric tensor $T^{\mu\nu}$ to be diagonal with components (ρ, p, p, p). What are the components of $T_{\mu\nu}$

(b) Frame O moves with speed v in the $+x$-direction with respect to O. What are the components of $T'^{\mu\nu}$ in frame O? What are the components of $T'_{\mu\nu}$? How can the *rest* frame be identified? Suppose that $p = -\rho$ in the original frame O, what is $T'^{\mu\nu}$ then? Make an insightful observation.

EP #23 - Newtonian Gravity

Poissons formulation of Newtonian gravity is

$$\nabla^2\varphi = 4\pi\rho \quad , \quad \vec{g} = -\nabla\varphi$$

where ρ is the matter density, φ is the gravitational potential and $\vec{g}$ is the acceleration due to gravity. Show that this gives the usual Newtonian formula for a point-like source.

EP #24 - Tides

Tides occur because the force of gravity is slightly different at two nearby points, such as a point at the earths surfaced and at its center.

(a) What is the difference between the gravitational acceleration induced by a mass M (the sun or the moon) evaluated at the center and at a point on the surface of a sphere of radius r (the earth) located a distance R from M (take $r << R$). Write the radial component of this difference at the surface as a function of the angle from the line joining the two objects. How many high/low tides are there in a day?

(b) If the earth were a perfect sphere covered with water, compute or estimate the height difference between high and low tides(ignoring complications such as rotation, friction, viscosity) for spring tides (directions of sun and moon aligned) and neap tides (sun and moon at right angles).

(c) A neutron star is a collapsed object of nuclear density with mass $M = 1.4M_{Sun}$,and radius $R = 10\,km$. In Larry Nivens short story Neutron Star(1966), tidal forces in the neighborhood of the title object prove fatal to the unwary. What is the tidal acceleration across the diameter of a person (say a distance of $1\,m$) at a distance of $100\,km$ from a neutron star?

EP #25 - An Invisible Sphere

A hollow sphere has density ρ, inner radius a and outer radius b. Find the gravitational field in the region $r < a$. Suppose now that the sphere were invisible. Could an observer at the center deduce its existence without leaving the region $r < a$?

EP #26 - Gravitational Fields

(a) Compute the gradient of the gravitational field $\partial g_i/\partial x_j$ (a nine component object) corresponding to a sphere of density ρ and radius R centered at the origin.

(b) Find a mass distribution $\rho(x, y, z)$ on a bounded domain, that is, zero whenever $x^2 + y^2 + z^2 > R^2$ for some positive constant R; uniformly bounded, i.e., $|\rho(x, y, z)| < C$ for some positive constant C independent of position; and for which at least one component of the gradient of the gravitational field is infinite at some point.

EP #27 - Riemann Tensor

Find an expression for

$$\nabla_c \nabla_d T^a_b - \nabla_d \nabla_c T^a_b$$

in terms of the Riemann tensor.

EP #28 - Isometries and Killing Vectors

(a) Define an isometry

(b) Define the Killing vector and show that it satisfies

$$\nabla_a k_b + \nabla_b k_a = 0$$

(c) Show how the Killing vector defines a constant along geodesics

(d) If k_a and l_a are Killing vectors, show that

$$[k, l]_a = k^b \nabla_b l_a - l^b \nabla_b k_a$$

is also a Killing vector. It is useful to recall the symmetry properties of the Riemann tensor $R_{abcd} = -R_{bacd}$ and $R_{abcd} = R_{cdab}$.

EP #29 - On a Paraboloid

A paraboloid in three dimensional Euclidean space

$$ds^2 = dx^2 + dy^2 + dz^2$$

is given by

$$x = u \cos\varphi \quad , \quad y = u \sin\varphi \quad , \quad z = u^2/2$$

where $u \geq 0$ and $0 \leq \varphi \leq 2\pi$.

(a) Show that the metric on the paraboloid is given by

$$ds^2 = (1 + u^2) du^2 + u^2 d\varphi^2$$

(b) Writing $x^1 = u$ and $x^2 = \varphi$ find the Christoffel symbols for this metric.

(c) Solve the equation for parallel transport

$$U^a \nabla_a V^b = 0$$

where

$$U^a = \frac{dx^a}{dt}$$

for the curve $u = u_0$ where u_0 is a positive constant and with initial conditions $V^1 = 1$ and $V^2 = 0$. HINT: the problem is simplified if you take $t = \varphi$. The equation of parallel transport will give you two coupled equations for U^1 and U^2, differentiating the $dU^1/d\varphi$ equation again allows you to decouple the U^1 equation.

EP #30 - A Two-Dimensional World

A certain two-dimensional world is described by the metric

$$ds^2 = \frac{dx^2 + dy^2}{\left[1 + \frac{x^2+y^2}{4a^2}\right]^2}$$

(a) Compute the connection coefficients Γ^i_{jk}

(b) Let $\vec{\xi} = -y\hat{e}_x + x\hat{e}_y$. Show that $\vec{\xi}$ is a solution of Killings equation.

(c) What is the conserved quantity that corresponds to this symmetry? Show from the geodesic equation that this quantity is indeed conserved.

(d) Compute the Riemann tensor R^{ij}_{kl}, the Ricci tensor R^i_j, and the Ricci scalar R. What is the shape of this world?

EP #31 - Timelike Geodesics

Find the timelike geodesics for the metric

$$ds^2 = \frac{1}{t^2}\left(-dt^2 + dx^2\right)$$

EP #32 - More Geodesics

Consider the 2-dimensional metric

$$ds^2 = a^2\left(d\chi^2 + \sinh^2\chi d\varphi^2\right)$$

(a) Compute the connection coefficients Γ^i_{jk}

(b) Compute all components of the Riemann tensor R^{ij}_{kl}, the Ricci tensor R^i_j, and the Ricci scalar R.

(c) A geodesic starts at $\chi = b$, $\varphi = 0$ with tangent $d\varphi/d\lambda = 1$, $d\chi/d\lambda = 0$. Find the trajectory $\chi(\varphi)$.

(d) A second geodesic starts at $\chi = b + \xi(\xi \ll 1)$, also initially in the ?φ-direction. How does the separation initially increase or decrease along the two curves.

(e) What is the shape of the geodesic trajectory as $a \to \infty$, $\chi \to 0$ with $r = a\chi$ fixed.

EP #33 - Parallel Transport on a Sphere

On the surface of a 2−sphere of radius a

$$ds^2 = a^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

Consider the vector $\vec{A}_0 = \vec{e}_\theta$ at $\theta = \theta_0$, $\varphi = 0$. The vector is parallel transported all the way around the latitude circle $\theta = \theta_0$ (i.e., over the range $0 \leq \varphi \leq 2\pi$ at $\theta = \theta_0$). What is the resulting vector $\vec{A}$? What is its magnitude $\left(\vec{A} \cdot \vec{A}\right)^{1/2}$? HINT: derive differential equations for A^θ and A^φ as functions of φ.

EP #34 - Curvature on a Sphere

(a) Compute all the nonvanishing components of the Riemann tensor R_{ijkl} $((i,j,k,l) \in (\theta,\varphi))$ for the surface of a 2−sphere.

