Physics 130 General Relativity Seminar Week-9 Fall 2008

Readings: Woodhouse - General Relativity
Chapter 8 - Schwarzschild Orbits
Walecka - General Relativity
Chapter 8 - Precession of Perihelion
Chapter 9 - Gravitational Redshift
PDF Files
GR Tests
Pioneer Anomaly

Present Ideas and Lead Discussion:

Woodhouse Chapter 8	_Margeret_
Walecka Chapter 8	_Eric_____
Walecka Chapter 9	_Ben G____

Present Problems:

Wo8.2 - Free particle worldlines	_Margeret_
Wo8.3 - Equatorial null geodesics	
Wa8.1 - Get the Hamiltonian	_Eric_____
Wa8.2 - Effective potential	
Wa8.4 - Conservation of angular momentum	_Ben P____
Wa8.6 - Deflection of light 1	_Chris____
Wa8.7 - Deflection of light 2	
Wa9.1 - Redshift near earth surface	_Robert___
Wa9.2 - Extend muon lifetime	
Wa9.3 - Mass to radius ratio	

EP33. Velocity of Light _Sam______

The Schwarzschild metric is given by

$$ds^2 = -\left(1-\frac{2GM}{c^2 r}\right)c^2 dt^2 + \left(1-\frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

As a function of r, what is the coordinate velocity of light in this metric (a) in the radial direction? (b) in the transverse direction? What are the physical consequences of these results.

EP34. Orbiting Photons _Erin_____

Consider a photon in orbit in a Schwarzschild geometry. For simplicity, assume that the orbit lies in the equatorial plane (i.e., $\theta = \pi/2$ is constant).

(a) Show that the geodesic equations imply that

$$\bar{E}^2 = \frac{1}{c^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{\bar{J}^2}{c^2 r^2}\left(1-\frac{2GM}{c^2 r}\right)$$

where $\bar{E}$ and $\bar{J}$ are constant of the motion and λ is an affine parameter.

(b) Define the effective potential

$$V_{eff} = \frac{\bar{J}^2}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right)$$

Our readings showed that the effective potential yields information about the orbits of massive particles. Employing similar considerations, show that for photons there exists an unstable circular orbit of radius $3r_s/2$, where $r_s = 2GM/c^2$ is the Schwarzschild radius. HINT: Make sure you check for minima and maxima.

(c) Compute the proper time for the photon to complete one revolution of the circular orbit as measured by an observer stationed at $r = 3r_s/2$.

(d) What orbital period does a very distant observer assign to the photon?

(e) The instability of the orbit can be exhibited directly. Show, by perturbing the geodesic in the equatorial plane, that the circular orbit of the photon at $r = 3r_s/2$ is unstable. HINT: in the orbit equation put $r = 3r_s/2 + \eta$, and deduce an equation for η. Keep only the first order terms in η, and solve the resulting equation.

EP35. Light Cones _Markus___

Consider the 2-dimensional metric

$$ds^2 = -x dw^2 + 2 dw dx$$

(a) Calculate the light cone at a point (w,x), i.e., find dw/dx for the light cone. Sketch a (w,x) spacetime diagram showing how the light cones change with x. What can you say about the motion of particles, and in particular, about whether they can cross from positive to negative x and vice versa.

(b) Find a new system of coordinates in which the metric is diagonal.

EP36. Circular Orbit (Graded #3) _Everyone__

An object moves in a circular orbit at Schwarzschild radius R around a spherically symmetric mass M. Show that the proper time t is related to coordinate time t by

$$\frac{\tau}{t} = \sqrt{1 - \frac{3M}{R}}$$

HINT: It is helpful to derive a relativistic version of Kepler's third law.

EP37. Space Garbage __Ben G____

In a convenient coordinate system, the spacetime of the earth is approximately

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1+\frac{2GM}{r}\right)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$
$$= -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1+\frac{2GM}{r}\right)\left[dx^2 + dy^2 + dz^2\right]$$

where M is the earth's mass. In the second version we remapped the spherical coordinates to cartesian coordinates in the usual way:

$$x = r\sin\theta\cos\phi \quad , \quad y = r\sin\theta\sin\phi \quad , \quad z = r\cos\theta$$

Note that the Cartesian form of the spacetime metric is conveniently written $g_{\alpha\beta} = \eta_{\alpha\beta} + 2\Phi\hat{I}$ where $\hat{I}$ = diag(1,1,1,1) and $\Phi = GM/r$. We can assume that $\Phi \ll 1$ throughout this problem.

