The bending of light and lensing in modified gravity

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Our modified gravity theory (MOG) was used successfully in the past to explain a range of astronomical and cosmological observations, including galaxy rotation curves, the CMB acoustic peaks, and the galaxy mass power spectrum. MOG was also used successfully to explain the unusual features of the Bullet Cluster 1E0657-558 without exotic dark matter. In the present work, we derive the relativistic equations of motion in the spherically symmetric field of a point source in MOG and, in particular, we derive equations for light bending and lensing. Our results also have broader applications in the case of extended distributions of matter, and they can be used to validate the Bullet Cluster results and provide a possible explanation for the merging clusters in Abell 520.

PACS numbers: 04.20.Cv,04.50.Kd,04.80.Cc,45.20.D-,45.50.-j,98.80.-k

I. INTRODUCTION

Our modified gravity theory (MOG), also known as Scalar-Tensor-Vector Gravity (STVG, [1]), is a theory based on the action principle. The action incorporates, in addition to the usual Einstein-Hilbert term associated with the metric $g_{\mu\nu}$, a massive vector field ϕ_μ, and three scalar fields representing running values of the gravitational constant G, the vector field's mass μ, and its coupling strength ω. The vector field is associated with a fifth force charge that is proportional to mass-energy. This fifth force is repulsive; for a large source mass, at large distances, gravity is stronger than that predicted by Newton or Einstein, but at short range, this stronger gravitational attraction is canceled by the fifth force field, leaving only Newtonian gravity. The theory has been used successfully to account for the rotation curves of galaxies [2, 3, 4, 5], the mass profiles of galaxy clusters [6], and cosmological observations [7, 8] without exotic dark matter.

The theory was also used to offer an explanation for the unusual features of the Bullet Cluster 1E0657-558 [9].In [9], a lensing formula was used that was deduced from the nonrelativistic acceleration law for test particles in the vicinity of a MOG point source.

In the present work, we derive a light bending formula using a fully relativistic approach, following the route described by Weinberg [10]. We also develop a formulation for extended distributions of sources, which is important not only for lensing, but also in future, planned work that includes astronomical N-body simulations using MOG.

We begin in Section II, reviewing the basic equations of MOG and the results of an exact numerical solution in the spherically symmetric case. We proceed by developing the equations of motion in the vicinity of a MOG point source in Section III. In Section IV, we obtain an exact treatment for light bending in the field of a point source. We generalize our discussion to extended sources and lensing in Section V. Lastly, in Section VI we conclude by exploring the possible consequences for the Bullet Cluster and for the merging clusters of galaxies Abell 520 [11].

II. MODIFIED GRAVITY THEORY

Our modified gravity theory is based on postulating the existence of a massive vector field, ϕ_μ. The choice of a massive vector field is motivated by our desire to introduce a *repulsive* modification of the law of gravitation at short range. The vector field is coupled universally to matter. The theory, therefore, has three constants: in addition to the gravitational constant G, we must also consider the coupling constant ω that determines the coupling strength between the ϕ_μ field and matter, and a further constant μ that arises as a result of considering a vector field of non-zero mass, and controls the coupling range. The theory promotes G, μ, and ω to scalar fields, hence they are allowed to run, resulting in the following action [1, 2]:

$$S = S_G + S_\phi + S_S + S_M, \tag{1}$$

where

$$S_G = -\frac{1}{16\pi}\int \frac{1}{G}\left(R + 2\Lambda\right)\sqrt{-g}\ d^4x, \tag{2}$$

$$S_\phi = -\int \omega\left[\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{2}\mu^2\phi_\mu\phi^\mu + V_\phi(\phi)\right]\sqrt{-g}\ d^4x, \tag{3}$$

$$S_S = -\int \frac{1}{G}\left[\frac{1}{2}g^{\mu\nu}\left(\frac{\nabla_\mu G\nabla_\nu G}{G^2} + \frac{\nabla_\mu\mu\nabla_\nu\mu}{\mu^2} - \nabla_\mu\omega\nabla_\nu\omega\right) + \frac{V_G(G)}{G^2} + \frac{V_\mu(\mu)}{\mu^2} + V_\omega(\omega)\right]\sqrt{-g}\ d^4x, \tag{4}$$

where S_M is the "matter" action, $B_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu$, while $V_\phi(\phi)$, $V_G(G)$, $V_\omega(\omega)$, and $V_\mu(\mu)$ denote the self-interaction potentials associated with the vector field and the three scalar fields. The symbol ∇_μ is used to denote covariant differentiation with respect to the metric $g^{\mu\nu}$, while the symbols R, Λ, and g represent the Ricci-scalar, the cosmological constant, and the determinant of the metric tensor, respectively. We define the Ricci tensor as

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\alpha\nu}. \tag{5}$$

Unless otherwise noted, our units are such that the speed of light, $c = 1$; we use the metric signature $(+, -, -, -)$.

