Lecture notes on general relativity

Petr Hadrava
Norges teknisk-naturvitenskapelige universitet NTNU,
N-7055 Dragvoll, Trondheim, Norway

May 1, 2000

Contents

Foreword

This document contains lecture notes on the course of general relativity FX2 H97 read in the fall semester 1997 at the Physics Institute of NTNU. It is not aimed to be a textbook, but rather a guide through the textbooks and other literature recommended to the students.

The basic textbook for this course is Part I of "General relativity" by R. M. Wald, [6].

It is highly recommended to the students to solve the exercises contained in the lecture notes, because some of them illustrate important applications of the main text.[1] Similar exercises (or some other, which can be found e.g. in [4] or [6]) will be used also at the examinations.

[1] There exist also solutions of the exercices, which can be requested at the author.

Chapter 1

Introduction

Classical non-relativistic physics is supposed to be the theory describing behaviour of particles, extended bodies or physical fields. It is assumed that the play of these physical entities takes place on the stage of the physical space with fixed properties well-described by Euclidean geometry and that it goes on in time, which is an even simpler one-dimensional space. Special relativity disclosed that the perception of space and time as different quantities is in fact an illusion dependent on the dynamical status of the observer, and that these quantities are actually only different projections of one entity – the spacetime – which has Minkowskian geometry. The subject of relativistic physics is thus the reformulation of classical physics in view of this fact, making its laws more symmetrical, i.e., simpler for those who are accustomed with this view. Relativity revealed that introducing gravity into its theory requires that one abandons the assumption of flat Euclidean space (or spacetime) and to accept its (pseudo-) Riemannian geometry. The geometry of the spacetime is thus no longer a background for physics, but one of the physical fields (i.e., the gravitational field) mutually interacting with others.

This demands more imagination to understand general relativity and more powerful mathematical tools to handle its calculations. The latter, which is first of all the differential geometry of pseudo-Riemannian manifolds, can be treated on different levels. The classical approach used by Einstein is based on tensor analysis using a notation with indices corresponding to components in a coordinate basis. This formalism is used in many textbooks, e.g., in [5] to which we will also refer or in [7]. More advanced formalisms using tetrads and differential forms have been developed to save at least a part of the tedious computations of spacetime geometry and to yield directly the view of local observers. This approach is used in 'the relativistic bible' – MTW [4] (which is, however, very extensive, so that it is better used as a supplementary source rather than a basic textbook) and in the textbook by Wald [6], which will be used as the basic one in this course. This approach is sometimes referred to as too formal to the detriment of the physical insight to the theory. However, an investment of effort to learn a more elegant form of mathematics can not only simplify tedious calculations but also facilitate the understanding of the physics. Certainly, the same help is yielded by physical intuition. It is thus a matter of individual choice which one of these parallel ways will be preferred. In any case, the underlying mathematics is only a tool, which will be continuously developed to handle the recognized physical nature. Nowadays a relativistic physicist should understand both approaches to be able to read the literature. To facilitate this understanding, a brief summary of differential geometry is added here in an appendix.

Chapter 2

Special relativity

In this chapter (which is supplementary to chapter 1 of [6]) will be discussed basic concepts of spacetime in special relativity and their consequences for general relativity, which can be deduced from the relativistic motion of accelerated observers. There is given only a short review of special-relativistic dynamics and electrodynamics. However, it is recommended to the students to repeat these disciplines in details in context of problems discussed here.

2.1 Spacetime

The Newtonian mechanics is invariant with respect to Galileo's group of transformations, i.e., its dynamical laws have the same form in any inertial frame. Let us demonstrate this fact on one-dimensional motions of point masses drawn in the space-time diagram, i.e., in a representation of the **spacetime**, the set of **events**, each one characterized by its position in space and time. On the left panel of Fig. 2.1, drawn with respect to an inertial frame S with coordinates $\{t, x\}$ the **worldline**, i.e., the subset of the spacetime corresponding to the events of occurrence of a particular body, of a reference point of the beginning of another inertial frame S' is a straight line $x = vt$. To transform this diagram to S', i.e.,

$$x' = x - vt \ , \qquad t' = t \ , \tag{2.1}$$

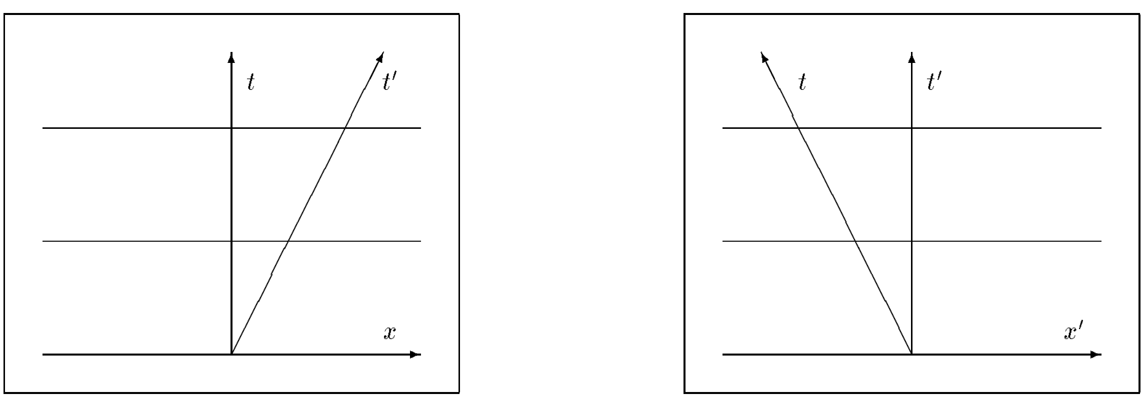

Figure 2.1: Galileo's transformation.

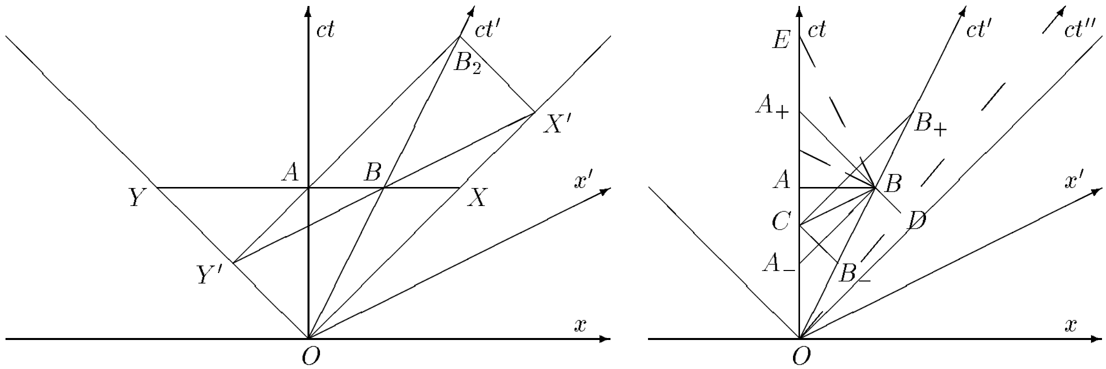

Figure 2.2: Relativity of simultaneity.

means to skew it by shifting its space slices (see the right panel). Note, that the Cartesian coordinates $t, \vec{x}$ determine both the topology of the spacetime as well as the metric properties of both space and time (i.e., the spatial distance Δs is given by $\Delta s^2 = \sum_{i=1}^{3} \Delta x^i \Delta x^i$ and the temporal distance by Δt). Because the force $\vec{F}$ and inertial mass m are supposed to be invariant with respect to Galileo's transformation (2.1), also the Newton's law

$$m\frac{d^2}{dt'^2}\vec{x}' = m\frac{d^2}{dt^2}\vec{x} = \vec{F} \tag{2.2}$$

is invariant. An example of a force (and its invariance) is the Newtonian gravitational force

$$\vec{F}_G = -G\frac{mM}{r^2}\vec{r}^0 \ . \tag{2.3}$$

Unlike Newtonian mechanics, classical electrodynamics is not invariant with respect to the Galileo's group of transformations (2.1). It can be seen e.g., from the fact that while two charges at rest interact by the electrostatic Coulomb force only, in an inertial frame S' each charge represents also an electric current inducing the magnetic field which exerts an additional magnetic component of the Lorentz force on the other charge. Obviously, also the Maxwell's equations are not invariant and, in particular their consequence of the speed of light equal to c could be valid only in an privileged 'rest frame of the ether' according to the Newtonian point of view. However, it has been verified experimentally by Michelson interferometer that the speed of light is the same in all inertial systems. It means that if two light signals will be radiated forward and backward from the event O (see Fig. 2.2, left) then, at the time t_A of the static observer, they will be found in equal distances $AX = AY$. However, another inertial observer at event B can find equal distances to the worldlines of both light signals only if he measures them in the direction $X'Y'$ oblique with respect to the space XY of events simultaneous according to the static observer. It means that each inertial observer finds a different cross-section of the spacetime to be his proper space, i.e., the set of simultaneous events. The simultaneity can be found e.g., by synchronization using light-ranging. On Fig. 2.2, left, the event B is simultaneous with events X' and Y' according to the observer S', because it is in the middle of the time-interval OB_2 (spent by light to travel from O to X' or Y' and back to the event B_2 on the worldline of the observer S'). Analogously on Fig. 2.2, right, B is simultaneous with A according to S (because A is in the middle between A_- and A_+) but B is simultaneous with C according to S' (because B is in the middle between B_- and B_+). Obviously, the obliquity of the proper space $X'Y'$ of the moving observer must be such to form

