Math for physicists: differential forms

Disclaimer: this is a first draft, so a few signs might be off.

1 Basic properties

Differential forms come up in various parts of theoretical physics, including advanced classical mechanics, electromagnetism, thermodynamics, general relativity, and quantum field theory. So they're well worth knowing about. This is supposed to be a self-contained exposition for someone who has some knowledge of multivariable calculus.

Forms are like infinitesimal objects, but this is (or can be made) a completely rigorous subject. The most basic object is exterior derivative, d. For a function $F(x_1, \ldots, x_n)$,

$$dF = \frac{\partial F}{\partial x_i} dx_i \tag{1}$$

by definition. Summation over i from 1 to n is implicit. In some subjects (particularly relativity), there is a preference to write x^i and/or dx^i, but I will eschew this here. Eq. (1) looks like an infinitesimal variation. The special thing about forms is that they involve an anti-commuting wedge product, $\wedge$, defined so that

$$dx_i \wedge dx_j = -dx_j \wedge dx_i \,. \tag{2}$$

A general p-form (for $p \leq n$) is

$$\omega = \frac{1}{p!} \omega_{i_1 \ldots i_p} dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p} \,. \tag{3}$$

Again there is an implicit sum on i_1 through i_p, each from 1 to n. The coefficient functions $\omega_{i_1 \wedge i_p}$ may be assumed to be antisymmetric in i_1 through i_p. So for instance, if $p = 2$, $\omega_{ij} = -\omega_{ji}$. The $1/p!$ is there in (3) because the sum contains $p!$ copies of each term. Again for the case of 2-forms, say now in 2 dimensions, we would have

$$\omega = \frac{1}{2} \omega_{ij} dx_i \wedge dx_j = \omega_{12} dx_1 \wedge dx_2 \,. \tag{4}$$

The middle expression, explicitly, is $\frac{1}{2}(\omega_{12} dx_1 \wedge dx_2 + \omega_{21} dx_2 \wedge dx_1)$, but the last expression is obviously equal to this.

The exterior derivative of a p-form can be defined as

$$d\omega = \frac{\partial \omega_{i_1 \ldots i_p}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_p} \,. \tag{5}$$

It's crucial to put the dx_j consistently where I did. You can verify the following product rule for d:

$$d(\mu_{(p)} \wedge \nu_{(q)}) = d\mu_{(p)} \wedge \nu_{(q)} + (-1)^p \mu_{(p)} \wedge d\nu_{(q)} \,, \tag{6}$$

where by $\mu_{(p)}$ I mean a p-form. And you can show that $d^2 = 0$, in the sense that if $\mu = d\omega$, then $d\mu = 0$. An important and non-trivial theorem says that if $d\mu = 0$, then $\mu = d\omega$ for some ω.[1]

Differential forms were invented for integration. Here's how it works. Suppose you have some integration region $\mathcal{M}$, which could be any smooth p-surface in $\mathbf{R}^n$, and suppose you have a set of smooth 1-1 functions, $(\xi_1, \ldots, \xi_p) \to (x_1, \ldots, x_n)$, which map a *fiducial* p-dimensional integration region $\mathcal{I}$ into $\mathcal{M}$. For instance, I could be $[0,1]^p$ in $\mathbf{R}^p$. Then, for any p-form ω, we define

$$\int_{\mathcal{M}} \omega = \int_{\mathcal{I}} \tfrac{1}{p!} \omega_{i_1 \ldots i_p} \frac{\partial(x_{i_1}, \ldots, x_{i_p})}{\partial(\xi_1, \ldots, \xi_p)} d^p \xi \,. \tag{7}$$

Here $\partial(x_{i_1}, \ldots, x_{i_p})/\partial(\xi_1, \ldots, \xi_p)$ denotes a Jacobian: that is, the determinant of the matrix of partial derivatives $\partial x_{i_j}/\partial \xi_k$, where j and k run from 1 to p. The definition (7) would follow, formally, if we wrote $dx_i = \frac{\partial x_i}{\partial \xi_j} d\xi_j$. (Try it!)