(b) Consider the parallel transport of a tangent vector $\vec{A} = A^\theta \hat{e}_\theta + A^\varphi \hat{e}_\varphi$ on the sphere around an infinitesimal parallelogram of sides $\hat{e}_\theta d\theta$ and $\hat{e}_\varphi d\varphi$. Using the results of part (a), show that to first order in $d\Omega = \sin\theta d\theta d\varphi$, the length of $\vec{A}$ is unchanged, but its direction rotates through an angle equal to $d\Omega$.

(c) Show that, if $\vec{A}$ is parallel transported around the boundary of any simply connected solid angle Ω, its direction rotates through an angle Ω. (*Simply connected* is a topological term meaning that the boundary of the region could be shrunk to a point; it tells us that there are no holes in the manifold or other pathologies). Using the result of part (b) and intuition from proofs of Stokes theorem, this should be an easy calculation. Compare with the result of EP #32.

EP #35 - Riemann Tensor for 1+1 Spacetimes

(a) Compute all the nonvanishing components of the Riemann tensor for the spacetime with line element

$$ds^2 = -e^{2\varphi(x)}dt^2 + e^{2\psi(x)}dx^2$$

(b) For the case $\varphi = \psi = \frac{1}{2}\ln|g(x - x_0)|$ where g and x_0 are constants, show that the spacetime is flat and find a coordinate transformation to globally flat coordinates $(\bar{t}, \bar{x})$ such that $ds^2 = -d\bar{t}^2 + d\bar{x}^2$.

EP #36 - About Vectors Tangent to Geodesics

Let $x^\mu(\tau)$ represent a timelike geodesic curve in spacetime, where τ is the proper time as measured along the curve. Then $u^\mu \equiv dx^\mu/d\tau$ is tangent to the geodesic curve at any point along the curve.

(a) If $g_{\mu\nu}$ is the metric of spacetime, compute the magnitude of the vector u^μ. Do not use units where $c = 1$, but keep any factors of c explicit. Compare your result with the one obtained in flat Minkowski spacetime. HINT: The magnitude of a timelike vector v^μ is given by $(-g_{\mu\nu}v^\mu v^\nu)^{1/2}$.

(b) Consider a contravariant timelike vector v^μ at a point P on the geodesic curve. Move the vector v^μ from the point P to an arbitrary point Q on the geodesic curve via parallel transport. Prove that the magnitude of the vector v^μ at the point Q equals the magnitude of the vector v^μ at point P.

(c) Suppose that at the point P on the geodesic curve, $v^\mu = u^\mu$. Now, parallel transport the vector v^μ along the geodesic curve to arbitrary point Q. Show that $v^\mu = u^\mu$ at the point Q. NOTE: This result implies that a vector tangent to a geodesic at a given point will always remain tangent to the geodesic curve when parallel transported along the geodesic.

EP #37 - Velocity of Light

The Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

As a function of r, what is the coordinate velocity of light in this metric (a) in the radial direction? (b) in the transverse direction? What are the physical consequences of these results.

EP #38 - Orbiting Photons

Consider a photon in orbit in a Schwarzschild geometry. For simplicity, assume that the orbit lies in the equatorial plane (i.e., $\theta = \pi/2$ is constant).

(a) Show that the geodesic equations imply that

$$\bar{E}^2 = \frac{1}{c^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{\bar{J}^2}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right)$$

where $\bar{E}$ and $\bar{J}$ are constant of the motion and λ is an affine parameter.

(b) Define the effective potential

$$V_{eff} = \frac{\bar{J}^2}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right)$$

The effective potential yields information about the orbits of massive particles. Show that for photons there exists an unstable circular orbit of radius $3r_s/2$, where $r_s = 2GM/c^2$ is the Schwarzschild radius. HINT: Make sure you check for minima and maxima.

(c) Compute the proper time for the photon to complete one revolution of the circular orbit as measured by an observer stationed at $r = 3r_s/2$.

(d) What orbital period does a very distant observer assign to the photon?

(e) The instability of the orbit can be exhibited directly. Show, by perturbing the geodesic in the equatorial plane, that the circular orbit of the photon at $r = 3r_s/2$ is unstable. HINT: in the orbit equation put $r = 3r_s/2 + \eta$, and deduce an equation for η. Keep only the first order terms in $\eta << 1$, and solve the resulting equation.

EP #39 - Light Cones

Consider the 2-dimensional metric

$$ds^2 = -xdw^2 + 2dwdx$$

(a) Calculate the light cone at a point (w, x), i.e., find dw/dx for the light cone. Sketch a (w, x) spacetime diagram showing how the light cones change with x. What can you say about the motion of particles, and in particular, about whether they can cross from positive to negative x and vice versa.

(b) Find a new system of coordinates in which the metric is diagonal.

EP #40 - Circular Orbit

An object moves in a circular orbit at Schwarzschild radius R around a spherically symmetric mass M. Show that the proper time τ is related to coordinate time t by

$$\frac{\tau}{t} = \sqrt{1 - \frac{3M}{R}}$$

HINT: It is helpful to derive a relativistic version of Keplers third law.

EP #41 - Space Garbage

In a convenient coordinate system, the spacetime of the earth is approximately

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 + \frac{2GM}{r}\right) \left[dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right] \\ &= -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 + \frac{2GM}{r}\right) \left[dx^2 + dy^2 + dz^2\right] \end{aligned}$$

where M is the earths mass. In the second version we remapped the spherical coordinates to cartesian coordinates in the usual way:

$$x = r\sin\theta\cos\varphi \quad , \quad y = r\sin\theta\sin\varphi \quad , \quad z = r\cos\theta$$

Note that the Cartesian form of the spacetime metric is conveniently written $g_{\alpha\beta} = \eta_{\alpha\beta} + 2\Phi\hat{I}$ where $\hat{I} = diag(1,1,1,1)$ and $\Phi = GM/r$. We can assume that $\Phi << 1$ throughout this problem.

The space shuttle orbits the earth in a circular $u^r = 0$), equatorial ($\theta = \pi/2$, $u^\theta = 0$) orbit of radius R.

(a) Using the geodesic equation, show that an orbit which begins equatorial remains equatorial: $du^\theta/dt = 0$ if $u^\theta = 0$ and $\theta = \pi/2$ at $t = 0$. HINT: Begin by computing the non-zero connection coefficients; use the fact that $\Phi << 1$ to simplify your answer. We now require that the orbit must remain circular: $du^r/dt = 0$. This has already been done in earlier problems and in the text.

(b) By enforcing this condition with the geodesic equation, derive an expression for the orbital frequency

$$\Omega = \frac{d\varphi/d\tau}{dt/d\tau}$$

Does this result look familiar? This has been done in Problem #41 The next part is most conveniently described in Cartesian coordinates; you may describe the shuttles orbit as

$$x = R\cos\Omega t \ , \ \ y = R\sin\Omega t$$

An astronaut releases a bag of garbage into space, spatially displaced from the shuttle by $\xi^i = x^i_{garbage} - x^i_{shuttle}$.