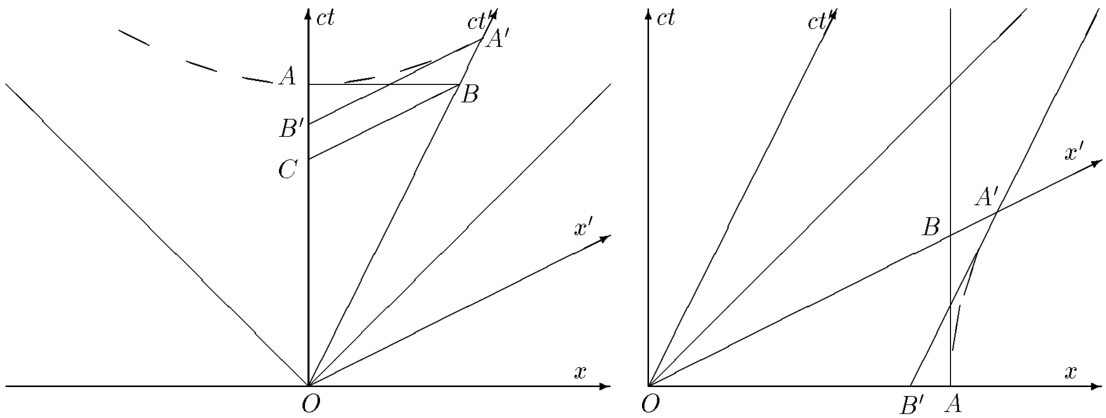

Figure 2.3: Relativity of spacetime units – dilatation of time (left) and contraction of lengths (right).

the other diagonal in the parallelogram $OX'B_2Y'$, i.e., by the same angle $\alpha \equiv \angle AOB = \angle XBX'$, given by relation $\tan\alpha = \beta \equiv \frac{v}{c}$, if the scales of axes x and t are chosen to give the slope of the worldline of photons equal to 45° (i.e., if the length ct is plotted on the t-axis and $OX'B_2Y'$ is thus a rectangle).[1] Consequently, the time-sequence of two events will be found the same by all inertial observers only if one of them is inside the (future or past) light-cone of the other event and they can thus be joined by a (subluminal) worldline. In the opposite case of a couple of events outside the light-cone their sequence or simultaneity is relative.

2.2 Lorentz transformation

If the simultaneity is relative, then also the length of time-unit must be relative: if the interval OA is chosen to be unity according to observer S, and if observer S' would choose OB to comply with S, then S' would find C simultaneous with B and thus the clock of S slower than his own (cf. Fig. 2.3 – left). To preserve the relativity of both observers an interval OA' must be attributed as time-unit of S' in such a way to satisfy $\frac{OA'}{OB} = \frac{OA}{OB'} \equiv \gamma$. Consequently,

$$\gamma^2 = \frac{OA'}{OB}\frac{OA}{OB'} = \frac{OB'}{OC}\frac{OA}{OB'} = \frac{OA}{OC} = \frac{OA}{OA - AC} = \frac{1}{1-\beta^2}\ . \tag{2.4}$$

The clocks of other observers thus seem to each of them to be slower than his own just in the ratio $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ (this is the so-called time dilatation). The set of all points A' defining the unit of time

[1] In relativity there is often used a so-called 'geometric' system of units, in which the units of time and mass are chosen to give the speed of light and the gravitational constant $G = 1 = c$. This choice makes many calculations simpler and the result can always be expressed again in the standard units. However, there are also good reasons not to do it in a textbook. The first one is to demonstrate that relativity naturally works also for nonrelativistic problems. Nonrelativistic physicists use to measure the spacetime quantities in projection to the frame-vectors connected with their laboratories. In these frames the time-vector ∂_t is usually chosen much longer than space-vectors ∂_x to fit the usual slowness of daily life. The 'generality' of the relativistic approach enables also this choice. Another reason to prefer here the use of frames which are only orhogonal instead of the more symmetric orthonormal ones, is to make obvious the geometric difference between these vectors and their covectors dt, dx. Finally, like in the computer physics, the use of real units instead of those fitting the problem in question by numbers of order of 1 yields a natural check of reasonability of the results.

for observers moving from the event O with different velocities forms the hyperbola $c^2t^2 - x^2 = 1$ (or actually a rotational hyperboloid $c^2t^2 - x^2 - y^2 - z^2 = 1$) with asymptotes on the light cone.

To keep the speed of light constant for all observers, the length of unit of spatial distances must be amplified in the same ratio $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. It is well known, that unlike the time dilatation, the length undergoes the relativistic contraction. However, this apparent asymmetry of space and time is caused by the more tricky way in which we compare the lengths of unit-rods of different observers (cf. Fig. 2.3 – right). If the world line of one end of the unit-rod of the unprimed observer coincides with the axis t, then the world line AB of its other end will intersect the axis x' at the event B. The primed observer will thus find its length OB γ-times shorter than the length OA' of his own unit-rod. On the other hand, the world lines t' and $B'A'$ of the primed unit-rod intersect the axis x in events O and B', so that also the unprimed observer will find it γ-times shorter than OA. The events in proper distances (with respect to different observers) from O equal to this unit thus also form the rotational hyperboloid $c^2t^2 - x^2 - y^2 - z^2 = -1$) (rotating around the t-axis) with asymptotes on the light cone. Note that the lengths in directions perpendicular to the velocity are unaffected by the motion, while the t- and x- dimensions change in an inversely proportional ratio, so that the spacetime volume is invariant.

The relativistic transformation (the Lorentz's transformation) between inertial frames moving in the x-direction thus reads

$$\begin{aligned} ct' &= \gamma(ct - \beta x) , \\ x' &= \gamma(-\beta ct + x) , \\ y' &= y , \\ z' &= z . \end{aligned} \tag{2.5}$$

It is easy to see that the so-called spacetime interval[2]

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{2.6}$$

is invariant with respect to this transformation (this is also why the proper space $t' = 0$ of each observer intersects in all directions the hyperboloid $s^2 = +1$ of unit distances just in the unit sphere $x'^2 + y'^2 + z'^2 = 1$). The invariance of this indefinite quadratic form implies also invariance of the scalar product defined in the coordinates t, x, y, z by the matrix (Minkowski tensor) $\eta_{\iota\kappa} = \mathrm{diag}(-c^2, +1, +1, +1)$. Lorentz transformations (in the same directions, boost) form a one-parametric group. Its parameter β satisfies the rule of addition of velocities

$$\beta = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} , \tag{2.7}$$

as it can be seen multiplying matrices corresponding to (2.5)

$$\Lambda(\beta) \equiv \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} = \gamma_1\gamma_2 \begin{pmatrix} 1 & -\beta_1 \\ -\beta_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\beta_2 \\ -\beta_2 & 1 \end{pmatrix} . \tag{2.8}$$

Exercise 1 *Explain the "car and garage paradox" (pp.9-10 of [6]): A car moves into garage of equal proper length and a doorman closes the door immediately as its back enters. According to him the Lorentz contracted car fits inside, while from the point of view of the driver the garage is too small. Who is right?*

Discuss the variant "fast man on a net" of this paradox: The car is relativistically contracted with respect to a net with holes of the same proper length. According to the driver, the holes are contracted. Will the car fall down through the net or not?

[2]Note that in the literature both signatures are used: $(-+++)$ e.g., in [4] or [6], $(+---)$ in [2] or [3].