The most important theorem in this subject is Stokes' Theorem, which says that if ω is a $(p-1)$-form and $\mathcal{M}$ is a p-suface, then

$$\int_{\mathcal{M}} d\omega = \int_{\partial \mathcal{M}} \omega \,, \tag{8}$$

where $\partial \mathcal{M}$ is the boundary of $\mathcal{M}$. Actually, there's something a little tricky about (8): both $\mathcal{M}$ and $\partial \mathcal{M}$ have *orientations* which must be specified, and picking them in some ways gives a minus sign in front of the right hand side of (8). To understand this orientation business, note that if you replaced ξ_1 by $-\xi_1$, or swapped ξ_1 and ξ_2, in (7), it would change the sign of the Jacobian, and hence the sign of the integral. It's not a priori obvious how to relate the orientations of $\mathcal{M}$ and $\partial \mathcal{M}$ in general, but this is getting beyond the scope of what we really need to know.

2 Div, grad, and curl

Let's now consider some examples. Start with a function f on $\mathbf{R}^3$. Its exterior derivative, $v = df$, can be written as $v = v_i dx_i$, where $\vec{v} = (v_1, v_2, v_3) = \nabla f$ is the gradient.

[1] This is true up to cohomology. So it's precisely true if we're working on $\mathbf{R}^n$, or on any other space that's contractable to a point.

So vectors can be turned into one-forms. Now take v to be *any* one-form, and you get

$$\begin{aligned} dv &= \frac{\partial v_i}{\partial x_j} dx_j \wedge dx_i \\ &= \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right) dx_2 \wedge dx_3 + \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right) dx_3 \wedge dx_1 + \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) dx_1 \wedge dx_2 \\ &= w_1 dx_2 \wedge dx_3 + w_2 dx_3 \wedge dx_1 + w_3 dx_1 \wedge dx_2 \equiv w \end{aligned} \tag{9}$$

where $\vec{w} = (w_1, w_2, w_3) = \nabla \times \vec{v}$. Observe that if $\vec{v} = \nabla f$, then $\vec{w} = \nabla \times \vec{v} = 0$. And recall that if a vector field has no curl, then it's the gradient of some function: that's why we can write $\vec{E} = -\nabla\Phi$ in electrostatics. These statements are special cases of $d^2 = 0$ and the deeper theorem that if $d\mu = 0$ then $\mu = d\omega$ for some ω. An important point is that a vector $\vec{w}$ is naturally associated to a 2-form, in the way we wrote in (9). This is a special property of three dimensions: in n dimensions, a vector would be naturally associated to either a 1-form or a $(n-1)$-form.

Continuing on, let's consider any $\vec{w}$ and the associated 2-form w. Differentiating once more gives

$$dw = \left(\frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2} + \frac{\partial w_3}{\partial x_3}\right) dx_1 \wedge dx_2 \wedge dx_3 = (\nabla \cdot \vec{w}) dx_1 \wedge dx_2 \wedge dx_3 \,. \tag{10}$$

Obviously, if $w = dv$, then $dw = 0$. This is precisely the statement that the curl of a vector field has no divergence. And recall also that if a vector field is divergence-free, then it's the curl of something: that's why we can always write $\vec{B} = \nabla \times \vec{A}$ in electromagnetism. Again, these statements are special cases of $d^2 = 0$ and $d\mu = 0 \quad\Rightarrow\quad \mu = d\omega$.

We can pursue our vector calculus examples further to get a couple of special cases of Stokes' Theorem. For instance, if we have a vector field $\vec{w}$ on $\mathbf{R}^3$, then we know how to integrate its divergence over a simply connected region $\mathcal{M}$ (simply connected just means, no holes):

$$\int_{\mathcal{M}} \nabla \cdot \vec{w}\, d^3x = \int_{\partial\mathcal{M}} \hat{n} \cdot \vec{w}\, \sqrt{g}\, d^2\xi \,, \tag{11}$$

