Einstein's Equations for the Robertson–Walker Spacetime.

The Friedman–Robertson–Walker Cosmology

The Robertson – Walker Metric

So in class, we deduced that the line element is:

$$ds^2 = -dt^2 + \mathrm{a(t)}^2 \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right]$$

Where a(t) is the scale factor of our universe, which is either flat and open (k=0), hyperbolic and open (k=−1), or spherical and closed (k=1). The point of this exercise is to derive the dynamics of this geometry of our universe from Einstein's equations. To this end, we should work out all of the components of all of the geometric quantities. We need then get the Einstein tensor $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$, and equate it to ($8\pi G$ times) the energy–momentum tensor for a fluid, which represents the matter and radiation content of the universe: $T^{\mu\nu} = (p(t) + \rho(t)) u^\mu u^\nu + p(t) g^{\mu\nu}$. (Here, p is scalar pressure and ρ is density, and because we are co-moving with the fluid, we have the four velocity as $u^\mu = [1, 0, 0, 0]$.)

...and so we'll write the metric components as:

$$g_{\mu\nu} = \begin{bmatrix} \frac{\mathrm{a}(t)^2}{1 - kr^2} & 0 & 0 & 0 \\ 0 & \mathrm{a}(t)^2 r^2 & 0 & 0 \\ 0 & 0 & \mathrm{a}(t)^2 r^2 \sin(\theta)^2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Christoffel symbol of the second kind (Metric Connection) (symmetric in first two indices) :

$$,_{r\ r}{}^{r} = -\frac{kr}{-1 + kr^2}$$

$$,_{r\ r}{}^{t} = -\frac{\mathrm{a}(t)(\frac{\partial}{\partial t}\mathrm{a}(t))}{-1 + kr^2}$$

$$,_{r\ \theta}{}^{\theta} = \frac{1}{r}$$

$$,_{r\ \phi}{}^{\phi} = \frac{1}{r}$$

$$, {}_{r\ t}{}^{r} = \frac{\frac{\partial}{\partial t}\mathrm{a}(t)}{\mathrm{a}(t)}$$
$$, {}_{\theta\ \theta}{}^{r} = (-1 + kr^2)r$$
$$, {}_{\theta\ \theta}{}^{t} = \mathrm{a}(t)r^2(\frac{\partial}{\partial t}\mathrm{a}(t))$$
$$, {}_{\theta\ \phi}{}^{\phi} = \frac{\cos(\theta)}{\sin(\theta)}$$
$$, {}_{\theta\ t}{}^{\theta} = \frac{\frac{\partial}{\partial t}\mathrm{a}(t)}{\mathrm{a}(t)}$$
$$, {}_{\phi\ \phi}{}^{r} = (-1 + kr^2)r\sin(\theta)^2$$
$$, {}_{\phi\ \phi}{}^{\theta} = -\sin(\theta)\cos(\theta)$$
$$, {}_{\phi\ \phi}{}^{t} = \mathrm{a}(t)r^2\sin(\theta)^2(\frac{\partial}{\partial t}\mathrm{a}(t))$$
$$, {}_{\phi\ t}{}^{\phi} = \frac{\frac{\partial}{\partial t}\mathrm{a}(t)}{\mathrm{a}(t)}$$

Covariant Riemann Tensor :

$$R_{r\ \theta\ r\ \theta} = -\frac{\mathrm{a}(t)^2r^2(k + (\frac{\partial}{\partial t}\mathrm{a}(t))^2)}{-1 + kr^2}$$
$$R_{r\ \phi\ r\ \phi} = -\frac{\mathrm{a}(t)^2r^2\sin(\theta)^2(k + (\frac{\partial}{\partial t}\mathrm{a}(t))^2)}{-1 + kr^2}$$
$$R_{r\ t\ r\ t} = \frac{\mathrm{a}(t)(\frac{\partial^2}{\partial t^2}\mathrm{a}(t))}{-1 + kr^2}$$
$$R_{\theta\ \phi\ \theta\ \phi} = \mathrm{a}(t)^2r^4\sin(\theta)^2k + \mathrm{a}(t)^2r^4(\frac{\partial}{\partial t}\mathrm{a}(t))^2\sin(\theta)^2$$
$$R_{\theta\ t\ \theta\ t} = -\mathrm{a}(t)r^2(\frac{\partial^2}{\partial t^2}\mathrm{a}(t))$$
$$R_{\phi\ t\ \phi\ t} = -\mathrm{a}(t)r^2\sin(\theta)^2(\frac{\partial^2}{\partial t^2}\mathrm{a}(t))$$