The space shuttle orbits the earth in a circular ($u^r = 0$), equatorial ($\theta = \pi/2, u^\theta = 0$) orbit of radius R.

(a) Using the geodesic equation, show that an orbit which begins equatorial remains equatorial: $du^\theta/dt = 0$ if $u^\theta = 0$ and $\theta = \pi/2$ at t = 0. HINT: Begin by computing the non-zero connection coefficients; use the fact that $\Phi \ll 1$ to simplify your answer.

We now require that the orbit must remain circular: $du^r/dt = 0$.

(b) By enforcing this condition with the geodesic equation, derive an expression for the orbital frequency

$$\Omega = \frac{d\phi/d\tau}{dt/d\tau}$$

Does this result look familiar?

The next part is most conveniently described in Cartesian coordinates; you may describe the shuttle's orbit as

$$x = R\cos\Omega t \ , \ y = R\sin\Omega t$$

An astronaut releases a bag of garbage into space, spatially displaced from the shuttle by $\xi^i = x^i_{garbage} - x^i_{shuttle}$.

(c) Using the equations of geodesic deviation, work out differential equations for the evolution of ξ^t , ξ^x , ξ^y , and ξ^z as a function of time. You may neglect terms in $(GM/r)^2$, and treat all orbital velocities as non-relativistic. You will need the Cartesian connection coefficients for this.

(d) Suppose the initial displacement is $\xi^x = \xi^y = 0$, $\xi^z = L$, $d\xi^i / d\tau = 0$. Further, synchronize the clocks of the garbage and the space shuttle: $\xi^0 = 0$, $\partial_t \xi^0 = 0$. Has the astronaut succeeded in getting rid of the garbage?

EP38. Astronauts in Orbit _Emma______

Consider a spacecraft in a circular orbit in a Scharzschild geometry. As usual, we denote the Schwarzschild coordinates by (ct, r, θ, ϕ) and assume that the orbit occurs in the plane where $\theta = \pi/2$. We denote two conserved quantities by

$$e = \left[1 - \frac{r_s}{r}\right]\frac{dt}{d\tau} \quad \text{and} \quad \ell = r^2 \frac{d\phi}{d\tau}$$

where $r_s = 2GM/c^2$ and τ is the proper time.

(a) Write down the geodesic equation for the variable r . Noting that r is independent of τ for a circular orbit, show that:

$$\frac{\ell}{c} = c\left(\tfrac{1}{2} r_s r\right)^{1/2}\left[1 - \frac{r_s}{r}\right]^{-1}$$

(b) Show that for a timelike geodesic, $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -c^2$, where $\dot{x}^\mu = dx^\mu / d\tau$. From this result, derive a second relation between ℓ and e for a circular orbit. Then, using the result of part (a) to eliminate e, obtain an expression for $d\tau/d\phi$ in terms of the radius r of the orbit.

(c) Using the result of part (b), determine the period of the orbit as measured by an observer at rest inside the orbiting spacecraft, as a function of the radius r of the orbit.

(d) Suppose an astronaut leaves the spacecraft and uses a rocket-pack to maintain a fixed position at radial distance r equal to the orbital radius and at fixed $\theta = \pi/2$ and $\phi = 0$.

The astronaut outside then measures the time it takes the spacecraft to make one orbital revolution. Evaluate the period as measured by the outside astronaut. Does the astronaut outside the spacecraft age faster or slower than the astronaut orbiting inside the spacecraft?

EP39. Weak Gravity __Markus___

In weak gravity, the metric of a mass M at rest at the origin is

$$ds^2 = -(1+2\phi)dt^2 + (1-2\alpha\phi)\delta_{ij}dx^i dx^j$$

where α is a constant and $\phi = -GM/r$.

(a) What is the value of α in general relativity?

(b) Instead of sitting at rest at the origin, the mass M moves in the +x-direction with speed v, passing through the origin at time t = 0, so that its position as a function of time is x = vt. What is the metric in this case?

(c) A photon moves along a trajectory originally in the +y-direction with offset b behind the y-axis, so that its undeflected trajectory is $x_0 = -b\hat{x} + t\hat{y}$. By what angle is the path of this test particle deflected?

(d) What is change in energy of deflected photon in part (c).