where $\hat{n}$ is the outward-pointing unit vector normal to the boundary $\partial\mathcal{M}$. This is sometimes called Gauss's Theorem. Saying that $\hat{n}$ points outward amounts to specifying the orientation of $\partial\mathcal{M}$. In (11), $\sqrt{g}\, d^2\xi$ is by definition the area element on $\partial\mathcal{M}$. So if $\mathcal{M}$ were a ball, and we used angular coordinates, then $\sqrt{g}\, d^2\xi = \sin\theta d\theta d\phi$. I've slipped in the $\sqrt{g}$ because something has to tell you that there's a non-trivial measure on the surface. A more concise way to phrase (11), which avoids needing to discuss complicated things like measures, is just

$$\int_{\mathcal{M}} dw = \int_{\partial\mathcal{M}} w \,. \tag{12}$$

The left side hardly needs explaining: we just plug in (10) for dw. On the right hand side, to integrate over $\partial\mathcal{M}$, we require a map $(\xi_1, \xi_2) \to (x_1, x_2, x_3)$, and then we use the definition (7). For the case where $\mathcal{M}$ is a unit ball, we could take $(\xi_1, \xi_2) = (\theta, \phi)$, and $\vec{x} = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta)$. With some work, you can show that the right hand side of (12) is the same as our previous expression in (11).[2]

Another example, in two dimensions, is Green's theorem: if we have a vector field $\vec{v} = (v_x, v_y)$, and a simply connected region $\mathcal{M}$, then

$$\int_{\mathcal{M}} \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) = \int_{\partial\mathcal{M}} \vec{v} \cdot d\vec{\ell} = \int_{\partial\mathcal{M}} (v_x dx + v_y dy) \,, \tag{13}$$

where we traverse $\partial\mathcal{M}$ counterclockwise (this is the specification of orientation). You should be able to convince yourself in short order that this can be concisely rephrased as $\int_{\mathcal{M}} dv = \int_{\partial\mathcal{M}} v$, where the right hand side is defined via a map that circles around $\partial\mathcal{M}$ counterclockwise as ξ ranges from 0 to 1. We could give a similar treatment of the classic statement of Stokes' Theorem (really another small special case of (8)) relating with a line integral over the boundary of a curved surface in three dimensions... but hopefully the basic ideas are by now obvious.

3 Partial derivatives and forms

An important aspect of differential forms is their relation to partial derivatives. Suppose we have different ways of coordinatizing the same space: say (x_1, x_2) and (y_1, y_2). If you like, the x coordinates could be Cartesian coordinates on $\mathbf{R}^2$ while the y coordinates are polar coordinates. But this is only an example. Another example is for the x coordinates to be q and p for a classical system with one degree of freedom, while the y coordinates are some other Q and P related by a canonical transformation. What we want in general is for there to be a smooth 1-1 map from the x coordinates to the y coordinates. In generic situations (that is, barring some "accident" like $x_1 = y_2$), we can choose *any* two of the four variables (x_1, x_2, y_1, y_2) to parametrize $\mathbf{R}^2$. So if we start with a function $f(x_1, x_2)$, we could write it instead as a function of y_1 and y_2, or x_1 and y_2, or whatever we please. With this in mind, we should write partial derivatives of f like this:

$$df = \left(\frac{\partial f}{\partial x_1} \right)_{x_2} dx_1 + \left(\frac{\partial f}{\partial x_2} \right)_{x_1} dx_2 \,, \tag{14}$$

[2]The best way to go about this is to say $(\xi_0, \xi_1, \xi_2) = (r, \theta, \phi)$, and then show $\partial(x_1, x_2, x_3)/\partial(\xi_0, \xi_1, \xi_2) = r^2 \sin\theta$. Now the result should be obvious at least for $w = f(r) \sin\theta d\theta \wedge d\phi$, which corresponds to $\vec{w}$ pointing in the radial direction.

where the $()_u$ notation means that u is held fixed. But it seems we could as well have written

$$df = \left(\frac{\partial f}{\partial y_1}\right)_{y_2} dy_1 + \left(\frac{\partial f}{\partial y_2}\right)_{y_1} dy_2 \,. \tag{15}$$