(c) Using the equations of geodesic deviation, work out differential equations for the evolution of ξ^t, ξ^x, ξ^y, and ξ^z as a function of time. You may neglect terms in $(GM/r)^2$, and treat all orbital velocities as non-relativistic. You will need the Cartesian connection coefficients for this.

(d) Suppose the initial displacement is $\xi^x = \xi^y = 0$, $\xi^z = L$, $d\xi^i/dt = 0$. Further, synchronize the clocks of the garbage and the space shuttle: $\xi^0 = 0$, $\partial_t\xi^0 = 0$. Has the astronaut succeeded in getting rid of the garbage?

EP #42 - Astronauts in Orbit

Consider a spacecraft in a circular orbit in a Scharzschild geometry. As usual, we denote the Schwarzschild coordinates by (ct, r, θ, φ) and assume that the orbit occurs in the plane where $\theta = \pi/2$. We denote two conserved quantities by

$$e = \left[1 - \frac{r_s}{r}\right]\frac{dt}{d\tau} \quad and \quad \ell = r^2\frac{d\varphi}{d\tau}$$

where $r_s = 2GM/c^2$ and τ is the proper time.

(a) Write down the geodesic equation for the variable r. Noting that r is independent of τ for a circular orbit, show that:

$$\frac{\ell}{c} = c\left(\tfrac{1}{2}r_s r\right)^{1/2}\left[1 - \frac{r_s}{r}\right]^{-1}$$

(b) Show that for a timelike geodesic, $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -c^2$, where $\dot{x}^\mu = dx^\mu/d\tau$. From this result, derive a second relation between ℓ and e for a circular orbit. Then, using the result of part (a) to eliminate e, obtain an expression for $d\tau/d\varphi$ in terms of the radius r of the orbit.

(c) Using the result of part (b), determine the period of the orbit as measured by an observer at rest inside the orbiting spacecraft, as a function of the radius r of the orbit.

(d) Suppose an astronaut leaves the spacecraft and uses a rocket-pack to maintain a fixed position at radial distance r equal to the orbital radius and at fixed $\theta = \pi/2$ and $\varphi = 0$. The astronaut outside then measures the time it takes the spacecraft to make one orbital revolution. Evaluate the period as measured by the outside astronaut. Does the astronaut outside the spacecraft age faster or slower than the astronaut orbiting inside the spacecraft?

EP #43 - Weak Gravity

In weak gravity, the metric of a mass M at rest at the origin is

$$ds^2 = -(1+2\varphi)dt^2 + (1-2\alpha\varphi)\delta_{ij}dx^i dx^j$$

where α is a constant and $\varphi = -GM/r$.

(a) What is the value of α in general relativity?

(b) Instead of sitting at rest at the origin, the mass M moves in the $+x-$direction with speed v, passing through the origin at time $t = 0$, so that its position as a function of time is $x = vt$. What is the metric in this case?

(c) A photon moves along a trajectory originally in the $+y-$direction with offset b behind the $y-$axis, so that its undeflected trajectory is $x_0 = -b\hat{x} + t\hat{y}$. By what angle is the path of this test particle deflected?

(d) What is change in energy of deflected photon in part (c).

EP #44 - Star with Constant Density

The metric of a star with constant density is

$$ds^2 = -\left(1 - \frac{2\mathrm{M}(r)}{r}\right)c^2dt^2 + \left(1 - \frac{2\mathrm{M}(r)}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2$$

where

$$M(r) = \begin{cases} M(r/R)^3 & 0 < r < R \\ M & R < r \end{cases}$$

is the mass interior to radius r, M is the total mass of the star, and R is the coordinate radius of the surface of the star. Assume $R > 2M$. We consider the orbits of photons where $g_{\mu\nu} u^\mu u^\nu = 0$.

(a) Are there any singularities (coordinate or otherwise) of the metric?

(b) Write the timelike and spacelike Killing vectors for this spacetime. There are actually two spacelike Killing vectors, but we will only need one since the photon orbits are planar. You may set $\theta = \pi/2$. Write out the associated conserved quantities.

(c) Derive an expression for $dr/d\lambda$ where λ is the affine parameter. Put your expression in the form

$$\frac{1}{b^2} = \frac{1}{l^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r)$$

and define b in terms of the constants of motion and W_{eff}.

(d) Sketch W_{eff} and describe the photon orbits. How do these differ from the photon orbits in the standard Schwarzschild geometry?

(e) Calculate the coordinate time t for a photon to travel from the center of the star at $r = 0$ to the surface at $r = R$.

(f) Assume $R >> M$ and find the approximate delay, i.e., the extra time relative to the result from special relativity ($t = R$) to leading order. What is the value for the Sun where $M = 1.5\,km$ and $R = 7.0 \times 10^3\,km$.

EP #45 - In the Schwarzschild Geometry

Consider a spacetime described by the Schwarzschild line element:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

(a) A clock at fixed (r, θ, φ) measures an (infinitesimal) proper time interval, which we denote by dT. Express dT (as a function of r) in terms of the coordinate time interval dt.

(b) A stationary observer at fixed (t, θ, φ) measures an (infinitesimal) radial distance, which we denote by dR. Express dR (as a function of r) in terms of the coordinate radial distance dr.

(c) Consider the geodesic equations for free particle motion in the Schwarzschild geometry. Write out explicitly the equation corresponding to the time component. The equations corresponding to the space components will not be required. The resulting equation can be used to determine $dt/d\tau$ (where τ is the proper time and $t = x^0/c$ is the coordinate time). In particular, show that the quantity

$$k = \left(1 - \frac{2GM}{c^2 r}\right) \frac{dt}{d\tau}$$

is a constant independent of τ. Using the time component of the geodesic equation obtained earlier, compute the values of $\Gamma^0_{\alpha\beta}$ for this geometry. Consider all possible choices of α and β.

(d) Consider a particle falling radially into the center of the Schwarzschild metric, i.e., falling in radially towards $r = 0$. Assume that the particle initially starts from rest infinitely far away from $r = 0$. Since this is force-free motion, the particle follows a geodesic. Using the results of part (c), evaluate the constant k and thereby obtain a unique expression for $dt/d\tau$ that is valid at all points along the radial geodesic path. HINT: What is the value of $dt/d\tau$ at $r \to \infty$ (where the initial velocity of the particle is zero)?

(e) Since $ds^2 = -c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$ it follows that

$$g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -c^2$$

In this problem, $g_{\mu\nu}$ is determined from the Schwarzschild line element. Using these results and the result obtained in part (d) for $dt/d\tau$, compute the particles inward coordinate velocity, $v = dr/dt$, as a function of the coordinate radial distance r. Invert the equation, and integrate from $r = r_0$ to $r = r_s$, where r_0 is some finite coordinate distance such that $r_0 > r_s$ and $r_s = 2GM/c^2$ is the Schwarzschild radius. Show that the elapsed coordinate time is infinite, independent of the choice of the starting radial coordinate r_0, i.e., it takes an infinite coordinate time to reach the Schwarzschild radius. HINT: For radial motion, θ and φ are constant independent of τ. Note that for inward radial motion $dt/d\tau$ is negative.