In the case of a spherically symmetric field in vacuum around a compact (point) source, we were able to derive an exact numerical solution [2]. We found that the scalar fields G, μ, and ω remain constant except in the immediate vicinity of the source. The spatial part of the vector field ϕ_μ is zero, while its t-component obeys a simple exponential relationship. Meanwhile, the metric is approximately the Reissner-Nordström metric of a charged source.

Specifically, given a spherically symmetric, static metric in the standard form

$$d\tau^2 = Bdt^2 - Adr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{6}$$

we found that, for a source mass M,

$$A \simeq B^{-1}, \tag{7}$$

$$B \simeq 1 - \frac{2G_N M}{r} + \frac{4\pi\omega G_0 Q_5^2}{r^2}, \tag{8}$$

$$G \simeq G_0 = G_N + (G_\infty - G_N)\frac{M}{(\sqrt{M} + E)^2}, \tag{9}$$

$$\mu \simeq \mu_0 = \frac{D}{\sqrt{M}}, \tag{10}$$

$$\omega \simeq \omega_0 = \frac{1}{\sqrt{12}}, \tag{11}$$

$$\phi_t \simeq -Q_5 \frac{e^{-\mu r}}{r}, \tag{12}$$

where G_N is Newton's constant of gravitation, $Q_5 = \kappa M$ is the fifth force charge associated with the source mass M, while G_∞, D and E are constants. Further,

$$\kappa = \sqrt{\frac{G_N}{\omega}}, \tag{13}$$

$$D \simeq 6250\ M_\odot^{1/2}\text{kpc}^{-1}, \tag{14}$$

$$E \simeq 25000\ M_\odot^{1/2}, \tag{15}$$

$$G_\infty \simeq 20G_N. \tag{16}$$

When r is large (that is, large relative to the Schwarzschild-radius $r_S = 2G_N M$ for a source mass M), the metric coefficients become

$$A \simeq B^{-1}, \tag{17}$$

$$B \simeq 1 - \frac{2G_N M}{r}. \tag{18}$$

III. EQUATIONS OF MOTION IN A SPHERICALLY SYMMETRIC FIELD

To develop an equation of motion for a point particle, and use it to derive a formula for light bending, we follow the approach presented in [10].

We begin with the point particle action in MOG, which is written in the form

$$\begin{aligned} S &= -\int (m + \alpha\omega q_5 \phi_\mu u^\mu)\ d\tau \\ &= -\int (m\sqrt{g_{\alpha\beta}u^\alpha u^\beta} + \alpha\omega q_5 \phi_\mu u^\mu)\ d\tau, \end{aligned} \tag{19}$$

where m is the point particle mass, and q_5 is its fifth force charge. The fifth force charge is assumed to be proportional to m, such that $q_5 = \kappa m$, where κ is a constant. In earlier work, we determined that $\kappa = \sqrt{G_N/\omega}$. To develop an equation of motion, we compute the derivatives of the Lagrangian with respect to positions and velocities:

$$\mathcal{L} = -m\sqrt{g_{\alpha\beta}u^\alpha u^\beta} - \alpha\omega q_5 \phi_\mu u^\mu, \tag{20}$$

$$\frac{\partial \mathcal{L}}{\partial x^\nu} = -\frac{1}{2} m g_{\alpha\beta,\nu} u^\alpha u^\beta - \alpha\omega q_5 \phi_{\mu,\nu} u^\mu - \alpha\omega_{,\nu} q_5 \phi_\mu u^\mu, \tag{21}$$

$$\frac{\partial \mathcal{L}}{\partial u^\nu} = -m g_{\alpha\nu} u^\alpha - \alpha\omega q_5 \phi_\nu, \tag{22}$$

$$\begin{aligned} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial u^\nu} &= -m g_{\alpha\nu}\frac{du^\alpha}{d\tau} - m u^\beta g_{\alpha\nu,\beta} u^\alpha \\ &\quad - \alpha\omega q_5 u^\beta \phi_{\nu,\beta} - \alpha\omega_{,\beta} q_5 u^\beta \phi_\nu. \end{aligned} \tag{23}$$