Exercise 2 *Phileas Fogg II. was asked to synchronize clocks in stations on the circum-equatorial railway. He used the method of light-ranging starting from the main station eastward. What was the result after coming back to the starting point?*

Exercise 3 *Derive the relation (2.4) for the Lorentz γ-factor from the assumption that the so-called Bondi K-factor, i.e., the ratio of proper times of receiving the light signal by one observer to the time of sending it by the other ($\frac{t'_B}{t_{A-}} \equiv K \equiv \frac{t_{A+}}{t'_B}$ in Fig 2.2, right) is the same for both S and S'. Derive the formula (2.7) using the K-factors for the transformations $S \to S'$, $S' \to S''$ and $S \to S''$ (investigate the light-ray DBA_+ in Fig. 2.2, right, from the point of view of each couple of systems).*

Exercise 4 *Find the general formulae (according to special relativity) for the Doppler effect and aberration of light. Hint: Compare the components of the 4-momentum of a photon as seen by two observers moving with different velocities not aligned with the direction of the photon path.*

2.3 Accelerated observers

A non-inertial observer moving along the worldline $x^\iota = x^\iota(s)$ (s being an arbitrary parameter), will measure on his (ideal) clock his proper time τ running at the same rate as the time of an inertial observer instantaneously comoving with him,

$$c\tau = \int \sqrt{-\eta_{\iota\kappa} dx^\iota dx^\kappa} \, . \tag{2.9}$$

He will thus measure the temporal variation of any scalar function $f(x^\iota)$ on the spacetime

$$\frac{d}{d\tau} f = \frac{dx^\iota}{d\tau} \frac{\partial}{\partial x^\iota} f(x) \, . \tag{2.10}$$

The differential operator $U \equiv \frac{d}{d\tau}$, named 4-velocity, is the vector tangent to the worldline (cf. Definition 7) and satisfying $(U, U) = -c^2$.

Let us investigate a one-dimensional accelerated motion $t = t(\tau)$, $x = x(\tau)$ of an observer who will measure acceleration $a = a(\tau)$ in his own space. After an infinitesimal increment $d\tau$ of his proper time, he will reach the velocity $dv' = a.d\tau$ with respect to the primed inertial observer comoving with him (with velocity v) at time τ. With respect to a basic reference inertial observer he will thus reach the velocity

$$v + dv = \frac{v + dv'}{1 + v.dv'/c^2} = v + (1 - \frac{v^2}{c^2})dv' + o(v^2) \, . \tag{2.11}$$

Expressing dv' from here and integrating, we arrive at

$$\int a d\tau = \int \frac{dv}{1 - \frac{v^2}{c^2}} = c.\text{argtanh}(\frac{v}{c}) \, . \tag{2.12}$$

From here

$$v = c.\text{tgh} \int \frac{a d\tau}{c} = \frac{dx}{d\tau} \frac{d\tau}{dt} = \frac{dx}{d\tau} \sqrt{1 - \frac{v^2}{c^2}} = \frac{dx}{d\tau} \frac{1}{\cosh \int a d\tau / c} \, , \tag{2.13}$$

and finally by integration

$$x = \frac{c^2}{a} \cosh \int \frac{a d\tau}{c} \tag{2.14}$$

and

$$t = \frac{c}{a}\sinh \int \frac{a d\tau}{c} \, . \tag{2.15}$$

It is possible to calculate from Eq. (2.13) also the coordinate acceleration observed by the static observer

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{d\tau}\frac{d\tau}{dt} = c\frac{a/c}{\cosh^2 \int a d\tau / c}\sqrt{1-\frac{v^2}{c^2}} = a\left(1-\frac{v^2}{c^2}\right)^{3/2} \, , \tag{2.16}$$

which is consistent with the special-relativistic dynamics according to which the inertial mass is a function of velocity

$$F \equiv m_0 a = \frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}\right) \, . \tag{2.17}$$

For a constant proper acceleration, the integration in Eqs. (2.14) and (2.15) reduces to a simple product $(a\tau/c)$. This result corresponds in the non-relativistic limit $c \to \infty$ to the standard result $t \simeq \tau$, $x \simeq x_0 + at^2/2$. The resulting worldline is (up to the scaling factor a and arbitrary shift in x) identical with the hyperbola AA' of events in unit distance from O in Fig. 2.3 right. The lines OA, OA' are thus proper spaces of the accelerated observer in different times. Worldlines in constant proper distances from this solution correspond to other solutions of (2.12) with initial (at $t = 0$) values of the acceleration $a = c^2/x$. The system of these worldlines (for the initial $x > 0$) fills only one quarter of the spacetime (above the past and below the future light cones of events $x = 0$, $t = 0$). Their parameters τ, a can be used as alternative curvilinear coordinates in this region of spacetime. These coordinates (proper times τ of the observers and a or, alternatively, their proper distances $x' = c^2/a$ from O) are not orthogonal with respect to the Minkowskian metric. A set of time-orthogonal comoving coordinates t', x' can be obtained introducing the 'coordinate' time $t' = a\tau/a_0$ (given by the proper time of one of the accelerated observers chosen as a reference one). The transform from the primed to the inertial coordinates reads

$$x = x'\cosh\frac{a_0 t'}{c} \, , \tag{2.18}$$

$$t = x'\sinh\frac{a_0 t'}{c} \, , \tag{2.19}$$

and the spacetime interval (in this so-called Rindler spacetime) reads

$$ds^2 = -c^2dt^2 + dx^2 = -\left(\frac{a_0 x'}{c^2}\right)^2 c^2 dt'^2 + dx'^2 \, . \tag{2.20}$$

The proper time τ for each fixed x' is thus running (compared to the coordinate time t') at the rate proportional to x', which can be interpreted by the accelerated observers as a consequence of the apparent gravitational acceleration in their noninertial system.

If a general dependence of gravitational acceleration on the distance (e.g., Newtonian one, $a = GM/x^2$) should be locally modelled by strips of the spacetime cut from Minkowski spacetime around worldlines of corresponding a in this solution, then they can not be glued together in a flat spacetime (cf. exercise 8).

Exercise 5 *Explain the twin paradox (calculate the proper times of world lines OAE and OBE in Fig. 2.2, right; investigate their instantaneous proper spaces).*

2.4 Electrodynamics

Electromagnetic field is in relativity described by a spacetime 2-form F (i.e., by an antisymmetric tensor field) which acts on a charged particle with charge q and rest mass m_0 (both are its scalar characteristics) and with 4-velocity $U = \frac{d}{d\tau}$ (τ being the proper time of the particle) by the Lorentz force resulting in its 4-acceleration

$$a^\iota \equiv \frac{D}{d\tau} U^\iota = \frac{q}{c m_0} F^\iota{}_\kappa U^\kappa \ . \tag{2.21}$$

This dynamical law is obviously consistent with the normalization condition $U^2 = -c^2$, because $\frac{d}{d\tau} U^2 = 2(Ua) = 0$ due to the antisymmetry of F.

It corresponds also to the non-relativistic view where in a chosen observers reference frame there is F described by two 3-vectors E of an electric and B of a magnetic field[3]

$$F_{\iota\kappa} = \left(\begin{array}{c|cccc} & {\scriptstyle \kappa=0} & {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} \\ \hline {\scriptstyle \iota=0} & 0 & -E^1 & -E^2 & -E^3 \\ {\scriptstyle 1} & E^1 & 0 & B^3 & -B^2 \\ {\scriptstyle 2} & E^2 & -B^3 & 0 & B^1 \\ {\scriptstyle 3} & E^3 & B^2 & -B^1 & 0 \end{array} \right) \ . \tag{2.22}$$

In this "3+1"-formalism we then get for components $U^\iota = \gamma(c, v^i)$ linearly dependent equations

$$\frac{d(\gamma v^i)}{d\tau} = \frac{\gamma q}{m_0} (E^i + \epsilon^{ikl} \frac{v_k}{c} B_l) \ , \tag{2.23}$$

$$\frac{d\gamma}{d\tau} = \frac{\gamma q}{c m_0} E^k v_k \ . \tag{2.24}$$

The components of F are interrelated between different reference frames by a usual Lorentz transformation. For a pure space rotation both E and B are transformed like vectors (because they are defined by the same u). On the other hand, for a pure boost in direction n (cf. Eq (2.8))

$$\Lambda^\iota{}_\kappa = \begin{pmatrix} \gamma & \gamma\beta n \\ \gamma\beta n & \mathbf{1} + (\gamma - 1) n \otimes n \end{pmatrix} \ , \tag{2.25}$$

they are mixed by the mutual induction

$$E' = [\gamma \mathbf{1} + (1-\gamma) n \otimes n] E + \gamma\beta n \times B \ , \tag{2.26}$$
$$B' = [\gamma \mathbf{1} + (1-\gamma) n \otimes n] B - \gamma\beta n \times E \ , \tag{2.27}$$

(where $\times$ is the vector product – cf. Definition 4). There can be found reference frames in which E and B are parallel, and

$$F = -e\, dx^0 \wedge dx^1 + b\, dx^2 \wedge dx^3 \ . \tag{2.28}$$

[3] These can be defined for a given 4-velocity u of the observer as 4-vectors perpendicular to it by

$$E^\iota \equiv \frac{1}{c} F^\iota{}_\kappa u^\kappa \ ,$$
$$B_\iota \equiv \frac{1}{2c} \epsilon_{\iota\kappa\lambda\mu} u^\kappa F^{\lambda\mu} \ .$$

In different systems of units these vectors are also measured by independent units.