(f) Compute the velocity dR/dT as measured by a stationary observer at a coordinate radial distance r. Verify that $|dR/dT| \to c$ as $r \to r_s$. HINT: Use the result for dR and dT obtained in parts (a) and (b).

EP #46 - Lightcones and Embedding

A certain spacetime is describe by the metric

$$ds^2 = -(1 - H^2r^2)dt^2 + (1 - H^2r^2)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

(a) Describe the lightcone structure in the (r, t) plane using both equations and a spacetime diagram. Think carefully about the lightcone structure for $r > H^{-1}$ versus $r < H^{-1}$.

(b) Construct an embedding diagram for this spacetime. The following steps will guide you through the process:

(1) Argue that it is sufficient to consider the 2-dimensional slice

$$d\Sigma^2 = (1 - H^2r^2)^{-1}dr^2 + r^2d\varphi^2$$

(2) Pick one of the three common forms for the 3-dimensional flat space line element:

$$ds_{3D}^2 = dx^2 + dy^2 + dz^2$$
$$ds_{3D}^2 = d\rho^2 + \rho^2d\varphi^2 + dz^2$$
$$ds_{3D}^2 = dw^2 + w^2(d\Theta^2 + \sin^2\Theta d\Phi^2)$$

and find the equations that describe the 2-dimensional surface corresponding to the 2-dimensional slice metric above. What is the geometry of this surface?

EP #47 - Time delay to Jupiter

The Solar System is accurately described by the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{c^2r}\right)c^2dt^2 + \left(1 - \frac{2GM}{c^2r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2$$

where M is the mass of the Sun, t the time coordinate, r the radial coordinate, and θ and φ are polar angles.

A radio pulse is sent from the Earth, reflected off a satellite of Jupiter (the satellite is a point), and received on Earth. Jupiter is a distance r_2 from the Sun, the Earth is a distance r_1. Assume that Jupiter is on the other side of the Sun relative to the Earth. Let r_0 be the distance of closest approach of the radio pulse to the Sun. Calculate the gravitational delay in the round-trip time of the radio pulse as a function of r_0, to lowest order in G. Estimate very roughly the magnitude of the effect, given that

$$\text{mass of Sun} \approx 2 \times 10^{33}gm$$
$$\text{radius of Sun} \approx 7 \times 10^{10}cm$$
$$\text{Sun} - \text{Earth distance} \approx 1.5 \times 10^{13}cm$$
$$\text{Sun} - \text{Jupiter distance} \approx 8 \times 10^{13}cm$$
$$G \approx 6.67 \times 10^{-8}cm^3/gm - \sec^2$$

EP #48 - Geodesic Effect

If in flat spacetime a spacelike vector λ^μ is transported along a timelike geodesic without changing its spatial orientation, then, in Cartesian coordinates, it satisfies $d\lambda^\mu/d\tau = 0$ where τ is the proper time along the geodesic. That is, λ^μ is parallel transported through spacetime along the geodesic. Moreover, if at some point λ^μ is orthogonal to the tangent vector $\dot{x}^\mu = dx^\mu/d\tau$ to the geodesic, then $\eta_{\mu\nu}\lambda^\mu\dot{x}^\nu = 0$, and this relationship is preserved under parallel transport. This orthogonality condition simply means that λ^μ has no temporal component in an instantaneous rest frame of an observer traveling along the geodesic. The corresponding criteria for transporting a spacelike vector λ^μ in this fashion in the curved spacetime of general relativity are, therefore,

$$\frac{d\lambda^\mu}{d\tau} + \Gamma^\mu_{\nu\sigma}\lambda^\nu\dot{x}^\sigma = 0 \quad , \quad g_{\mu\nu}\lambda^\mu\dot{x}^\nu = 0$$

(a) Explain why these are the correct equations.

(b) Consider a spinning particle (perhaps a gyroscope) moving in a gravitational field. No non-gravitational forces are present. Write down and explain the equation which governs the behavior in time of the spin(vector) of the particle.

(c) Consider a slowly rotating thin spherical shell of mass M, radius R and rotation frequency ω. The metric of the field due to this shell can be written as

$$ds^2 = -c^2H(r)dt^2 + \frac{1}{H(r)}\left[dr^2 + r^2d\theta^2 + r^2\sin^2\theta(d\varphi - \Omega dt)^2\right]$$

where $\Omega = 4GM\omega/3Rc^2$ for $r < R$, $\Omega \to 0$ for $r \to \infty$, and

$$H(r) = \begin{cases} 1 - \frac{2GM}{rc^2} & r > R \\ 1 - \frac{2GM}{Rc^2} & r < R \end{cases}$$

This form of the metric is valid if $GM/Rc^2 << 1$. Consider a spinning particle at rest at the center of the sphere($r = 0$). Using the equation from part (b), with what frequency will the spin of the particle precess? What is the precession frequency quantitatively, if ω is the rotational frequency of the Earth and M and R, the mass and radius of the Earth, are $M \approx 6.0\times10^{27}\,gm$ and $R \approx 6.4\times10^3\,km$? A rough estimate is enough.

EP #49 - Kruskal Coordinates

Consider the Schwarzschild metric, which in (t, r, θ, φ) coordinates is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)c^2dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

(a) Show that if we define

$$u' = \left(\frac{r}{2M} - 1\right)^{1/2} e^{(r+t)/4M} \quad , \quad v' = -\left(\frac{r}{2M} - 1\right)^{1/2} e^{(r-t)/4M}$$

the metric in $u', v', \theta, \varphi)$ coordinates (Kruskal coordinates) is

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} du' dv' + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

(b) Find the locations in the (u,v) plane where this metric has singularities.

(c) What are the possible u,v values for events that can send signals to an event at ($u = u_0$,$v = v_0$)?

(d) What are the possible u,v values for events that can receive signals to an event at ($u = u_0$,$v = v_0$)?

(e) Consider a timelike observer in a cicular orbit at $r = 6M$. How is this described in Kruskal coordinates?

(f) What part of the spacetime cannot send signals to this observer? What part of the spacetime cannot receive signals from this observer?

EP #50 - Perturbing Circular Orbits

A particle is in a circular orbit around a black hole. It is perturbed so that the angular momentum is the same, but the energy is slightly increased so that there is a small velocity component outwards. Describe and sketch the resulting behavior, for initial radii $3M$, $4M$, $5M$, $6M$ and $7M$. HINT: you need to consider both the stability of the circular orbit and whether the particle has sufficient energy to escape to infinity.

EP #51 - Null Geodesics in Strange Metric

Consider the metric

$$ds^2 = -dt^2 + (1 - \lambda r^2)dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

where λ is a positive constant. Consider the null geodesics, and choosing coordinates so that the geodesics lie in the plane $\theta = \pi/2$, show that they satisfy

$$\left(\frac{dr}{d\varphi}\right)^2 = r^2(1 - \lambda r^2)\left(\mu r^2 - 1\right)$$

where μ is a constant. Integrate this and show that the paths of light rays are ellipses.