Equating these two reproduces the well-known multi-variable chain rule:

$$\begin{pmatrix} \partial f/\partial y_1 \\ \partial f/\partial y_2 \end{pmatrix} = \begin{pmatrix} \partial x_1/\partial y_1 & \partial x_2/\partial y_1 \\ \partial x_1/\partial y_2 & \partial x_2/\partial y_2 \end{pmatrix} \begin{pmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{pmatrix} , \tag{16}$$

where now to avoid clutter I just leave it implicit that the partials on the left hand side hold either y_1 or y_2 fixed, whereas the ones on the right hand side hold either x_1 or x_2 fixed. A less obvious application of forms is to prove the relation

$$\left(\frac{\partial f}{\partial x_1}\right)_{y_1} = \left(\frac{\partial f}{\partial x_1}\right)_{x_2} - \left(\frac{\partial f}{\partial x_2}\right)_{x_1} \frac{(\partial y_1/\partial x_1)_{x_2}}{(\partial y_1/\partial x_2)_{x_1}} \tag{17}$$

To prove this, we can set

$$dy_1 = \left(\frac{\partial y_1}{\partial x_1}\right)_{x_2} dx_1 + \left(\frac{\partial y_1}{\partial x_2}\right)_{x_1} dx_2 = 0 \,, \tag{18}$$

solve for dx_2, and plug back into (14) to get

$$df = \left[\left(\frac{\partial f}{\partial x_1}\right)_{x_2} - \left(\frac{\partial f}{\partial x_2}\right)_{x_1} \frac{(\partial y_1/\partial x_1)_{x_2}}{(\partial y_1/\partial x_2)_{x_1}}\right] dx_1 \qquad \text{with } dy_1 = 0. \tag{19}$$

Now dividing by dx_1 gives (17). That wasn't quite rigorous, since I switched from regarding df and dx_1 as one-forms to regarding them as infinitesimals. A more rigorous method would be to equate (14) to

$$df = \left(\frac{\partial f}{\partial x_1}\right)_{y_1} dx_1 + \left(\frac{\partial f}{\partial y_1}\right)_{x_1} dy_1 \tag{20}$$

and then set $dy_1 = 0$, using (18). A special case of (17) is to set $f = x_2$; then we obtain

$$\left(\frac{\partial x_2}{\partial x_1}\right)_{y_1} = -\frac{(\partial y_1/\partial x_1)_{x_2}}{(\partial y_1/\partial x_2)_{x_1}} = -\left(\frac{\partial x_2}{\partial y_1}\right)_{x_1} \left(\frac{\partial y_1}{\partial x_1}\right)_{x_2} . \tag{21}$$

Plugging back into (17), we obtain the simpler identity

$$\left(\frac{\partial f}{\partial x_1}\right)_{y_1} = \left(\frac{\partial f}{\partial x_1}\right)_{x_2} + \left(\frac{\partial f}{\partial x_2}\right)_{x_1} \left(\frac{\partial x_2}{\partial x_1}\right)_{y_1} \tag{22}$$

4 Applications to physics

The relations (21) and (22) are quite useful in thermodynamics. For instance, we could them to relate specific heats at fixed volume versus fixed pressure:

$$\begin{aligned} C_P = T\left(\frac{\partial S}{\partial T}\right)_P &= T\left(\frac{\partial S}{\partial T}\right)_V + T\left(\frac{\partial S}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_P \\ &= C_V + T\left(\frac{\partial P}{\partial T}\right)_V\left(\frac{\partial V}{\partial T}\right)_P = C_V - T\left(\frac{\partial P}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_P^2 . \end{aligned} \tag{23}$$