We can now construct the Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial x^\nu} - \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial u^\nu} = m g_{\alpha\nu}\frac{du^\alpha}{d\tau} + m\Gamma_{\alpha\beta\nu} u^\alpha u^\beta + \alpha\omega q_5 u^\beta(\phi_{\nu,\beta} - \phi_{\beta,\nu}) - \alpha\omega_{,\nu} q_5 \phi_\mu u^\mu + \alpha\omega_{,\alpha} q_5 \phi_\nu u^\alpha = 0, \tag{24}$$

or, after rearranging terms,

$$m\left(\frac{du^\nu}{d\tau}+\Gamma^\nu_{\alpha\beta}u^\alpha u^\beta\right)=\alpha q_5\left[\omega u^\beta g^{\nu\alpha}B_{\alpha\beta}+\omega_{,\alpha}\phi_\mu(g^{\alpha\nu}u^\mu-g^{\mu\nu}u^\alpha)\right], \quad (25)$$

where we used $B_{\alpha\beta}=\phi_{\beta,\alpha}-\phi_{\alpha,\beta}$. This is the same as Eqs. (31–32) in [1].

If ω is constant (as confirmed by the numerical solution in [2]), we get

$$m\left(\frac{du^\nu}{d\tau}+\Gamma^\nu_{\alpha\beta}u^\alpha u^\beta\right)=\alpha q_5\omega u^\beta g^{\nu\alpha}B_{\alpha\beta}. \quad (26)$$

The Christoffel-symbols associated with the spherically symmetric metric (6) are

$$\Gamma^r_{tt}=\frac{B'}{2A},\quad \Gamma^t_{tr}=\frac{B'}{2B}, \quad (27)$$

$$\Gamma^r_{rr}=\frac{A'}{2A},\quad \Gamma^\theta_{r\theta}=\Gamma^\phi_{r\phi}=\frac{1}{r}, \quad (28)$$

$$\Gamma^r_{\theta\theta}=-\frac{r}{A},\quad \Gamma^\phi_{\theta\phi}=\cot\theta, \quad (29)$$

$$\Gamma^r_{\phi\phi}=-\frac{r\sin^2\theta}{A},\quad \Gamma^\theta_{\phi\phi}=-\cos\theta\sin\theta. \quad (30)$$

The equations of motion read, using $q_5=\kappa m$ and dividing through by m,

$$\frac{d^2t}{d\tau^2}+\frac{B'}{B}\frac{dt}{d\tau}\frac{dr}{d\tau} = \alpha\kappa\omega u^\beta g^{tt}B_{t\beta}, \quad (31)$$

$$\frac{d^2r}{d\tau^2}+\frac{B'}{2A}\left(\frac{dt}{d\tau}\right)^2+\frac{A'}{2A}\left(\frac{dr}{d\tau}\right)^2-\frac{r\sin^2\theta}{A}\left(\frac{d\phi}{d\tau}\right)^2 = \alpha\kappa\omega u^\beta g^{rr}B_{r\beta}, \quad (32)$$

$$\frac{d^2\theta}{d\tau^2}+\frac{2}{r}\frac{dr}{d\tau}\frac{d\theta}{d\tau}-\cos\theta\sin\theta\left(\frac{d\phi}{d\tau}\right)^2 = \alpha\kappa\omega u^\beta g^{\theta\theta}B_{\theta\beta}, \quad (33)$$

$$\frac{d^2\phi}{d\tau^2}+\frac{2}{r}\frac{dr}{d\tau}\frac{d\phi}{d\tau}+2\cot\theta\frac{d\theta}{d\tau}\frac{d\phi}{d\tau} = \alpha\kappa\omega u^\beta g^{\phi\phi}B_{\phi\beta}. \quad (34)$$

We can set $\theta=\pi/2$ without loss of generality. We can also recognize that the only non-zero components of $B_{\alpha\beta}$ are $B_{tr}=-B_{rt}=\partial_t\phi_r-\partial_r\phi_t=-\phi'_t$. We get

$$\frac{d^2t}{d\tau^2}+\frac{B'}{B}\frac{dt}{d\tau}\frac{dr}{d\tau} = -\frac{\alpha\kappa\omega}{B}\frac{dr}{d\tau}\phi'_t, \quad (35)$$

$$\frac{d^2r}{d\tau^2}+\frac{B'}{2A}\left(\frac{dt}{d\tau}\right)^2+\frac{A'}{2A}\left(\frac{dr}{d\tau}\right)^2-\frac{r}{A}\left(\frac{d\phi}{d\tau}\right)^2 = -\frac{\alpha\kappa\omega}{A}\frac{dt}{d\tau}\phi'_t, \quad (36)$$

$$\frac{d^2\phi}{d\tau^2}+\frac{2}{r}\frac{dr}{d\tau}\frac{d\phi}{d\tau} = 0. \quad (37)$$