EP #52 - A Charged Black Hole

The metric for the spacetime around a static spherically symmetric source of mass M and charge Q(in appropriate units) is

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

This is called the Reissner-Nordstrom metric.

(a) Show that if $Q > M$, this metric is only singular at $r = 0$.

(b) For $Q < M$, the metric in this coordinate system is also singular at $r = r_\pm (r_+ > r_-)$. Find $r_\pm$ in terms of Q, M.

(c) Define a new coordinate u (analogous to Eddington-Finkelstein coordinates) so that the metric in (u, r, θ, φ) coordinates is regular at r_+.

EP #53 - Do Not Touch Anything!

An astronaut in command of a spaceship equipped with a powerful rocket motor enters the horizon $r = r_s$ of a Schwarzschild black hole.

(a) Prove that in proper time no larger than $r_s\pi/2$, the astronaut reaches the singularity at $r = 0$.

(b) Prove that in order to avoid the singularity for as long as possible, the astronaut ought to move in a purely radial direction. HINT: For purely radial motion, with $dr < 0$ and $dt = d\varphi = d\theta = 0$, show that the increment in proper time is

$$d\tau = -\frac{dr}{\sqrt{\frac{r_s}{r} - 1}} \quad , \quad for\ r \leq r_s$$

and then integrate this between $r = r_s$ and $r = 0$ to obtain

$$\Delta\tau = \frac{\pi r_s}{2}$$

Finally, check that if dt, $d\varphi$, $d\theta$ are different from zero, then the increment $d\tau$, for a given value of $-dr$, is necessarily smaller than the value given above.

(c) Show that in order to achieve the longest proper time the astronaut must use her rocket motor in the following way: outside the horizon, she must brake her fall so as to arrive at $r = r_s$ with nearly zero radial velocity; inside the horizon she must shut off her motor and fall freely. HINT: show that $\Delta\tau = \pi r_s/2$ corresponds to free fall from $r = r_s$ (do not do anything!).

EP #54 - Escape from Black Hole by Ejecting Mass

A spaceship whose mission is to study the environment around black holes is hovering at the Schwarzschild radius coordinate R outside a spherical black hole of mass M. To escape back to infinity, the crew must eject part of the rest mass of the ship to propel the remaining fraction to escape velocity. What is the largest fraction f of the rest mass that can escape to infinity? What happens to this fraction as R approaches the Schwarzschild radius of the black hole?

EP #55 - Gravitational Wave Stuff

(a) Explain briefly why in Einsteins theory of general relativity it is impossible to have monopole or dipole gravitational radiation.

(b) Suppose two compact stars, each of one solar mass, are in circular orbit around each other with a radius of one solar radius. What is the approximate rate of energy loss due to gravitational radiation from this system? What is the time scale for decay for this orbit? Take

$$\begin{aligned} &solar\ mass = 2 \times 10^{33} gm \\ &solar\ radius = 7 \times 10^{10} cm \end{aligned}$$

EP #56 - Waves from Masses on a Spring

Two equal masses M are at the ends of a massless spring of unstretched length L and spring constant k. The masses started oscillating in line with the spring with an amplitude A so that their center of mass remains fixed.

(a) Calculate the amplitude of gravitational radiation a long distance away from the center of mass of the spring as a function of the angle θ from the axis of the spring to lowest non-vanishing order in A.

(b) Analyze the polarization of the radiation.

(c) Calculate the angular distribution of power radiated in gravitational waves.

EP #57 - Waves from Accelerating Particle

A particle of mass m moves along the $z-$axis according to $z(t) = gt^2/2$ (g is a constant) between times $t = -T$ and $t = +T$ and is otherwise moving with constant speed. Calculate the gravitational wave metric perturbations at a large distance L along the positive z-axis.

EP #58 - Waves from Colliding Battleships

In a desperate attempt to generate gravitational radiation artificially, we take two large battleships of $70,000$ tons each, and we make them collide head-on at $40\,km/h$. Assume that during the collision the battleships decelerate at a constant rate and come to rest in $2.0\,sec$.

(a) Estimate the gravitational energy radiated during the collision. Treat the battleships as point masses.

(b) Could we detect these waves?

EP #59 - Waves from a Cannon

A cannon placed at the origin of coordinates fires a shot of mass $50\,kg$ in a horizontal direction. The barrel of the cannon has a length of $2.0\,m$; the shot has a uniform acceleration while in the barrel and emerges with a muzzle velocity of $300\,m/s$. Calculate the gravitational radiation field generated by the shot at a point P on the $z-$axis at a vertical distance of $20\,m$ above the cannon. What is the maximum value of the wave field? Ignore the gravitational field of the Earth.

EP #60 - Plane Wave Properties

Show that there is a coordinate choice so that the linearized vacuum Einstein equations are

$$\partial^2 h_{ab} = 0$$

where

$$g_{ab} = \eta_{ab} + h_{ab}$$

Find the plane wave solutions to this equation and explain why there are only two polarizations. The transverse trace-free polarization has basis

$$e_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad and \quad e_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

By taking the e_+ polarization, describe the physical effect of a gravitational plane wave.

EP #61 - Robertson-Walker = Minkowski

The Robertson-Walker line element for absolutely empty space, $T_i^j = 0$ and $\Lambda = 0$, is

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1+r^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right)$$

with $a(t) \propto t$. Show that this describes flat space and find the coordinate transformation that brings it to the Minkowski form.

EP #62 - Red Shift in Model Galaxy

Assume that the universe is isotropic and spatially flat. The metric then takes the form

$$ds^2 = -dt^2 + a^2(t)\left(dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right)$$

where r, θ, and φ are co-moving coordinates. By this is meant any galaxy will have constant values of r, θ, φ(peculiar motions of galaxies are neglected). The universe is assumed to be matter-dominated with matter density $\rho(t)$ at time t.

(a) Under this circumstance show that the Einstein equations are

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 \quad and \quad \ddot{a} = -\frac{4\pi G}{3}\rho a$$

(b) From the fact that light propagates along null geodesics, show that the cosmological red shift of spectral lines emitted at time t_e and received at time t_0, defined as

$$Z = \frac{wavelength\ of\ received\ line\ -\ wavelength\ of\ emitted\ line}{wavelength\ of\ emitted\ line}$$

is

$$Z = \frac{a_0}{a_e} - 1$$

where $a_0 = a(t_0)$, $a_e = a(t_e)$.

(c) In the cosmological model under discussion a given galaxy will decrease in angular size with increasing distance from the observer - up to a critical distance. Beyond this the angular size will increase with distance. What is the red shift Z_{crit} corresponding to the minimum in angular size?

EP #63 - Expanding Universe

The metric of the expanding universe has the form

$$ds^2 = dt^2 - R^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

where the possible curvature of space has been neglected. The detailed form of $R(t)$ depends on the matter content of the universe.

(a) A particle of mass m has energy E_0 and momentum p_0 at time t_0; assume $R(t_0) = R_0$. The particle thereafter propagates freely except for the effects of the above metric. Calculate the energy and momentum as a function of time.