There exist two scalar characteristics of F (which are thus Lorentz invariant)

$$|F|^2 = F^{\iota\kappa} F_{|\iota\kappa|} = B^2 - E^2 \ , \tag{2.29}$$

and

$$\frac{1}{4}\langle {}^*F, F\rangle = \epsilon^{\iota\kappa\lambda\mu} F_{|\iota\kappa|} F_{|\lambda\mu|} = (EB) \ , \tag{2.30}$$

where the dual 2-form ${}^*F$ has components

$${}^*F_{\iota\kappa} = \epsilon_{\iota\kappa\lambda\mu} F^{\lambda\mu} = \left(\begin{array}{c|cccc} & {\scriptstyle \kappa=0} & {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} \\ \hline {\scriptstyle \iota=0} & 0 & B^1 & B^2 & B^3 \\ {\scriptstyle 1} & -B^1 & 0 & E^3 & -E^2 \\ {\scriptstyle 2} & -B^2 & -E^3 & 0 & E^1 \\ {\scriptstyle 3} & -B^3 & E^2 & -E^1 & 0 \end{array}\right) \ . \tag{2.31}$$

The dynamical law for F are the Maxwell's equations which can be written using the n-forms formalism in a compact form

$$dF = 0 \ , \tag{2.32}$$

$$\delta F \equiv {}^*d^*F = 4\pi j \ , \tag{2.33}$$

where j is the 4-current density and $\delta \equiv {}^*d^*$ is the 'interior derivative'. The first of these equations can be satisfied if the 4-potential A is introduced so that

$$F = dA \ . \tag{2.34}$$

The 4-potential is determined by this equation up to the gauge transformation $A \to A'$,

$$A' = A + df \ , \tag{2.35}$$

where f is an arbitrary scalar function. Maxwell's equations (2.32) and (2.33) can be written in components as

$$F_{\iota\kappa,\lambda} + F_{\kappa\lambda,\iota} + F_{\lambda\iota,\kappa} = 0 \ , \tag{2.36}$$

$$F^{\iota\kappa}{}_{,\kappa} = 4\pi j^\iota \ , \tag{2.37}$$

and in "3+1"-formalism

$$(\vec{\nabla}.\vec{B}) = 0 \ , \qquad \partial_0 \vec{B} + \vec{\nabla} \times \vec{E} = 0 \ , \tag{2.38}$$

$$(\vec{\nabla}.\vec{E}) = 4\pi\rho \ , \qquad \partial_0 \vec{E} - \vec{\nabla} \times \vec{B} = 4\pi\vec{j} \ , \tag{2.39}$$

where $\rho \equiv j^0$ is the charge density. Eq. (2.34) reads in these formalisms

$$F_{\iota\kappa} = A_{\iota,\kappa} - A_{\kappa,\iota} \ , \tag{2.40}$$

$$\vec{B} = -\vec{\nabla} \times \vec{A} \ , \qquad \vec{E} = \partial_0 \vec{A} - \vec{\nabla}\varphi \ . \tag{2.41}$$

These equations can be obtained from a variational principle with the Lagrange density of free electromagnetic field

$$\mathcal{L} = -\frac{1}{16\pi c} F^{\iota\kappa} F_{\iota\kappa} \ , \tag{2.42}$$

where the variable is the 4-potential A, so that (2.36) is implicitly satisfied and the explicit form of the Lagrangian is

$$\mathcal{L} = -\frac{1}{8\pi c}\left(A^{\iota,\kappa} A_{\iota,\kappa} - A^{\kappa,\iota} A_{\iota,\kappa}\right) ; . \tag{2.43}$$

According to (B.22) there follows (2.37).

The stress-energy tensor can be chosen symmetric as

$$T_\iota{}^\kappa = \frac{\partial \mathcal{L}}{\partial A_{\lambda,\kappa}} A_{\lambda,\iota} - \delta_\iota^\kappa \mathcal{L} + \frac{1}{4\pi c} F^{\lambda\kappa} A_{\iota,\lambda} \ , \tag{2.44}$$

i.e.,

$$T_\iota{}^\kappa = \frac{1}{4\pi c} \left(-F^{\lambda\kappa} F_{\lambda\iota} + \frac{1}{4} \delta_\iota^\kappa F^{\lambda\mu} F_{\lambda\mu} \right) \ . \tag{2.45}$$

In "3+1" formalism

$$4\pi c T^{\iota\kappa} = \left(\begin{array}{c|cccc} & {\scriptstyle \kappa=0} & {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} \\ \hline {\scriptstyle \iota=0} & -\frac{E^2+B^2}{2} & & E \times B & \\ {\scriptstyle 1} & & & & \\ {\scriptstyle 2} & E \times B & & E \otimes E + B \otimes B - \mathbf{1}\frac{E^2+B^2}{2} & \\ {\scriptstyle 3} & & & & \end{array} \right) \ . \tag{2.46}$$

The interaction Lagrangian is $\frac{q}{c}(Aj)$.

Chapter 3

Basic concepts of general relativity

In this chapter will be summarized the basic concepts of the general relativity. First, in Section 3.1, some reasons for the use, and illustrations of the consequences of differential geometry will be presented (differential geometry is reviewed in Appendix A). Together, this section and Appendix A thus correspond to chapters 2 and 3 (and appendix B) of [6]. In the following Sections (corresponding mainly to chapter 4 of [6]) will be explained the physical motivation for identifying the gravitational field with the geometry of spacetime, the formulation of the interaction of gravity with other physical fields and particles, and finally its correspondence with the Newton theory will be proven.

3.1 Geometry of curved spacetime

The concept of manifold (Definition 6) is used to handle effectively the natural topology of spacetime (or other physical spaces). General relativity forces us to limit the use of coordinate systems only as names (and arrangement) of events, and to treat the distances as an additional structure on the spacetime. However, even in flat spacetime or in nonrelativistic physics it is often advantageous to use some curvilinear coordinates to fit the symmetry of the problem in question, despite that the expression for distance will then be more complicated (cf. Cartesian vs. geographic spherical coordinates on the Earth).

Because of this limited significance of coordinates, it is advantageous to define all geometrical objects independently of the existence and choice of a particular coordinate system. This is the case of Definition 7 of vectors as differential operators. Having this definition of vectors, we arrive at their natural expression in any coordinate system

$$X = X^i \frac{\partial}{\partial x^i} = X'^j \frac{\partial}{\partial x'^j} \tag{3.1}$$

and the transformation properties of their components (cf. Eq. (A.18) for the relation of bases)

$$X^i = X'^j \frac{\partial x^i}{\partial x'^j} \ . \tag{3.2}$$

In the classical approach, quantities with components obeying just these transformation rules are defined as the 'contravariant vectors' unlike the so-called 'covariant vectors' obeying the rule (cf. Eq. (A.26))

$$\omega_i = \omega'_j \frac{\partial x'^j}{\partial x^i} \ . \tag{3.3}$$

However, to avoid a confusion, it is better to keep in mind that these are the co-vectors or 1-forms, i.e., elements of the dual vector space (see Definition 1). They can be spanned by exterior derivatives (differentials) of scalars – see Eq. (A.23) in Definition 10. Differentials dx^i of coordinates form a natural coordinate-induced basis, which is dual to the partial derivatives $\frac{\partial}{\partial x^i}$,

$$\omega = \omega_i dx^i \tag{3.4}$$

(from which the above transformation rule of covariant components easily follows).

Following Definition 5, there is a one-to-one correspondence between vectors and covectors defined by the scalar product (i.e., by the metric). In the physics (and geometry) of spacetime and other spaces of interest, we usually have a fundamental metric with respect to which we can identify these spaces and to take contravariant and covariant components as different expressions of the same physical quantity. It is important to remember that the values of these components can differ (in pseudo-Riemannian spaces at least by sign even in orthonormal bases). This is why upper and lower indices are distinguished in general relativity. As an example we can take the 4-velocity $U \equiv \frac{d}{d\tau}$ which has contravariant components $U^\iota = (\gamma, \gamma v^i)$ in the coordinate basis $\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i}\}$ and covariant $U_\iota = (-\gamma c^2, \gamma v^i)$ in the dual basis $\{dt, dx^i\}$ in which $ds^2 = -c^2 dt^2 + (dx^i)^2$. There are physical problems for which it is reasonable to introduce some additional metrics. The physical meaning of vectors and covectors and their correspondence must be then investigated carefully.