Eq. (35) can be rearranged after multiplying with B:

$$B\frac{d^2t}{d\tau^2}+\frac{dB}{d\tau}\frac{dt}{d\tau}=-\alpha\kappa\omega\frac{d\phi_t}{d\tau}, \quad (38)$$

which can be integrated to yield

$$\frac{dt}{d\tau}=\frac{C-\alpha\kappa\omega\phi_t}{B}. \quad (39)$$

Since $B\to 1$ and $\phi_t\to 0$ as $r\to\infty$, an asymptotically flat spacetime requires $C=1$.

Eq. (37) can be integrated directly:

$$r^2\frac{d\phi}{d\tau}=J, \quad (40)$$

where J is a constant of integration, which we identify as the angular momentum per unit mass.

Using these results in Eq. (36) yields

$$\frac{d^2r}{d\tau^2}+\frac{A'}{2A}\left(\frac{dr}{d\tau}\right)^2-\frac{J^2}{Ar^3}+\frac{B'}{2A}\left(\frac{1-\alpha\kappa\omega\phi_t}{B}\right)^2=-\frac{\alpha\kappa\omega(1-\alpha\kappa\omega\phi_t)}{AB}\phi_t'. \tag{41}$$

In the case of general relativity, the right-hand side of this equation is zero, and the particle moves along a geodesic. This is not the case here, but we can still integrate our equation. Multiplication by $2Adr/d\tau$ leads to

$$\frac{d}{d\tau}\left[A\left(\frac{dr}{d\tau}\right)^2+\frac{J^2}{r^2}-\frac{(1-\alpha\kappa\omega\phi_t)^2}{B}\right]=0. \tag{42}$$

Integration yields

$$A\left(\frac{dr}{d\tau}\right)^2+\frac{J^2}{r^2}-\frac{(1-\alpha\kappa\omega\phi_t)^2}{B}=-\mathcal{E}, \tag{43}$$

where $\mathcal{E}$ is another constant of integration. After using (39), we get

$$\frac{A}{B^2}\left(\frac{dr}{dt}\right)^2+\frac{J^2}{(1-\alpha\kappa\omega\phi_t)^2}\frac{1}{r^2}-\frac{1}{B}=-\frac{\mathcal{E}}{(1-\alpha\kappa\omega\phi_t)^2}. \tag{44}$$

This is an exact result. From this result, we can develop the equation of motion for a non-relativistic particle in the usual form. For this, let us assume that we are far from a source, and the metric is that of Schwarzschild, in accordance with (17) and (18). Then, our equation of motion becomes

$$\frac{1}{(1-2GM/r)^3}\left(\frac{dr}{dt}\right)^2+\frac{J^2}{(1-\alpha\kappa\omega\phi_t)^2}\frac{1}{r^2}-\frac{1}{1-2GM/r}=\frac{-\mathcal{E}}{(1-\alpha\kappa\omega\phi_t)^2}. \tag{45}$$

Multiplying both sides with $(1-2GM/r)^3$ gives

$$\left(\frac{dr}{dt}\right)^2+\frac{J^2}{(1-\alpha\kappa\omega\phi_t)r^2}\left(1-\frac{2GM}{r}\right)^3-\left(1-\frac{2GM}{r}\right)^2=\frac{-\mathcal{E}}{(1-\alpha\kappa\omega\phi_t)^2}\left(1-\frac{2GM}{r}\right)^3. \tag{46}$$

Fully differentiating with respect to t, dividing through with $2dr/dt$ and then rearranging terms yields

$$\frac{d^2r}{dt^2}-\frac{(1-2GM/r)^3}{1-\alpha\kappa\omega\phi_t}\frac{J^2}{r^3}+\frac{(1-2GM/r)^2}{2(1-\alpha\kappa\omega\phi_t)^2}\frac{J^2}{r^2}\alpha\kappa\omega\phi_t'+\frac{(1-2GM/r)^2}{1-\alpha\kappa\omega\phi_t}\frac{3J^2GM}{r^4}-\left(1-\frac{2GM}{r}\right)\frac{2GM}{r^2}$$

$$=\frac{-\mathcal{E}(1-2GM/r)^3}{(1-\alpha\kappa\omega\phi_t)^3}\alpha\kappa\omega\phi_t'-\frac{3\mathcal{E}(1-2GM/r)^2}{(1-\alpha\kappa\omega\phi_t)^2}\frac{GM}{r^2}. \tag{47}$$