(b) Suppose that the early universe contained a gas of non-interacting massless particle (perhaps photons) subject to gravitational effects only. Show that if at time t_0 they were in a thermal distribution at temperature T_0, they remained in a thermal distribution later, but with a temperature that depends on time in a fashion you should determine. HINT: EP60 shows that

$$photon\ frequencies\ change\ like: \frac{\nu'}{\nu} = \frac{R(t)}{R(t')}$$
$$volumes\ change\ like: \frac{V(t')}{V(t)} = \frac{R^3(t')}{R^3(t)}$$

(c) Show that, instead, a gas of non-interacting massive particles initially in a thermal distribution would not remain in a thermal distribution under the influence of the expansion of the universe.

(d) Suppose that the early universe contained a non-interacting gas of massless photons and also a non-interacting gas of massive particles of mass m (massive neutrinos to be definite). Suppose that at some early time the photons and neutrinos were both in a thermal distribution with a temperature $kT = mc^2$ (m being the neutrino mass) for both photons and neutrinos. It has been observed that in todays universe the photons are in a thermal distribution with kT about $3 \times 10^{-4}\, eV$. In terms of the neutrino mass, what (roughly) would be the typical velocity and kinetic energy of a neutrino today? Assume $m >> 3 \times 10^{-4}\, eV$.

EP #64 - Homogeneous, Isotropic Universe

Consider a homogeneous, isotropic cosmological model described by the line element

$$ds^2 = -dt^2 + \left(\frac{t}{t_*}\right)\left(dx^2 + dy^2 + dz^2\right)$$

where t_* is a constant.

(a) Is the model open, closed or flat?

(b) Is this a matter-dominated universe? Explain.

(c) Assuming the Friedmann equation holds for this universe, find $\rho(t)$.

EP #65 - Matter-Dominated RW Universe

Suppose that a galaxy is observed to have a red-shift $z = 1$. Assuming a matter-dominated RW cosmology, at what fraction t/t_0 of the present age of the universe did the light leave this galaxy?

EP #66 - Flat Dust Universe

Consider a flat dust universe with zero cosmological constant.

(a) Solve the cosmological equations and derive the time evolution of the scale parameter $a(t)$.

(b) By considering light emitted at time t, and received at the present time t_0, show that the distance to a star of red-shift z is given by

$$s = 3t_0\left(1 - \frac{1}{\sqrt{1+z}}\right)$$

(c) Explain why a flat universe with zero cosmological constant containing a mixture of dust and radiation will eventually be dominated by the dust.

EP #67 - Particle Horizon in Flat Dust Universe

The particle horizon is the radius of the sphere of all particles that could be seen by us. It is the maximum straight line distance that could be travelled by a light ray since the beginning of the universe. Obviously, in a static universe this would be t_0. What is it for a $k = 0$ dust universe?

EP #68 - The Horizon inside a Collapsing Shell

Consider the collapse of a spherical shell of matter of very small thickness and mass M. The shell describes a spherical three-surface in spacetime. Outside the surface, the geometry is the Schwarzschild geometry with this mass. Inside make the following assumptions:

1. The worldline of the shell is known as a function $r(\tau)$ going to zero at some finite proper time.
2. The geometry inside the shell is flat.
3. The geometry of the three-surface of the collapsing shell is the same inside as outside.

(a) Draw two spacetime diagrams: one an Eddington-Finkelstein diagram and the other corresponding to the spacetime inside in a suitable set of coordinates. Draw the worldline of the shell on both diagrams and indicate how points on the inside and outside correspond. Locate the horizon inside the shell as well as outside.

(b) How does the area of the horizon inside the shell change moving along the light rays which generate it?

EP #69 - Two Observers on a Kruskal Diagram

Two observers in two rockets are hovering above a Schwarzschild black hole of mass M. They hover at fixed radius R such that

$$\left(\frac{R}{2M} - 1\right)^{1/2} e^{R/4M} = \frac{1}{2}$$

and fixed angular position. (In fact $R \approx 2.16M$). The first observer leaves this position at $t = 0$ and travels into the black hole on a straight line in a Kruskal diagram until destroyed in the singularity at the point where the singularity crosses $u = 0$. The other observer continues to hover at R.

(a) On a Kruskal diagram sketch the worldlines of the two observers.

(b) Is the observer who goes into the black hole following a timelike worldline?

(c) What is the latest Schwarzschild time after the first observer departs that the other observer can send a light signal which will reach the first before being destroyed in the singularity?

EP #70 - $k = 1$ Robertson-Walker Spacetime

Suppose that the universe is described by a $k = 1$ Robertson-Walker spacetime with metric

$$ds^2 = -dt^2 + R^2(t)dx^2 + \sin^2 x \left(d\theta^2 + \sin^2 \theta d\varphi^2\right)$$

with $R(t) = R_0 t^{2/3}$ at the present epoch. An observer at $t = t_1$ observes a distant galaxy of proper size D perpendicular to the line of sight at $t = t_0$.

(a) What is the observed red shift in terms of R_0, t_0, t_1?

(b) What is the angular diameter of the galaxy, δ, in terms of the red shift?

(c) Show that as the red shift increases δ reaches a minimum for fixed D and then starts to increase.

EP #71 - General Robertson-Walker Spacetime

The Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right)$$

where $\kappa = 0, +1, -1$, according to whether the 3-dimensional space has zero, positive or negative curvature, respectively, gives rise to the first order Einstein field equation

$$\dot{a}^2 + \kappa = \frac{8\pi G}{3} a^2 \quad , \quad \rho a^3 = \text{constant}$$

for a matter-dominated universe of density ρ.

(a) Derive the above field equation.

(b) Calculate the distance $L_r(t)$ from the origin($r = 0$) to a particle with coordinate r at time t, in terms of r, $a(t)$.

Alternatively, we can formulate the theory in purely classical Newtonian terms by ignoring curvature inside a spherical volume of sufficiently small radius, i.e., assume that the space is flat inside the sphere and that any isotropic distribution of matter outside has no effect on curvature inside.

(c) Write down Newtons equation for the acceleration of a particle towards the origin at a distance L away. HINT: Consider a uniform distribution of matter inside a sphere of radius L.

(d) To conserve matter, we must also have $\rho a^3 =$ constant. Combine this with your result in (c) to determine the equations satisfied by the expansion parameter $a(t)$ and compare your answer with the cosmological one.

EP #72 - Spaceship in Robertson-Walker Spacetime

Assume that the geometry of the universe is described by Robertson-Walker metric (c=1)

$$ds^2 = -dt^2 + R^2(t)\left(\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right)$$

A spaceship sets out with velocity v relative to cosmological observers. At a later time when the universe has expanded by a scale factor $(1+z)$, find the velocity v with respect to cosmological observers.

EP #73 - Equation of State

(a) The equation of state is often written in adiabatic form where p is pressure and ρ is density and $0 \leq \gamma \leq 2$ is the adiabatic index with $\gamma = 0$ for dust and $\gamma = 4/3$ for radiation. Calculate $\rho(a)$ for general γ. For $k = 0$, calculate $a(t)$. Find the age of the universe for $k = 0$ and general γ.