In practice, it is often advantageous to use a basis other than the coordinate bases both for vectors and covectors. For instance, in the geographic coordinates φ, λ, in which the distance element ds reads (on a spherical Earth)

$$d^2 s = r^2 (d^2\varphi + \cos^2\varphi d^2\lambda) \ , \tag{3.5}$$

we use to express the vectors of ship velocities (i.e., derivatives with respect to their time) in east-west and north-south components V^1, V^2, resp., in the same units, i.e., with respect to an orthonormal basis

$$e_1 = \frac{1}{r}\frac{\partial}{\partial\varphi} \ , \qquad e_2 = \frac{1}{r\cos\varphi}\frac{\partial}{\partial\lambda} \ , \tag{3.6}$$

or even in polar coordinates in the tangent space, i.e., by the azimuth and the absolute value $|V|$ (e.g., in knots), which can be calculated from the components using the simple metric

$$g_{ij} = \mathrm{diag}(1, 1) \ . \tag{3.7}$$

The basis vectors need not commute (in the sence of Eq. (A.20), e.g., in our case

$$[e_1, e_2] = [\frac{\partial}{r\partial\varphi}, \frac{\partial}{r\cos\varphi\partial\lambda}] = \frac{\sin\varphi}{r^2\cos^2\varphi}\frac{\partial}{\partial\lambda} = \frac{\tan\varphi}{r} e_2 \ , \tag{3.8}$$

but they are more appropriate to the local physics. The dual basis to (3.6) reads

$$\theta^1 = r.d\varphi \ , \qquad \theta^2 = r\cos\varphi.d\lambda \ , \tag{3.9}$$

and its exterior derivatives

$$d\theta^1 = 0 \ , \qquad d\theta^2 = -r\sin\varphi d\varphi \wedge d\lambda = -\frac{\tan\varphi}{r}\theta^1 \wedge \theta^2 \ . \tag{3.10}$$

Because the components of the metric tensor in the orthonormal basis are constant, it follows from Eq. (A.50) that

$$\omega_{11} = \omega_{22} = 0 \ , \quad \text{and} \quad \omega_{12} = -\omega_{21} \ . \tag{3.11}$$

Following the first equation of structure (A.47), the 1-forms of the torsion-free metric connection thus satisfy

$$0 = \omega_{12} \wedge \theta^2 \ , \qquad 0 = -\frac{\tan\varphi}{r}\theta^1 \wedge \theta^2 + \omega_{21} \wedge \theta^1 \ . \tag{3.12}$$

It is obvious from the first of these equations that ω_{12} must be proportional to θ^2 only, and from the second one we thus find

$$\omega_{12} = -\omega_{21} = -\frac{\tan\varphi}{r}\theta^2 \ . \tag{3.13}$$

The equations of the geodesics thus read

$$\frac{dV^1}{dt} = \frac{\tan\varphi}{r}V^2V^2 \ , \qquad \frac{dV^2}{dt} = -\frac{\tan\varphi}{r}V^1V^2 \ . \tag{3.14}$$

It can be verified from here that the value of the velocity is constant. Following the second equation of structure (A.48), the 2-forms of curvature on the globe are

$$\Omega_{ij} = \epsilon_{ij} d\omega_{12} = \epsilon_{ij} d(\sin\varphi d\lambda) = \epsilon_{ij}\cos\varphi d\varphi \wedge d\lambda = \epsilon_{ij}\frac{1}{r^2}\theta^1 \wedge \theta^2 \ , \tag{3.15}$$

where $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$ are components of a unit antisymmetric matrix. This result is consistent with the fact that the curvature is constant on a sphere.

In general relativity, similar orthonormal frames correspond to local inertial observers – they lay out in the spacetime the directions of the observer's proper time and space. It is thus advantageous to describe the physical quantities which are of vector or tensor character with respect to just these frames (so that their components are identical with those found by local observers in their laboratory measurements) and to use the relation of these frames to the coordinate system only for studying the change of the quantities in the course of their propagation through the spacetime.

It is worth noting that a measure (element of volume, i.e., integration, which is needed to postulate variational principles for fields) on an n-dimensional manifold is determined by the choice of the unit n-form. A metric is thus not necessary, however, it is natural for a given metric to choose as the unit n-form the exterior product of some orthonormal dual basis (e.g., in the above example $r^2\cos\varphi d\varphi \wedge d\lambda = \theta^1 \wedge \theta^2$).

Exercise 6 *Prove that a tensor $F \in T_2^0$, which for all $X \in T$ gives $F(X,X) = 0$, must be antisymmetric, i.e., it must be a 2-form.*

Exercise 7 *Investigate the parallel transport (affine connection) on a general two-dimensional surface (and in a plane and on a sphere in particular) defined by a compass in vector H of a magnetic field with a general magnetic deviation.*

Exercise 8 *Calculate both in coordinate and orthonormalized bases of vectors the (torsion-free) metric connection and corresponding curvature of two-dimensional spacetime with the metric given by*

$$ds^2 = -f(x)^2 dt^2 + dx^2 \ .$$

Discuss the results in connection with the problem of accelerated observers in special relativity.

Exercise 9 *Calculate affine connections and curvature in a two-dimensional conformally flat spacetime*

$$ds^2 = f^2(t,x)\left(-dt^2 + dx^2\right) \ .$$

Investigate the particular case $f = 1/(\cos(x+t)\cos(x-t))$, and explain the result.

Exercise 10 *For a general affine connection ∇ prove Theorem 4 that (i) the torsion Q and (ii) the curvature R defined by equations Eqs. (A.41) and (A.42) are tensors.*

Exercise 11 *Using the equations of structure (A.47) and (A.48) prove Theorem 6 that (i) for a metric connection (for which Eq. (A.50) is valid) the components of the tensor of curvature satisfy the antisymmetry (A.53), (ii) that for a torsion-free connection they satisfy the symmetry (A.54), and (iii) the Bianchi identities (A.55).*

3.2 Principles of general relativity

General relativity is the relativistic theory of gravitation. It is required from any new physical theory to satisfy the so-called **principle of correspondence**, i.e., to give the same results as the previous theories to be revised in the limiting cases when the assumptions of these theories are satisfied and they successfully explain the observed facts (cf. Newtonian mechanics as limit of special relativity for $c \to \infty$ or classical mechanics as the limit of quantum mechanics for $h \to 0$). Historically, the reason for revising the Newtonian theory of gravitation was not its contradiction with experimental results,[1] but its inconsistency with special relativity, which has to be accepted due to the experimental results in electrodynamics. The Newton law of gravitation (2.3) supposes instantaneous interaction at a distance. This is inconsistent with the relativity of simultaneity. The Newtonian theory of gravitation is often equivalently formulated as a field theory with a scalar gravitational potential Φ determining the gravitational force

$$\vec{F}_G = -m\vec{\nabla}\Phi \ , \tag{3.16}$$

and being determined by the mass M of a point-like source

$$\Phi = -G\frac{M}{r} \ , \tag{3.17}$$

or as a linear superposition of such fields corresponding to several sources. Φ can also be determined in differential form by the distribution of mass density ρ according to the Poisson equation

$$\Delta\Phi = 4\pi G\rho \ , \tag{3.18}$$

which is equivalent, because $\frac{1}{r}$ is the Green's function for the operator Δ.

This field formulation of Newton's gravity could, in principle, be adapted (e.g., replacing here the Laplacian operator $\Delta \equiv \nabla^2$ by the d'Alembertian $\Box \equiv -c^{-2}\partial_t^2 + \Delta$, i.e., integrating the retarded potential along the past light-cone) to carry the gravitational information with the speed of light (or some other velocity, but then a privileged 'rest frame' of the field must be postulated, which would violate the equivalence of all inertial systems).