In the third equality we used a Maxwell relation, $(\partial S/\partial V)_T = (\partial P/\partial T)_V$, which follows from the fact that both sides can be expressed as $-\partial^2 F/\partial V \partial T$, where $F = F(V,T)$ is the Helmholtz free energy. Defining the isothermal compressibility, $K_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$, and the expansion coefficient $\beta_0 = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P$, we find (c.f. Mandl, p. 123ff)

$$C_P - C_V = TV\beta_0^2/K_T \,, \tag{24}$$

which has been called "the most beautiful relation in thermodynamics" (I think because it's hard to remember how to derive it). The crucial step in the derivation is the second equality in (23), which came from parametrizing the possible equilibrium states of the system either by (V,T) (the canonical ensemble) or by (P,T) (which we might call the Gibbs ensemble since it's associated with the Gibbs free energy—but don't get confused between this and the grand canonical ensemble, where particle number fluctuates). Specializing to an ideal gas, we have $PV = NkT$, and (24) becomes

$$C_P = C_V + Nk \,, \tag{25}$$

which you probably learned in high school. Incidentally, the equipartition theorem applies most directly to C_V.

After all this, hopefully the exposition of canonical transformations given in class makes better sense. It's worth making a few additional comments. The Poisson bracket structure is equivalent in information content to specifying a so-called "symplectic form" on phase space:

$$\omega = \sum_{k=1}^{N} dq_k \wedge dp_k \,. \tag{26}$$

Canonical transformations by definition preserve the symplectic form: that is, if the canonically transformed coordinates are Q_k and P_k, then

$$\omega = \sum_{k=1}^{N} dQ_k \wedge dP_k \tag{27}$$

is the same form as we had in (26). Once we realize that Hamiltonian flow generates canonical transformations, we can give an intuitive proof of Liouville's Theorem in just a few lines. Observe that

$$\mathrm{vol} = \frac{1}{N!} \overbrace{\omega \wedge \ldots \wedge \omega}^{N \text{ times}} = \pm\, dq_1 \wedge \ldots \wedge dq_N \wedge dp_1 \wedge \ldots \wedge dp_N \,. \tag{28}$$

Ignoring the sign (which only depends on N), we see that vol is precisely the volume element on phase space. It is preserved by Hamiltonian flow because ω is. This is the essential content of Liouville's Theorem: phase space volume is invariant.

Differential forms give a very natural formulation of electromagnetism. In $\mathbf{R}^{3,1}$ (Minkowski space), suppose we construct

$$\begin{aligned} F &= dt \wedge (E_x dx + E_y dy + E_z dz) - B_x dy \wedge dz - B_y dz \wedge dx - B_z dx \wedge dy \\ \tilde{F} &= dt \wedge (B_x dx + B_y dy + B_z dz) + E_x dy \wedge dz + E_y dz \wedge dx + E_z dx \wedge dy \\ J &= \rho dx \wedge dy \wedge dz - dt \wedge (J_x dy \wedge dz + J_y dz \wedge dx + J_z dx \wedge dy) \\ A &= \Phi dt - A_x dx - A_y dy - A_z dz \,. \end{aligned} \tag{29}$$

Here $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields; ρ and $\vec{J}$ are charge and current density; and Φ and $\vec{A}$ are the electrostatic potential and vector potential. F is called the field strength tensor; $\tilde{F}$ is its "Hodge dual;" and A is called the gauge potential. It's straightforward to verify that the basic equations of electromagnetism (in conventions where $c = 1$—c.f. Jackson p. 548ff),

$$\begin{aligned} &\nabla \cdot \vec{E} = 4\pi\rho && \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \\ &\nabla \cdot \vec{B} = 0 && \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 4\pi \vec{J} \\ &\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t} && \vec{B} = \nabla \times \vec{A} \\ &\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0 \,, \end{aligned} \tag{30}$$

can be rephrased as

$$\begin{aligned} dF &= 0 \qquad & d\tilde{F} &= 4\pi J \\ F &= dA \qquad & dJ &= 0 \,. \end{aligned} \tag{31}$$

Note that the second line is entirely a consequence of the first, given $d^2 = 0$ and $d\mu = 0 \quad \Rightarrow \quad \mu = d\omega$. Incidentally, it's also obvious now that sending $A \to A + d\lambda$, for any function $\lambda(t, \vec{x})$, doesn't change F at all. This is a gauge transformation.