From (43), taking the large-r limit, we get

$$\mathcal{E}=1-v^2, \tag{48}$$

where $v=dr/d\tau\simeq dr/dt$ is the velocity of the particle at infinity. For photons, $\mathcal{E}=0$, for material particles in unbound orbits, $0<\mathcal{E}<1$, and for bound particles, $\mathcal{E}\geq 1$.

For a non-relativistic particle, $v^2\ll 1$. Further, in the weak field limit, $1-2GM/r\simeq 1$, $1-\alpha\kappa\omega\phi_t\simeq 1$, and $J^2/2r^2\ll 1$, so (III) becomes

$$\frac{d^2r}{dt^2}-\frac{J^2}{r^3}=-\alpha\kappa\omega\phi_t'-\frac{GM}{r^2}. \tag{49}$$

IV. THE BENDING OF LIGHT

To calculate light bending, we return to the exact result in (43). The first step is eliminating $d\tau$ using (40), since we are interested in the shape of the orbit, not its time evolution:

$$A\left(\frac{dr}{d\phi}\frac{d\phi}{d\tau}\right)^2+\frac{J^2}{r^2}-\frac{(1-\alpha\kappa\omega\phi_t)^2}{B}=-\mathcal{E}, \tag{50}$$

or,

$$\frac{A}{r^4}\left(\frac{dr}{d\phi}\right)^2+\frac{1}{r^2}=-\frac{\mathcal{E}}{J^2}+\frac{(1-\alpha\kappa\omega\phi_t)^2}{J^2B}. \tag{51}$$

From this we get

$$\phi = \pm \int \frac{A^{1/2}}{r^2 \left(\frac{(1-\alpha\kappa\omega\phi_t)^2}{J^2 B} - \frac{\mathcal{E}}{J^2} - \frac{1}{r^2} \right)^{1/2}} \, dr. \tag{52}$$

At closest approach to a source, $r = r_0$ and $dr/d\phi$ vanishes. Then, (51) becomes

$$J = r_0 \sqrt{\frac{[1 - \alpha\kappa\omega\phi_t(r_0)]^2}{B} + v^2 - 1}. \tag{53}$$

Putting this into (52) gives

$$\phi = \phi_\infty + \int_r^\infty \frac{A^{1/2}}{r^2 \left[\frac{1}{r_0^2} \left\{ \frac{[1-\alpha\kappa\omega\phi_t]^2}{B} + v^2 - 1 \right\} \left\{ \frac{[1-\alpha\kappa\omega\phi_t(r_0)]^2}{B(r_0)} + v^2 - 1 \right\}^{-1} - \frac{1}{r^2} \right]^{1/2}} \, dr. \tag{54}$$

The deflection for a particle coming from infinity to r_0 and then off to infinity is twice this angle:

$$\Delta\phi = 2|\phi - \phi_\infty| - \pi. \tag{55}$$

For a photon, $v = 1$ and

$$\Delta\phi_\gamma = 2 \left| \int_{r_0}^\infty \frac{A^{1/2}}{r^2 \left[\frac{1}{r_0^2} \left\{ \frac{[1-\alpha\kappa\omega\phi_t]^2}{B} \right\} \left\{ \frac{[1-\alpha\kappa\omega\phi_t(r_0)]^2}{B(r_0)} \right\}^{-1} - \frac{1}{r^2} \right]^{1/2}} \, dr \right| - \pi. \tag{56}$$

In the case of weak fields, $1 - \alpha\kappa\omega\phi_t \simeq 1$ and we get

$$\Delta\phi_\gamma = 2 \left| \int_{r_0}^\infty \frac{1}{r} \left[\frac{r^2}{r_0^2} \frac{B(r_0)}{AB} - \frac{1}{A} \right]^{-1/2} dr \right| - \pi. \tag{57}$$