(b) In the same notation as (a), find γ so that the expansion rate is constant. With this value of find $a(t)$ for $k = 1$ and $k = -1$.

(c) In the same notation as (a), show

$$\dot{\Omega} = (2 - 3\gamma)H\Omega(1 - \Omega)$$

Define the logarithmic scale factor $s = \log(a)$ and write an equation for $d\Omega/ds$. Notice that this formula gives a clear idea how Ω behaves.

EP #74 - Flat Universe with Period of Inflation

Consider a simplified model of the history of a flat universe involving a period of inflation. The history is split into four periods

1. $0 < t < t_3$ radiation only
2. $t_3 < t < t_2$ vacuum energy dominates with an effective cosmological constant $\Lambda = 3/(4t_3^2)$
3. $t_2 < t < t_1$ a period of radiation dominance
4. $t_1 < t < t_0$ matter domination

(a) Show that in (3) $\rho(t) = \rho_r(t) = 3/(32\pi t^2)$ and in (4) $\rho(t) = \rho_m(t) = 1/(6\pi t^2)$. The functions ρ_r and ρ_m are introduced for later convenience.

(b) Give simple analytic formulas for $a(t)$ which are approximately true in the four epochs.

(c) Show that during the inflationary epoch the universe expands by a factor

$$\frac{a(t_2)}{a(t_3)} = \exp\left(\frac{t_2 - t_3}{2t_3}\right)$$

(d) Show that

$$\frac{\rho_r(t_0)}{\rho_m(t_0)} = \frac{9}{16}\left(\frac{t_1}{t_0}\right)^{2/3}$$

(e) If $t_3 = 10^{-35}$ seconds, $t_2 = 10^{-32}$ seconds, $t_1 = 10^4$ years and $t_0 = 10^{10}$ years, give a sketch of $\log(a)$ versus $\log(t)$ marking any important epochs.

(f) Define what is meant by the particle horizon and calculate how it behaves for this model. Indicate this behavior on the sketch you made. How does inflation solve the horizon problem?

EP #75 - Worm-Hole Metric

Consider the *worm-hole* metric

$$ds^2 = dt^2 - dr^2 - (b^2 + r^2)d\Omega^2$$

Try and work out why this curve is known as a warp-drive.

(a) Find the Christoffel symbols for this geometry.

(b) Find the geodesic equations for this geometry.

EP #76 - Alcubierre Warp-Drive Spacetime

Consider the spacetime known as the Alcubierre Warp-Drive. The coordinates are t, x, y, z and consider a (not necessarily time-like) trajectory given by $x = x_s(t), y = 0, z = 0$. Then the warp-drive spacetime is given by the following metric

$$ds^2 = dt^2 - [dx - v_s(t)f(r_s)dt]^2 - dy^2 - dz^2$$

where $v_s(t) = dx_s(t)/dt$ is the velocity associated with the curve and $r_s^2 = \left[(x - x_s(t))^2 + y^2 + z^2\right]$ determines the distance of any point from the curve. The function f is smooth and positive with $f(0) = 1$ and vanishes whenever $r_s > R$ for some R. Notice that if we restrict to a curve with constant t, then the the metric is flat and that the metric is flat whenever a spacetime point is sufficiently far away from $x_s(t)$.

(a) Find the null geodesics $ds^2 = 0$ for this spacetime and draw a space-time diagram with some forward and backward light cones along the path $x_s(t)$.

(b) Check that the curve $x_s(t)$ is a geodesic and show that at every point along this curve the 4−velocity of the ship lies within the forward light cone.

(c) Consider the path $x_s(t)$ that connects coordinate time 0 with coordinate time T. How much time elapses for a spaceship traveling along $x_s(t)$?

(d) Calculate the components of a 4−vectors normal to a surface of constant t.

(e) Show that

$$T_{\alpha\beta}\eta^{\alpha}\eta^{\beta} = -\frac{1}{8\pi}\frac{v_s^2\left(y^2+z^2\right)}{2r_s^2}\left(\frac{df}{dr_s}\right)^2$$

This is the energy density measured by observers at rest with respect to the surfaces of constant t. The fact that it is negative means that the warp-drive spacetime cannot be supported by ordinary matter!

EP #77 - General Relativistic Twins

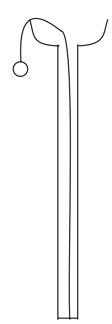
Paul is orbiting a neutron star at a distance of $4GM/c^2$, in circular orbit. Patty, his twin sister, is fired radially outward from the surface with less than escape velocity. Her path crosses Paul's orbit just as Paul comes by, so they synchronize their clocks. On her way back down, Patty again encounters Paul as their trajectories cross, Paul having completed 10 orbits between their meetings. They again compare clocks. How much do their clocks disagree?

EP #78 - Hollow Ball in a Bucket

A hollow plastic ball is held at the bottom of a bucket of water and then released. As it is released, the bucket is dropped over the edge of a cliff. What happens to the ball as the bucket falls?

EP #79 - Einstein's Birthday Present

A version of this device (call an *equivalence principle device* was constructed as a birthday present for Albert Einstein. Simplified, the device consists of a hollow tube with a cup at the top, together with a metal ball and an elastic string as shown below.



When the tube is held vertical, the ball can rest in the cup. The ball is attached to one end of the elastic string, which passes through the hole in the bottom of the cup, and down the hollow center of the tube to the bottom, where the other end is secured. You hold the tube vertical, with your hand at the bottom, the cup at the top, and with the ball out of the cup, suspended on the elastic string. The tension in the string is not quite sufficient to draw the ball back into the cup. The problem is to find an *elegant* way to get the ball back into the cup.

EP #80 - Accelerating Pendulum

A pendulum consists of a light rod and a heavy bob. Initially it is at rest in a vertical stable equilibrium. The upper end is then made to accelerate down a straight line which makes an angle α with the horizontal with constant acceleration f. Show that in the subsequent motion, the pendulum oscillates between the vertical and the horizontal positions if $g = f(\cos\alpha + \sin\alpha)$. This problem is very easy if you apply the equivalence principle and think abou the direction of the apparent gravitational field in an appropriate frame.

EP #81 - What is going on?

For each of the following, either write out the equation with the summation signs included explicitly or say in a few words why the equation is ambiguous or does not make sense.

(i) $x^a = L^a_b M^b_c \hat{(x)}^c$

(ii) $x^a = L^b_c M^c_d \hat{x}^d$

(iii) $\delta^a_b = \delta^a_c \delta^c_d \delta^d_b$

(iv) $\delta^a_b = \delta^a_c \delta^c_c \delta^c_b$

(v) $x^a = L^a_b \hat{x}^b + M^a_b \hat{x}^b$

(vi) $x^a = L^a_b \hat{x}^b + M^a_c \hat{x}^c$

(vii) $x^a = L^a_c \hat{x}^c + M^b_c \hat{x}^c$

EP #82 - Does It Transform Correctly?