However, there are additional features in Newton's gravitation which are taken into account in the formulation of the relativistic theory. First is the equivalence of inertial mass (m in Eq. (2.2) defined as the coefficient of proportion of the force acting on a body to the body's acceleration with respect to an inertial frame caused by this force) and the passive gravitational mass (m in Eq. (3.16) defining the force by which a given gravitational field acts on any body). Because there is also no known way of screening the gravitational interaction, all bodies (free of any other force) fall with the same acceleration in the same field, and there is no local experiment distinguishing the

[1] This is due to insufficient precision of previous measurements, which must be increased to test the tiny quantitative differences between predictions of Newtonian gravitation and general relativity within the solar system or to enable observations of distant objects in some of which the differences are large.

presence of a gravitational field from the noninertiality of the reference frame.[2] This experimental evidence has been tested starting from the historical experiment of Galileo in Pisa up to the present cosmic experiments. This (up to now confirmed) experimental fact is known as the **principle of equivalence**, which has several alternative formulations differing in their strength. A weak formulation of the equivalence principle says that gravity is a kinematic force, i.e., it can be described as a non-inertiality of the reference frame, and all bodies free of any other external interaction (with identical initial conditions) will follow in the same gravitational field the same worldline independent of their inner structure.[3] A strong formulation of the equivalence principle says that any local physical experiments have the same outcome (i.e., the physical laws are the same) in any locally inertial (i.e., freely falling) reference frame.[4]

If we accept the equivalence of the gravitational field and the noninertiality of the reference frame, we see (e.g., from the example shown by Eq. (2.20)) that the gravitational field must be described by the components of the metric tensor which give the relation between the coordinates and measurements of space and time distances. The physical meaning of these components can be understood from the following experiment. Imagine a source of monochromatic light with a frequency ν placed in a homogeneous (and stationary) gravitational field with gravitational acceleration g. Let an absorber of the light be placed on a different gravitational potential $\Phi_{abs} = \Phi_{em} + lg$ (l being their vertical distance; imagine for the sake of simplicity the absorber to be just above the emitter). Because of the stationarity of the problem, the absorber will register the same number of waves in the same (synchronized) interval of coordinate time. The frequency ν of the emitted photons must remain constant with respect to the inertial frame which was at rest at the instant of emission, i.e., to the reference frame which just started to fall with an acceleration g at that moment. The photon will reach the absorber after the time $t = l/c$ when the inertial frame will move with the velocity $v = gt = (\Phi_{abs} - \Phi_{em})/c$. The absorber is thus receding with respect to this inertial frame and will see the photon (gravitationally) red-shifted for $\Delta\nu/\nu \sim \Delta\Phi/c^2$.[5] It means that the proper time τ of the absorber is running with a higher rate than the coordinate time just in the ratio

$$d\tau/dt = \sqrt{|g_{00}|} \simeq 1 + \Phi/c^2 \,, \tag{3.19}$$

if the coordinate time is chosen to be the proper time of an observer who is at rest at infinity, where $\Phi = 0$. This is consistent with the relation between the metric of the accelerated observer and the potential of his apparent gravitational field – see Eq. (2.20)). The components of the metric tensor have thus the meaning of gravitational potentials, their derivatives (the components of 1-forms of connections) the meaning of gravitational acceleration, and their second derivatives meaning of gradients of intensity of the gravitational field, i.e., they specify the tidal forces.

Another idea which played an important heuristic role in the development of general relativity was **Mach's principle**, according to which the preference of the inertial frames (in the Newtonian mechanics) must be due to the interaction of local bodies with the matter in distant parts of the universe. This idea, which has not been convincingly mathematically incorporated into

[2]A non-local experiment may consist in ranging the position of a test particle with respect to another one far away from the gravitating body, where the gravity vanishes, or at least with respect to a slightly distant particle, where the gravitational field has different intensity and direction.

[3]The distribution of mass in bodies extending on a scale comparable to inhomogeneities of the gravitational field is not in this sense treated as their inner structure but rather as several mutually interacting bodies in different fields. Several mechanisms based on this principle have been proposed to change orbits of satellites or the spin-orbital interaction in binaries is another example of this mechanism.

[4]Speaking about 'the same shape' of physical laws in different frames, the number of dynamical variables must be specified. Obviously, by adding new parameters, any law can be generalized to any wider class of frames.

[5]This experiment has been performed in practice historically first by measurement the gravitational redshift of spectral lines in the solar atmosphere and recently also on the Earth using the Mössbauer effect (with l of the order of meters).

the Newton's theory, is confirmed by the relativistic effect of the dragging of inertial frames (cf. exercise 14).

3.3 Motion in curved spacetime

In general relativity we suppose that the spacetime is a 4-dimensional pseudo-Riemannian[6] manifold with the signature $(-+++)$, i.e., it locally resembles the Minkowskian spacetime and an orthonormal frame of basis vectors can locally correspond to the inertial systems in the Newtonian physics. The metric torsion-less connection (i.e., the Riemannian connection) determines how the directions of inertial coordinate axes should be prolonged to the neighbouring events. It means also that the metric connection determines how the 4-momentum of free particles (proportional to their 4-velocities $U = \frac{d}{d\tau}$ and thus being tangent to their worldlines) should be transported along the worldlines

$$0 = \frac{D}{d\tau}U = \left(\frac{d}{d\tau}U^{\iota} + U^{\kappa}\omega^{\iota}{}_{\kappa}(\frac{d}{d\tau})\right) e_{\kappa} \ , \tag{3.20}$$

so that they are thus 'straight'. An alternative view on how a particle can recognize its proper way in curved spacetime is yielded by Feynman's approach to quantum mechanics, using the integrals on paths. The wave-function is a superposition of contributions of all paths

$$\psi = \int \exp \frac{iS}{h} \ , \tag{3.21}$$

but, due to the rapidly varying phases, only the contributions of paths close to the extremal of action S are increased by their interference, while others mutually cancel. In this view, free particles in spacetime (the action of which is given, in agreement with Eq. (A.56), by their proper time) spread along the geodesics, i.e., the extremal line.[7] Following Theorem 7 the resulting equation of motion again is (3.20).

If we follow different geodesic worldlines $x = x(\tau, \epsilon)$ labeled by a 'comoving' coordinate ϵ, each one parametrized by its proper time τ, the 4-velocity $U = \frac{\partial}{\partial\tau}$ commutes with the vector $\xi = \frac{\partial}{\partial\epsilon}$ of displacement of the geodesic (because they are both coordinate basis-vectors in the coordinates τ, ϵ). Hence in the torsion-free connection

$$0 = [U, \xi] = \nabla_U \xi - \nabla_\xi U \ , \tag{3.22}$$

where $v = \nabla_U \xi$ is the relative velocity of the worldlines. Their relative acceleration thus is

$$a = \nabla_U v = \nabla_U \nabla_U \xi = \nabla_U \nabla_\xi U = ([\nabla_U, \nabla_\xi] - \nabla_{[U,\xi]})U = \langle R, U \otimes U \otimes \xi \rangle \tag{3.23}$$

(because the terms added to correspond to the definition (A.42) are zero due to (3.22) and the equation of geodesic motion $\nabla_U U = 0$). According to this equation of geodesic deviation the relative acceleration of nearby geodesic worldlines thus is determined by the curvature of spacetime. This corresponds to the relative acceleration of freely falling particles due to the tidal forces in Newtonian theory, $\nabla a = -\nabla\nabla\Phi$.

[6] Following the Theorem 6 we mean by Riemannian geometry the torsion-free metric connection and the corresponding curvature. By 'pseudo-' we distinguish the case of the indefinit metric.

[7] For the time-like lines it is the longest line, because the proper time of an accelerated particle with relativistic velocities will be smaller like in the twin-paradox.

3.4 Field equations

If we already know that the spacetime metric (or its equivalents in non-coordinate vector bases) plays the role of gravitational potentials and that it forces (by the components of 1-forms of connection) the particles to move along its geodesics, we need to find the field equations to complete the theory. These equations must be a generalization of Eq. (3.18) and they must determine the influence of the sources (bodies) on the metric. In Newton's theory the source of the gravitation is the active gravitational mass, which is postulated to be equal also to the passive one and to the inertial mass. This is necessary to satisfy the conservation of momentum of an isolated self-gravitating system. Because in relativity the inertial mass is given by the total energy of the bodies (including their kinetic and even the bounding potential energy), the source term in the field equations cannot be given simply as a sum of charges of individual particles (e.g., their rest masses). However, the density of energy is not a scalar quantity, but only one of the components of the stress-energy tensor. The value of the energy density with respect to a particular reference frame of an observer is determined by the projection of the stress-energy tensor into the direction of his 4-velocity. It is thus reasonable to suppose that the whole stress-energy tensor $T_{\mu\nu}$ will be the source term in a frame-independent formulation of the field equations.[8]

The left-hand side of the field equations can be constructed, in principle, in an infinite number of ways. Only some of them will lead to a reasonable theory yielding, in accordance with the principle of correspondence, the Newtonian approximation as their special case. There have been found several different generalizations predicting slightly or more different post-Newtonian corrections. The choice between them is, first of all, a subject of their experimental verification. Einstein's standard general relativity has been verified to within the observational limits, and this theory is remarkable amongst others by its simplicity. Experimentally, there cannot be excluded also those generalizations, which can for a special choice of their free parameters explain the same results as the Einstein's theory. However, it is thus practical to prefer just this simplest theory. Several alternative ways of reasoning can be used to derive the equations of Einstein's theory and to demonstrate its predominance consisting in its mathematical beauty. Nevertheless, it is important to remember that, as with any other physical theory, the crucial decision is on the nature, and our aesthetic criteria would be modified to understand and justify the theory approved by experience. We will thus not make as though the equations here are derived in a unique possible way. Instead, we will postulate it (with only a few explanatory remarks) and we will show how these equations give the Newtonian approximation.