This formula is formally identical to the light bending formula in the weak field limit of general relativity, with one notable difference: instead of $G = G_N$, we are using $G = G_\infty = (1 + \alpha)G_N$ in the Schwarzschild coefficients A and B. From this formula, the approximate deflection can be calculated as [1, 10]:

$$\Delta\phi = \frac{4GM}{r_0} = \frac{4(1 + \alpha)G_N M}{r_0}. \tag{58}$$

If ϕ_t cannot be ignored, we can use the form [2]:

$$\phi_t = -Q_5 \frac{e^{-\mu r}}{r} = -\kappa M \frac{e^{-\mu r}}{r}, \tag{59}$$

where $Q_5 = \kappa M$ is the fifth force charge of the source with mass M. This yields the formula for light bending in the strong field of a point source in the form

$$\Delta\phi_\gamma = 2 \left| \int_{r_0}^\infty \frac{1}{r} \left[\frac{[r + \alpha G_N M \exp(-\mu r)]^2}{[r_0 + \alpha G_N M \exp(-\mu r_0)]^2} \frac{B(r_0)}{AB} - \frac{1}{A} \right]^{-1/2} dr \right| - \pi, \tag{60}$$

where we used $\kappa^2\omega = G_N$.

V. EXTENDED SOURCE DISTRIBUTIONS

The nonrelativistic equation of motion (49) can be further simplified when only radial motion is considered, such that $J = 0$:

$$\ddot{r} = -\frac{G_N M}{r^2} \left[1 + \alpha - \alpha(1 + \mu r)e^{-\mu r} \right]. \tag{61}$$

This corresponds to the potential

$$\Phi(r) = -\frac{G_N M}{r}\left[1+\alpha-\alpha e^{-\mu r}\right], \qquad (62)$$

such that ($r = |\mathbf{r}|$):

$$\ddot{\mathbf{r}} = -\nabla\Phi(\mathbf{r}). \qquad (63)$$

We write the potential as the sum of two constituents:

$$\Phi(\mathbf{r}) = \Phi_N(\mathbf{r}) + \Phi_Y(\mathbf{r}), \qquad (64)$$

where

$$\Phi_N(\mathbf{r}) = -\frac{(1+\alpha)G_N M}{r} \qquad (65)$$

and

$$\Phi_Y(\mathbf{r}) = \frac{\alpha G_N M}{r}e^{-\mu r}. \qquad (66)$$

We find that Φ_N is the solution of the Poisson equation:

$$\nabla^2\Phi_N(\mathbf{r}) = 4\pi(1+\alpha)G_N\rho(\mathbf{r}), \qquad (67)$$

with $\rho(\mathbf{r}) = M\delta^3(\mathbf{r})$, where δ is Dirac's delta function.

For Φ_Y, we consider the inhomogeneous Helmholtz equation:

$$(\nabla^2 + k^2)G(\mathbf{r}) = -\delta^3(\mathbf{r}). \qquad (68)$$

This equation is solved by

$$G(\mathbf{r}) = \frac{e^{ikr}}{4\pi r}. \qquad (69)$$

Replacing k with $i\mu$, we find that

$$(\nabla^2 - \mu^2)\Phi_Y(\mathbf{r}) = -4\pi\alpha G_N\rho(\mathbf{r}). \qquad (70)$$

Adding (67) and (70), we get

$$\nabla^2\Phi(\mathbf{r}) = 4\pi G_N\rho(\mathbf{r}) + \mu^2\Phi_Y(\mathbf{r}). \qquad (71)$$

If $\mu = 0$, we get back the Poisson equation for Newtonian gravity, as expected.

If ρ is not a point source but a general continuous distribution of matter, the solution of the inhomogeneous Helmholtz equation is in the form

$$\Phi_Y(\mathbf{r}) = \alpha G_N\int\frac{e^{-\mu|\mathbf{r}-\tilde{\mathbf{r}}|}}{|\mathbf{r}-\tilde{\mathbf{r}}|}\rho(\tilde{\mathbf{r}})\ d^3\tilde{\mathbf{r}}. \qquad (72)$$

Thus,

$$\nabla^2\Phi(\mathbf{r}) = 4\pi G_N\rho(\mathbf{r}) + \alpha\mu^2 G_N\int\frac{e^{-\mu|\mathbf{r}-\tilde{\mathbf{r}}|}}{|\mathbf{r}-\tilde{\mathbf{r}}|}\rho(\tilde{\mathbf{r}})\ d^3\tilde{\mathbf{r}}. \qquad (73)$$

Or, since $\ddot{\mathbf{r}} = -\nabla\Phi(\mathbf{r})$, we can write

$$\nabla\cdot\ddot{\mathbf{r}} = -4\pi G_N\rho(\mathbf{r}) - \alpha\mu^2 G_N\int\frac{e^{-\mu|\mathbf{r}-\tilde{\mathbf{r}}|}}{|\mathbf{r}-\tilde{\mathbf{r}}|}\rho(\tilde{\mathbf{r}})\ d^3\tilde{\mathbf{r}}. \qquad (74)$$

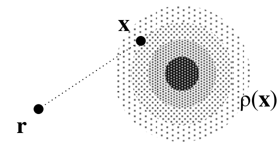

FIG. 1: The position $\mathbf{r}$ of a test particle relative to a point $\mathbf{x}$ inside an extended mass distribution characterized by $\rho(\mathbf{x})$.

