Physics 130 General Relativity Seminar Week-13 Fall 2008

Readings: Walecka - General Relativity
Chapter 13 - Cosmological Constant; Inflation

Present Ideas and Lead Discussion:

Walecka Chapter 13	__Chris___
Paper: Inflation.pdf	__Erin____
Paper: Dark_Energy.pdf	__Emma____

Present Problems:

Wa13.3 - Proof of density equation in RW	__Eric____
Wa13.4 - Scalar curvature in RW	
Wa13.5 - Cold, matter-dominated cosmology	__Chris___
Wa13.6 - Cosmological constant only	
Wa13.7 - Scalar field Lagrangians	__Robert__
Wa13.8 - Hamilton's principle and EOM	
Wa13.9 - Approximate solutions	__Ben P___
Wa13.11 - More scalar field theory	
Wa13.12 - RW with everything present	_Margaret_
Wa13.13 - A fluid model	
Wa13.14 - Planck scale	

EP65. k=1 Robertson-Walker Spacetime _Eric_____

Suppose that the universe is described by a k=1 Robertson-Walker spacetime with metric

$$ds^2 = -dt^2 + R^2(t)dx^2 + \sin^2 x\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

with $R(t) = R_0 t^{2/3}$ at the present epoch. An observer at t = t_1 observes a distant galaxy of proper size D perpendicular to the line of sight at t = t_0.

(a) What is the observed red shift in terms of R_0, t_0, t_1?

(b) What is the angular diameter of the galaxy, δ, in terms of the red shift?

(c) Show that as the red shift increases δ reaches a minimum for fixed D and then starts to increase.

EP66. General Robertson-Walker Spacetime __Emma___

The Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right)$$

where $\kappa = 0, +1, -1$, according to whether the 3-dimensional space has zero, positive or negative curvature, respectively, gives rise to the first order Einstein field equation

$$\dot{a}^2 + \kappa = \frac{8\pi G}{3} a^2 \quad , \quad \rho a^3 = \text{constant}$$

for a matter-dominated universe of density ρ.

(a) Derive the above field equation.

(b) Calculate the distance $L_r(t)$ from the origin(r=0) to a particle with coordinate r at time t, in terms of r, a(t).

Alternatively, we can formulate the theory in purely classical Newtonian terms by ignoring curvature inside a spherical volume of sufficiently small radius, i.e., assume that the space is flat inside the sphere and that any isotropic distribution of matter outside has no effect on curvature inside.

(c) Write down Newton's equation for the acceleration of a particle towards the origin at a distance L away. HINT: Consider a uniform distribution of matter inside a sphere of radius L.

(d) To conserve matter, we must also have $\rho a^3 = \text{constant}$. Combine this with your result in (c) to determine the equations satisfied by the expansion parameter a(t) and compare your answer with the cosmological one.

EP67. Spaceship in Robertson-Walker Spacetime __Sam_____

Assume that the geometry of the universe is described by Robertson-Walker metric (c=1)

$$ds^2 = -dt^2 + R^2(t)\left(\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right)$$

A spaceship sets out with velocity v relative to cosmological observers. At a later time when the universe has expanded by a scale factor (1+z), find the velocity v' with respect to cosmological observers.

EP68. Equation of State __Markus__

(a) The equation of state is often written in adiabatic form where p is pressure and ρ is density and $0 \le \gamma \le 2$ is the adiabatic index with $\gamma = 0$ for dust and $\gamma = 4/3$ for radiation. Calculate $\rho(a)$ for general γ. For k=0, calculate a(t). Find the age of the universe for k=0 and general γ.

(b) In the same notation as (a), find γ so that the expansion rate is constant. With this value of find a(t) for k=1 and k=-1.

(c) In the same notation as (a), show

$$\dot{\Omega} = (2-3\gamma)H\Omega(1-\Omega)$$

Define the logarithmic scale factor $s = \log(a)$ and write an equation for $d\Omega / ds$. Notice that this formula gives a clear idea how Ω behaves.