If we suppose that the components of the metric tensor should substitute the potential in Eq. (3.18), it is worth noting that the equation is no longer linear in these variables, because the (contravariant, i.e., inverse) metric is implicitly included also in the differential operator ($\Delta = g^{ij}\nabla_i\nabla_j$). If we suppose the field equations to be also differential of the second order, we can look for equations linear in the second derivatives, but nonlinear in the first derivatives and the values of the components. Because the stress-energy tensor (which is of the type (0,2)) is supposed to determine the source term, we can try simply to equate it to a tensor of the same order constructed from the geometry (despite that this is not the only possible construction). A tensor which can be constructed from the metric as linear in its second derivatives is the Riemann tensor $R^{\iota}{}_{\kappa\lambda\mu}$, i.e., the tensor of curvature of the torsion-free metric connection. This tensor is of type (0,4), however, its contraction (e.g., in the first and the third index, which is named the Ricci tensor $R_{\kappa\mu} = R^{\iota}{}_{\kappa\iota\mu}$) is just of the type (0,2) as is the stress-energy tensor. The contraction of the Ricci tensor is the scalar curvature, which is also linear in the second derivatives, so that multiplied by the metric itself it is also of the desired type. There can also be a term proportional to the metric only (not

[8] Just the mixed space- and time- components T_{0k}, representing the momentum density of the matter can influence the solution in terms of Mach's principle.

to its derivatives). Such a term (named for its possible significance the cosmological term) can also be treated as the stress-energy tensor of the vacuum and added to the right-hand side; we will thus omit it for now. The expected form of the field equations thus is

$$R_{\iota\kappa} + aRg_{\iota\kappa} = kT_{\iota\kappa} \ . \tag{3.24}$$

The constant a can be determined by the following reasoning. The Riemann tensor identically satisfies the Bianchi identities

$$R^{\iota}{}_{\kappa\lambda\mu;\nu} + R^{\iota}{}_{\kappa\nu\lambda;\mu} + R^{\iota}{}_{\kappa\mu\nu;\lambda} = 0 \ , \tag{3.25}$$

(see Theorem 6) which, contracted e.g., in pairs of indices ι, λ and $\kappa\mu$, yield

$$R_{,\nu} - 2R^{\mu}{}_{\nu;\mu} = 0 \ . \tag{3.26}$$

Consequently, if we choose $a = \frac{1}{2}$, the left-hand side of the field equations (3.24) will be given by the so-called Einstein tensor

$$G_{\iota\kappa} = R_{\iota\kappa} - \frac{1}{2}Rg_{\iota\kappa} \ , \tag{3.27}$$

the divergence of which vanishes identically. The conservation of the 4-momentum of the mass

$$0 = T^{\iota\kappa}{}_{;\kappa} \tag{3.28}$$

(i.e., of the non-gravitational fields on the right-hand side) which, following Noether's[9] theorem, should be consequence of a symmetry of the spacetime, will thus follow from the field equations determining the spacetime geometry.

The constant k, which should be proportional to the gravitational constant, must be found according to the principle of correspondence from the weak-field approximation. This approximation must also verify if the field equations in this form give a viable generalization of Newton's theory of gravity. Let us thus bring first the final result, which will then be verified in the next Section. The Einstein's field equations read

$$G_{\iota\kappa} \equiv R_{\iota\kappa} - \frac{1}{2}Rg_{\iota\kappa} = \frac{8\pi G}{c^4}T_{\iota\kappa} \ . \tag{3.29}$$

Exercise 12 *Using the symmetries of the Riemann tensor*

$$R_{(ik)lm} = R_{ik(lm)} = R_{i(klm)} = 0 \ ,$$

prove the symmetry of the Ricci tensor.

Exercise 13 *Using the symmetries of the Riemann tensor*

$$R_{(ik)lm} = R_{ik(lm)} = R_{i(klm)} = 0 \ ,$$

show that the number of its algebraically independent components in an n-dimensional manifold is $n^2(n^2-1)/12$.

[9] According to Noether's theorem, each law of conservation corresponds to a symmetry of the problem in question.

3.5 Weak-field approximation

Let us suppose that in the coordinate frame dx^ι of nearly Minkowskian coordinates (we now choose $x^0 = ct$) the spacetime metric is described by a small smooth perturbation $\gamma_{\iota\kappa}$ of the Minkowskian metric $\eta_{\iota\kappa} = \mathrm{diag}(-1,+1,+1,+1)$,

$$g_{\iota\kappa} = \eta_{\iota\kappa} + \gamma_{\iota\kappa} \ . \tag{3.30}$$

By the smallness we understand that in a Taylor's expansion of any function of the metric, the higher order terms can be neglected, and by the smoothness that also the derivatives of the perturbation are negligible of the same order. The contravariant, i.e., the inverse metric[10] thus reads

$$(g^{-1})^{\iota\kappa} = \eta^{\iota\kappa} - \gamma^{\iota\kappa} \ , \tag{3.31}$$

where the raising (or lowering) of indices is performed by η. For the metric connection

$$d\gamma_{\iota\kappa} = \omega_{\iota\kappa} + \omega_{\kappa\iota} \ , \tag{3.32}$$

in the coordinate frame, the first equations of structure

$$0 = \omega_{\iota\kappa} \wedge dx^\kappa \tag{3.33}$$

have solutions (cf. Eq. (A.52))

$$\omega_{\iota\kappa} = \frac{1}{2}\left[d\gamma_{\iota\kappa} + (\gamma_{\lambda\iota,\kappa} - \gamma_{\kappa\lambda,\iota})dx^\lambda\right] \ . \tag{3.34}$$

In the second equations of structure the quadratic term can be neglected, hence

$$\Omega_{\iota\kappa} = d\omega_{\iota\kappa} = \frac{1}{2}(\gamma_{\lambda\iota,\kappa\mu} - \gamma_{\kappa\lambda,\iota\mu})dx^\mu \wedge dx^\lambda \ . \tag{3.35}$$

The Ricci tensor can be obtained by contraction with $\frac{\partial}{\partial x^\iota}$ in the form

$$R_{\kappa\lambda} = \frac{1}{2}\left(\gamma_{\lambda\iota,\kappa}{}^{\iota} + \gamma_{\kappa\iota,\lambda}{}^{\iota} - \gamma^{\iota}{}_{\iota,\kappa\lambda} - \gamma_{\kappa\lambda,\iota}{}^{\iota}\right) \ , \tag{3.36}$$

and the scalar curvature is

$$R = \gamma_{\mu\iota,}{}^{\mu\iota} - \gamma^{\mu}{}_{\mu,}{}^{\iota}{}_{\iota} \ . \tag{3.37}$$

The Einstein tensor thus reads

$$\begin{aligned} G_{\kappa\lambda} &= \frac{1}{2}\left(\gamma_{\lambda\iota,\kappa}{}^{\iota} + \gamma_{\kappa\iota,\lambda}{}^{\iota} - \gamma^{\iota}{}_{\iota,\kappa\lambda} - \gamma_{\kappa\lambda,\iota}{}^{\iota} - \eta_{\kappa\lambda}\gamma_{\mu\iota,}{}^{\mu\iota} + \eta_{\kappa\lambda}\gamma^{\mu}{}_{\mu,}{}^{\iota}{}_{\iota}\right) = \\ &= \frac{1}{2}\left(\bar\gamma_{\lambda\iota,\kappa}{}^{\iota} + \bar\gamma_{\kappa\iota,\lambda}{}^{\iota} - \bar\gamma_{\kappa\lambda,\iota}{}^{\iota} - \eta_{\kappa\lambda}\bar\gamma_{\mu\iota,}{}^{\mu\iota}\right) \ , \end{aligned} \tag{3.38}$$

where the so-called trace-reversed perturbation reads

$$\bar\gamma_{\iota\kappa} = \gamma_{\iota\kappa} - \frac{1}{2}\eta_{\iota\kappa}\gamma^{\lambda}{}_{\lambda} \ . \tag{3.39}$$

[10]Note that the exponent -1 is usually omitted only for brevity, when no confusion can arise in the presence of only one metric. However, according to Definition 5 the covariant and contravariant metric tensors are really inverse functions and should be distinguished.