In the case of a point source with mass M, μ is given by (10), while $\alpha = (G_0 - G_N)/G_N$ is determined by (9):

$$\alpha = \frac{M}{(\sqrt{M}+E)^2}\left(\frac{G_\infty}{G_N} - 1\right). \qquad (75)$$

Clearly, in the case of an extended mass distribution, these expressions must be modified. Presently, we do not have solutions to the MOG field equations for extended matter distributions. Therefore, we treat this problem phenomenologically. We seek an effective mass function $\mathcal{M}(\mathbf{x},\mathbf{r})$, to be used in place of M in (75) and (10), that determines an "effective mass", to be used in the formulae for α and μ, when calculating the gravitational influence of matter at $\mathbf{x}$ on a test particle located at $\mathbf{r}$ (see Figure 1). The function should yield the mass of the source in the case of a point source, and a mass proportional to volume in the case of a constant distribution. One function that satisfies these criteria is in the form,

$$\mathcal{M}(\mathbf{x},\mathbf{r}) = \int\rho(\bar{\mathbf{x}})\exp\left(-\xi\frac{|\bar{\mathbf{x}}-\mathbf{x}|}{|\mathbf{r}-\mathbf{x}|}\right)\ d^3\bar{\mathbf{x}}, \qquad (76)$$

with the coefficient ξ to be determined from observation, e.g., by comparison with the Bullet Cluster data. For a constant distribution $\rho(\mathbf{x}) = \rho_0$, this function gives $\mathcal{M}(\mathbf{x},\mathbf{r}) \propto |\mathbf{r}-\mathbf{x}|^3$. If ρ is not constant, we get the following expressions for α and μ:

$$\alpha(\mathbf{x},\mathbf{r}) = \frac{\mathcal{M}(\mathbf{x},\mathbf{r})}{(\sqrt{\mathcal{M}(\mathbf{x},\mathbf{r})}+E)^2}\left(\frac{G_\infty}{G_N}-1\right), \qquad (77)$$

$$\mu(\mathbf{x},\mathbf{r}) = \frac{D}{\sqrt{\mathcal{M}(\mathbf{x},\mathbf{r})}}. \qquad (78)$$

This means that in (73), we must now move α and μ under the integral sign:

$$\nabla^2\Phi(\mathbf{r}) = 4\pi G_N\rho(\mathbf{r}) \qquad (79)$$
$$+\ G_N\int\alpha(\tilde{\mathbf{r}},\mathbf{r})\mu^2(\tilde{\mathbf{r}},\mathbf{r})\frac{e^{-\mu(\tilde{\mathbf{r}},\mathbf{r})|\mathbf{r}-\tilde{\mathbf{r}}|}}{|\mathbf{r}-\tilde{\mathbf{r}}|}\rho(\tilde{\mathbf{r}})\ d^3\tilde{\mathbf{r}}.$$

Given that $\mathcal{M}$ depends only on $\mathbf{x}$ and $r_0 = |\mathbf{x}-\mathbf{r}|$, we can also write (76) in the form

$$\mathcal{M}(\mathbf{x},r_0) = \int\rho(\bar{\mathbf{x}})\exp\left(-\xi\frac{|\bar{\mathbf{x}}-\mathbf{x}|}{r_0}\right)\ d^3\bar{\mathbf{x}}, \qquad (80)$$

and α and μ can also be written as functions of $\mathbf{x}$ and r_0.