Covariant Ricci Tensor :

$$R_{\mu\nu} = \left[\begin{array}{c} -\frac{2k + 2(\frac{\partial}{\partial t}\mathrm{a}(t))^2 + \mathrm{a}(t)(\frac{\partial^2}{\partial t^2}\mathrm{a}(t))}{-1 + kr^2}, 0, 0, 0 \\ 0, 2kr^2 + 2r^2(\frac{\partial}{\partial t}\mathrm{a}(t))^2 + \mathrm{a}(t)r^2(\frac{\partial^2}{\partial t^2}\mathrm{a}(t)), 0, 0 \\ 0, 0, 2kr^2\sin(\theta)^2 + 2r^2\sin(\theta)^2(\frac{\partial}{\partial t}\mathrm{a}(t))^2 + \mathrm{a}(t)r^2\sin(\theta)^2(\frac{\partial^2}{\partial t^2}\mathrm{a}(t)), 0 \\ 0, 0, 0, -3\frac{\frac{\partial^2}{\partial t^2}\mathrm{a}(t)}{\mathrm{a}(t)} \end{array}\right]$$

Ricci Scalar :

$$R = 6\frac{k + (\frac{\partial}{\partial t}\mathrm{a}(t))^2 + \mathrm{a}(t)(\frac{\partial^2}{\partial t^2}\mathrm{a}(t))}{\mathrm{a}(t)^2}$$

Covariant Einstein Tensor :

$$G_{\mu\nu} = \left[\begin{array}{c} \frac{k + (\frac{\partial}{\partial t}\mathrm{a}(t))^2 + 2\mathrm{a}(t)(\frac{\partial^2}{\partial t^2}\mathrm{a}(t))}{-1 + kr^2}, 0, 0, 0 \\ 0, -kr^2 - r^2(\frac{\partial}{\partial t}\mathrm{a}(t))^2 - 2\mathrm{a}(t)r^2(\frac{\partial^2}{\partial t^2}\mathrm{a}(t)), 0, 0 \\ 0, 0, -kr^2\sin(\theta)^2 - r^2\sin(\theta)^2(\frac{\partial}{\partial t}\mathrm{a}(t))^2 - 2\mathrm{a}(t)r^2\sin(\theta)^2(\frac{\partial^2}{\partial t^2}\mathrm{a}(t)), 0 \\ 0, 0, 0, 3\frac{k + (\frac{\partial}{\partial t}\mathrm{a}(t))^2}{\mathrm{a}(t)^2} \end{array}\right]$$

So in the end, equating this all to the energy-momentum tensor for the fluid, as discussed in the beginning (see the lectures):

$$3\left(\frac{\dot{\mathrm{a}}(t)^2 + k}{\mathrm{a}(t)^2}\right) = 8\pi G\rho(t) \tag{1}$$

$$\frac{2\ddot{\mathrm{a}}(t) + \dot{\mathrm{a}}(t)^2 + k}{\mathrm{a}(t)^2} = -8\pi G p(t)\ . \tag{2}$$

(A dot denotes a time derivative.) The first equation we will show follows from a Newtonian argument as an energy equation, while the second (after using the first to elimiate some terms) is the equation of state that we already deduced in class.

This is the Friedman–Robertson–Walker (FRW) Cosmology, a sort of "Standard Model" of cosmology.

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