EP69. Flat Universe with Period of Inflation __Erin____

Consider a simplified model of the history of a flat universe involving a period of inflation. The history is split into four periods

(1) $0 < t < t_3$ radiation only

(2) $t_3 < t < t_2$ vacuum energy dominates with an effective cosmological constant $\Lambda = 3t_3^2 / 4$.

(3) $t_2 < t < t_1$ a period of radiation dominance

(4) $t_1 < t < t_0$ matter domination

(a) Show that in (3) $\rho(t) = \rho_r(t) = 3\pi t^2 / 32$ and in (4) $\rho(t) = \rho_m(t) = \pi t^2 / 6$. The functions ρ_r and ρ_m introduced for later convenience.

(b) Give simple analytic formulas for a(t) which are approximately true in the four epochs.

(c) Show that during the inflationary epoch the universe expands by a factor

$$\frac{a(t_2)}{a(t_3)} = \exp\left(\frac{t_2 - t_3}{2t_3}\right)$$

(d) Show that

$$\frac{\rho_r(t_0)}{\rho_m(t_0)} = \frac{9}{16}\left(\frac{t_1}{t_0}\right)^{2/3}$$

(e) $t_3 = 10^{-35}$ seconds, $t_2 = 10^{-32}$ seconds, $t_1 = 10^4$ years and $t_0 = 10^{10}$ years, give a sketch of $\log(a)$ against $\log(t)$ marking any important epochs.

(f) Define what is meant by the particle horizon and calculate how it behaves for this model. Indicate this behavior on the sketch you made. How does inflation solve the horizon problem?

EP70. Inflation due to a Scalar Field __Markus__

The universe undergoes a period of inflation driven by a scalar field with potential $V(\phi) = m^2\phi^2 / 2$. Write down the (slow-roll - you need to look this idea up) equations and find the solutions for the scale factor a and the field ϕ for initial conditions $a = a_i$

and $\phi = \phi_i$ at t=0. For what range of ϕ values is this solution inflationary? What condition must be satisfied by the initial scalar field ϕ_i, to ensure that an expansion by at least 10^{30} occurs.

EP71. Worm-Hole Metric __Ben G___

Consider the "worm-hole" metric

$$ds^2 = dt^2 - dr^2 - (b^2 + r^2)d\Omega^2$$

Try and work out why this curve is known as a warp-drive.

(a) Find the geodesic equations for this geometry.

(b) Find the Christoffel symbols for this geometry.

EP72. Alcubierre Warp-Drive Spacetime __Ben G___

Consider the spacetime known as the Alcubierre Warp-Drive. The coordinates are t,x,y,z and consider a (not necessarily time-like) trajectory given by $x=x_s(t)$, y=0, z=0. Then the warp-drive spacetime is given by the following metric

$$ds^2 = dt^2 - \left[dx - v_s(t) f(r_s) dt\right]^2 - dy^2 - dz^2$$

where $v_s(t) = dx_s(t)/dt$ is the velocity associated with the curve and $r_s^2 = \left[(x - x_s(t))^2 + y^2 + z^2\right]$ determines the distance of any point from the curve. The function f is smooth and positive with $f(0) = 1$ and vanishes whenever $r_s > R$ for some R. Notice that if we restrict to a curve with constant t, then the the metric is flat and that the metric is flat whenever a spacetime point is sufficiently far away from $x_s(t)$.

(a) Find the null geodesics $ds^2 = 0$ for this spacetime and draw a space-time diagram with some forward and backward light cones along the path $x_s(t)$.

(b) Check that the curve $x_s(t)$ is a geodesic and show that at every point along this curve the 4-velocity of the ship lies within the forward light cone.

(c) Consider the path $x_s(t)$ that connects coordinate time 0 with coordinate time T. How much time elapses for a spaceship traveling along $x_s(t)$?

(d) Calculate the components of a 4-vectors normal to a surface of constant t.

(e) Show that

$$T_{\alpha\beta}\eta^\alpha\eta^\beta = -\frac{1}{8\pi}\frac{v_s^2\left(y^2+z^2\right)}{2r_s^2}\left(\frac{df}{dr_s}\right)^2$$

This is the energy density measured by observers at rets with respect to the surfaces of constant t. The fact that it is negative means that the warp-drive spacetime cannot be supported by ordinary matter!