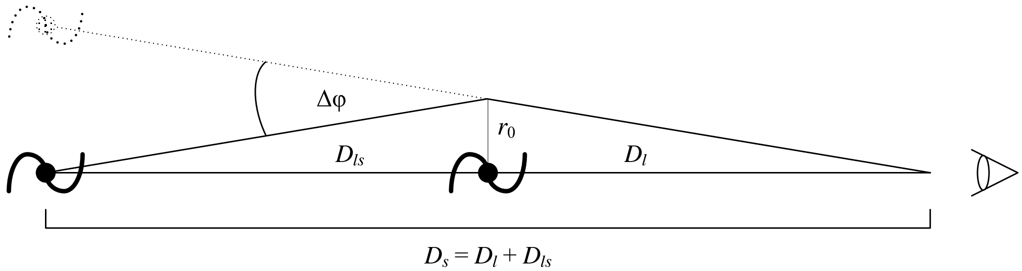

FIG. 2: The geometry of thin lensing: a foreground object, at distance D_l from the observer, deflects light from a background object, at distance D_s, by an angle $\Delta\phi$.

For lensing, we can use the κ-convergence formula [12, 13]:

$$\kappa(x,y) = \int \frac{4\pi G(\mathbf{x})}{c^2}\frac{D_l D_{ls}}{D_s}\rho(\mathbf{x})\ dz, \tag{81}$$

where D_l is the distance from the observer to the lensing plane, D_s is the distance from the observer to the distant light source, $D_{ls} = D_s - D_l$ (see Figure 2), and a rectilinear coordinate system is used such that $\mathbf{x} = (x, y, z)$, and the z-axis coincides with the line connecting the observer with the distant light source. In our case, G is given by

$$G(\mathbf{x}) = [1 + \alpha(\mathbf{x}, r_0)]G_N, \tag{82}$$

where r_0 now represents the distance between the point where a straight line connecting the light source with the observer intersects the lensing plane, and the point where the bent ray of light intersects the lensing plane. For a point mass, this distance can be calculated as

$$r_0 = \frac{D_l D_{ls}}{D_s}\sin\Delta\phi \simeq \frac{D_l D_{ls}}{D_s}\Delta\phi. \tag{83}$$

In the case of extended distributions, $\Delta\phi$ can be calculated by

$$\Delta\phi(x,y) = \frac{2}{c^2}\left|\int \nabla_\perp \Phi(\mathbf{x})\ dz\right|, \tag{84}$$

where $\nabla_\perp$ represents the gradient operator in the (x, y) lensing plane. $\Phi(\mathbf{x})$ would, in the case of general relativity, be given by the Poisson equation $\nabla^2\Phi(\mathbf{x}) = 4\pi G_N \rho(\mathbf{x})$. Our result for light bending (58) makes it clear that in the case of MOG, we cannot use the nonrelativistic equation (71). Instead, the correct equation to use for the effective potential Φ is

$$\nabla^2\Phi(\mathbf{x}) = 4\pi[1 + \alpha(\mathbf{x}, r_0)]G_N\rho(\mathbf{x}), \tag{85}$$

or, after substituting (83) and (84), we get

$$\begin{aligned}\nabla^2\Phi(\mathbf{x}) &= 4\pi G_N \rho(\mathbf{x}) \qquad (86)\\ &\times \left[1 + \alpha\left(\mathbf{x}, \frac{D_l D_{ls}}{D_s}\frac{2}{c^2}\left|\int \nabla_\perp\Phi(\mathbf{x})\ dz\right|\right)\right],\end{aligned}$$

an equation that is solvable for Φ.

VI. CONCLUSIONS

We derived an exact formulation for light bending in the spherically symmetric field of a point source in modified gravity (MOG). Introducing a phenomenological approach to model extended distributions of matter, we showed how the theory can be used to compute lensing. This work is directly applicable to the case of the Bullet Cluster 1E0657-558.

The merging of three clusters of galaxies in Abell 520 has created a mystery regarding the nature of dark matter and its role in merging clusters [11]. In contrast to the Bullet Cluster, the data show that the dark matter does not separate from the baryon plasma situated at approximately the center of the merging clusters, whereas the galaxies and stars have separated to the sides of the cluster. Moreover, these galaxies do not reveal the existence of dark matter haloes. In our derivation of the κ-convergence lensing, we believe that MOG can provide a natural explanation for these results without exotic dark matter, by modifying the mass profile at the center of the merging clusters. This will be the subject of future research.

The phenomenological formalism we developed for modeling extended distributions of matter may also be useful in future work, including N-body simulations using MOG, and cosmological computations.

Acknowledgments

We thank Joel Brownstein for helpful discussions.

The research was partially supported by National Research Council of Canada. Research at the Perimeter Institute for Theoretical Physics is supported by the Government of Canada through NSERC and by the Province of Ontario through the Ministry of Research and Innovation (MRI).

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