Show that if X and Y are vector fields on a manifold, then so is

$$Z^a = X^b \partial_b Y^a - Y^b \partial_b X^a$$

i.e., show that Z transforms correctly under a change of coordinates.

EP #83 - Closed Static Universe

Einstein proposed the following metric as a model for a closed static universe

$$ds^2 = dt^2 - dr^2 - \sin^2 r(d\theta^2 + \sin^2\theta d\phi^2)$$

Find the geodesic equation of the metric from Lagrange's equations and hence write down the Christoffel symbols (take $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$). Show that there are geodesics on which r and θ are constant and equal to $\pi/2$.

EP #84 - Strange Metric

Write down the geodesic equations for the metric

$$ds^2 = dudv + \log(x^2 + y^2)du^2 - dx^2 - dy^2$$

($0 < x^2 + y^2 < 1$). Show that $K = x\dot{y} - y\dot{x}$ is a constant of the motion.

By considering an equivalent problem in Newtonian mechanics, show that no geodesic on which $K \neq 0$ can reach $x^2 + y^2 = 0$.

EP #85 - Clocks in Schwarzschild Spacetime

A clock is said to be *at rest* in the Schwarzschild space-time if the r, θ, and ϕ coordinates are constant. Show that the coordinate time and the proper time along the clock's worldline are related by

$$\frac{dt}{d\tau} = \left(1 - \frac{2m}{r}\right)^{-1/2}$$

Note that the worldline is not a geodesic.

Show that along a radial null geodesic, that is, one on which only t and r are varying, that

$$\frac{dt}{dr} = \frac{r}{r - 2m}$$

Two clocks C_1 and C_2 are at rest at (r_1, θ, ϕ) and (r_2, θ, ϕ). A photon is emitted from C_1 at event A and arrives at C_2 at event B. A second photon is emitted from C_1 at event A' and arrives at C_2 at event B'. Show that the coordinate time interval Δt between A and A' is the same as the coordinate time interval between B and B'. Hence show that the time interval $\Delta\tau_1$ between A and A' measured by C_1 is related to the time interval $\Delta\tau_2$ between B and B' measured by C_2 by

$$\Delta\tau_1 \left(1 - \frac{2m}{r_1}\right)^{-1/2} = \Delta\tau_2 \left(1 - \frac{2m}{r_2}\right)^{-1/2}$$

If you wear two watches, one on your wrist and one on your ankle, and you synchronize them at the beginning of the year, by how much is the watch on your wrist faster or slower than the one on your ankle at the end of the year? (Assume that you spend the whole year standing upright without moving. In general units, you must replace m/r by Gm/rc^2).

EP #86 - Particle Motion in Schwarzschild Spacetime

Show that along free particle worldlines in the equatorial plane of the Schwarzschild metric, the quantities

$$J = r^2\dot{\phi} \quad \text{and} \quad E = \left(1 - \frac{2m}{r}\right)\dot{t}$$

are constant. Remember the dot is the derivative with respect to proper time. Explain why the particle cannot escape to infinity if $E < 1$.

Show that

$$\dot{r}^2 + \left(1 + \frac{J^2}{r^2}\right)\left(1 - \frac{2m}{r}\right) = E^2$$

$$\ddot{r} + \frac{m}{r^2} - \frac{J^2}{r^3} + 3\frac{mJ^2}{r^4} = 0$$

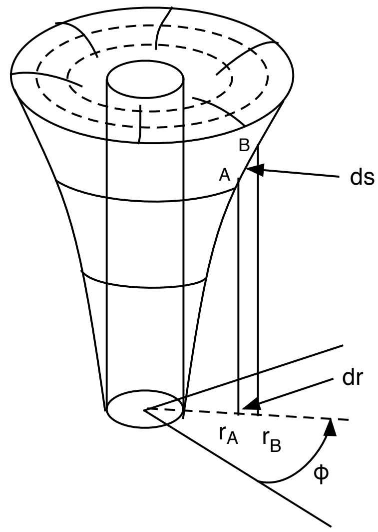
For a circular orbit at radius $r = R$, show that

$$J^2 = \frac{mR^2}{R - 3m} \quad , \quad \frac{d\phi}{dt} = \left(\frac{m}{R^3}\right)^{1/2}$$

Show by letting $r(\tau) = R + \epsilon(\tau)$, with ϵ small, that the circular orbit is stable if and only if $R > 6m$.

EP #87 - Meter Stick Near Black Hole

A standard meter stick lies on the surface shown below(AB). The surface is the two-dimensional riemannian surface defined by the Schwarzschild metric with two coordinates held constant (t = constant, $\theta = \pi/2$ = constant) as viewed(embedded) in three-dimensional euclidean space. The meter stick is oriented in the radial direction. It is then slid inward toward the symmetry axis. The location of its two ends at a given instant in t are reported to a record keeper who plots the the two points shown in the (r, ϕ) plane.



(a) What does the record keeper actually see?

(b) How is the record keeper able to keep track of what is happening physically?

EP #88 - Clocks and Rockets

A rocket of proper length$L$ leaves the earth vertically at speed $4c/5$. A light signal is sent vertically after it which arrives at the rocket's tail at $t = 0$ according to both the rocket and earth based clocks. When does the signal reach the nose of the rocket according to (a) the rocket clocks and (b) according to the earth clocks?

EP #89 - Events in Two Frames

In an inertial frame two events occur simultaneously at a distance of 3 meters apart. In a frame moving with to the inertial or laboratory frame, one event occurs later than the other by $10^{-8}\,sec$. By what spatial distance are the two events separated in the moving frame? Solve this problem in two ways: first by finding the Lorentz boost that connects the two frames, and second by making use of the invariance of the spacetime interval between two events.

EP #90 - Geometry in a Curved Space

In a certain spacetime geometry the metric is

$$ds^2 = -(1 - Ar^2)^2 dt62 + (1 - Ar^2)^2 dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

(a) Calculate the proper distance along a radial line from the center $r = 0$ to a coordinate radius $r = R$.

(b) Calculate the area of a sphere of coordinate radius $r = R$.

(c) Calculate the three-volume of a sphere of coordinate radius $r = R$.

(d) Calculate the four-volume of a four-dimensional tube bounded by the sphere of coordinate radius R and two $t =$ constant planes separated by time T.

EP #91 - Rotating Frames

The line element of flat spacetime in a frame (t, x, y, z) that is rotating with an angular velocity Ω about the $z-$axis of an inertial frame is

$$ds^2 = -[1 - \Omega^2(x^2 + y^2)]dt^2 + 2\Omega(ydx - xdy)dt + dx^2 + dy^2 + dz^2$$

(a) Find the geodesic equations for x, y, and z in the rotating frame.

(b) Show that in the non-relativistic limit these reduce to the usual equations of Newtonian mechanics for a free particle in a rotating frame exhibiting the centrifugal force and the Coriolis force.

EP #92 - Negative Mass

Negative mass does not occur in nature. But just as an exercise analyze the behavior of radial light rays in a Schwarzschild geometry with a negative value of mass M. Sketch the Eddington-Finkelstein diagram showing these light rays. Is the negative mass Schwarzschild geometry a black hole?