Up to now we hold the coordinate system and its coordinate basis as given. However, physically the same geometry

$$ds^2 = g_{\iota\kappa}dx^\iota dx^\kappa = \tilde{g}_{\iota\kappa}d\tilde{x}^\iota d\tilde{x}^\kappa \simeq \tilde{g}_{\iota\kappa}(dx^\iota dx^\kappa + d\xi^\iota dx^\kappa + dx^\iota d\xi^\kappa) \tag{3.40}$$

is described by different metric tensors

$$\tilde{g}_{\iota\kappa} = g_{\iota\kappa} - \frac{\partial\xi^\lambda}{\partial x^\iota}g_{\lambda\kappa} - g_{\iota\lambda}\frac{\partial\xi^\lambda}{\partial x^\kappa} \ , \tag{3.41}$$

if the coordinates (and the vector bases) are pertubed

$$d\tilde{x}^\iota = dx^\iota + d\xi^\iota(x) \ . \tag{3.42}$$

For the same order of magnitude of the coordinate perturbation and the perturbation of the Minkowskian metric

$$\tilde{\gamma}_{\iota\kappa} = \gamma_{\iota\kappa} - \xi_{\kappa,\iota} - \xi_{\iota,\kappa} \ , \tag{3.43}$$

and

$$\tilde{\bar{\gamma}}_{\iota\kappa} = \bar{\gamma}_{\iota\kappa} - \xi_{\kappa,\iota} - \xi_{\iota,\kappa} + \eta_{\iota\kappa}\xi^\lambda{}_{,\lambda} \ . \tag{3.44}$$

The divergence of this expression reads

$$\tilde{\bar{\gamma}}^\iota{}_{\kappa,\iota} = \bar{\gamma}^\iota{}_{\kappa,\iota} - \xi_{\kappa,}{}^\iota{}_\iota \ . \tag{3.45}$$

For a known divergence $\bar{\gamma}^\iota{}_{\kappa,\iota}$ it is thus possible to solve this Laplace equations for ξ_κ to get the divergence on the left-hand side vanishing. In this so-called Lorentz gauge all but the third term in the Einstein tensor (3.38) vanish, and it becomes

$$G_{\kappa\lambda} = -\frac{1}{2}\bar{\gamma}_{\kappa\lambda,\iota}{}^\iota = -\frac{1}{2}\Box\bar{\gamma}_{\kappa\lambda} \ , \tag{3.46}$$

where we have already omitted the mark ˜ of the new coordinate system, however, we suppose implicitely that our choice of the coordinate system ensures the condition

$$\bar{\gamma}^\iota{}_{\kappa,\iota} = 0 \ . \tag{3.47}$$

The Newtonian approximation should correspond to Eq. (3.18) in the case of a static distribution of the graviting mass. For coherent dust, the stress-energy tensor reads

$$T^{\iota\kappa} = \rho U^\iota U^\kappa \ , \tag{3.48}$$

hence in the comoving coordinate system the stress-energy tensor has the only non-zero component $T^{00} = \rho c^2$. Supposing the field equations in the form

$$-\frac{1}{2}\Box\bar{\gamma}_{\iota\kappa} \simeq G_{\iota\kappa} = kT_{\iota\kappa} \ , \tag{3.49}$$

we thus find that the only non-trivial component of $\bar{\gamma}_{\iota\kappa}$ (i.e., which is not a solution of the homogeneous, i.e., vacuum equations) is $\bar{\gamma}_{00}$. Consequently,

$$\gamma_{00} = \bar{\gamma}_{00} - \frac{1}{2}\eta_{00}\bar{\gamma}^\iota{}_\iota = \frac{1}{2}\bar{\gamma}_{00} \ , \qquad \gamma_{ii} = -\frac{1}{2}\eta_{ii}\bar{\gamma}^\iota{}_\iota = \frac{1}{2}\bar{\gamma}_{00} \ . \tag{3.50}$$

Following Eq. (3.19), the component γ_{00} should correspond to the gravitational potential ($g_{00} = -1 + \gamma_{00} = -1 - 2\Phi/c^2$) and in the static case it obeys the equation

$$\frac{2}{c^2}\Delta\Phi = -\Delta\gamma_{00} = -\frac{1}{2}\Delta\bar{\gamma}_{00} = kT_{00} = kc^2\rho \ , \tag{3.51}$$

which is identical with Newton's one just if $k = \frac{8\pi G}{c^4}$ (or 8π in the geometric units $c = 1 = G$) as it was supposed in Eq. (3.29).

Note that in addition to the predicted slowing down of the time in the gravitational field by the term γ_{00}, there is perturbed also the geometry of the space by the terms γ_{ii}. These terms have negligible influence on the motion much slower than the speed of light, but they are comparable for relativistic velocities. This is why the relativistic bending of light predicted by Einstein is twice higher than that predicted by Soldner on the basis of the corpuscular theory of light and Newtonian gravity.

The Poisson equation (3.51) for γ_{00} (or Φ) can be integrated using the corresponding Green's function (cf. Eq. (3.17)). It can be done also with the general weak-field equations

$$-\frac{1}{2}\Box\bar{\gamma}_{\iota\kappa} = \frac{8\pi G}{c^4}T_{\iota\kappa} \tag{3.52}$$

valid also for the non-stationary and non-static case (i.e., if there are also non-zero time derivatives on the left-hand side and other components of T on the right-hand side). The corresponding Green's function is given, in analogy with the classical electrodynamics, by the retarded potentials, i.e., we have to integrate along the past light-cone. Unlike Newton theory, in the weak-field limit of general relativity, the momentum or flow of energy T^{0i} and the stress T^{ij} produce the gravity (in a similar way as in electrodynamics where the currents induce the magnetic field in addition to the electric field produced by charges in electrostatics).

Exercise 14 *Calculate in the weak-field approximation (i.e., according to Eqs. (3.30), (3.39) and (3.49)) the metric inside a rotating homogeneous mass shell and investigate its influence on motion of test particles (cf. Problem 3 on p. 89 of [6]).*

Exercise 15 *Show that in linearized theory there is no gravitational interaction between two parallel beams of photons.*

3.6 Gravitational radiation

3.6.1 Newtonian preliminaries

In the Newtonian theory of gravitation a point-like body of mass m generates a scalar gravitational potential

$$\varphi(r) = -\frac{Gm}{r} \ . \tag{3.53}$$

If we split the mass like a dumbbell into two equal parts, each one shifted for a distance x in opposite directions, then we will get a deeper total potential in this direction

$$\varphi(r) = -\frac{Gm}{2(r-x)} - \frac{Gm}{2(r+x)} = -\frac{Gmr}{r^2-x^2} \simeq -\frac{Gm}{r}\left(1+\frac{x^2}{r^2}\right) \ , \tag{3.54}$$

and a shallower one in the perpendicular directions

$$\varphi(r) = -\frac{Gm}{\sqrt{r^2+x^2}} \simeq -\frac{Gm}{r}\left(1-\frac{x^2}{2r^2}\right) \ . \tag{3.55}$$

The perturbation

$$\delta\varphi(r) \simeq \frac{Gmx^2}{r^3} , \tag{3.56}$$

is the quadrupole term in the multipole expansion of the gravitational field of an isolated graviting system. In principle, we can thus construct a gravitational telegraph, if we will measure by some extremely sensitive torsion weight the change of gravitational acceleration

$$\delta g(r) = \nabla\delta\varphi \simeq \nabla\frac{Gmx^2}{r^3} \simeq \frac{Gmx^2}{r^4} , \tag{3.57}$$

caused by turning the source-dumbbell along or perpendicularly to the line of sight of the detector. To avoid the necessity of having fixed the distance r of the detector, we can also use a gradientometer, which will measure the change of relative accelerations of two test masses separated for a distance 2ξ

$$\delta g(r+\xi) - \delta g(r-\xi) = 2\xi\frac{d^2}{dr^2}\delta\varphi \simeq \xi\frac{Gmx^2}{r^5} . \tag{3.58}$$

The information about the change of orientation of the source is carried by some energy, which can be extracted from the change of gravitational field, e.g. if we let fall down a test mass μ in the phase of stronger gravitational field and we rise it up again in the weaker gravitational field. Or, to solve the need of exchange of momentum of the oscillating body, we can use again the two masses ($\frac{\mu}{2}$ each) in distance 2ξ as the absorber and turn them properly with the phase of variations of the gravitational field. The amount of energy absorbed by one absorber in one cycle of the field variation is thus of the order

$$\delta E = \frac{\mu}{2}\delta(2\varphi(r) - \varphi(r+\xi) - \varphi g(r-\xi)) \simeq \mu\xi^2\frac{d^2}{dr^2}\delta\varphi \simeq \frac{Gmx^2\mu\xi^2}{r^5} . \tag{3.59}$$

There can be many absorbers in such a gravitational energy transmitter, however, their own quadrupole moments will screen the variations of the field of source if they will be of the same order. The maximum energy which can be lost in one cycle by the source due to its interaction with the absorbers is thus of the order

$$\Delta E \simeq \frac{Gm^2x^4}{r^5} . \tag{3.60}$$

The rate of energy loses in such processes[11] is obviously dependent on the distribution of the absorbers in space. Their interaction with the source is instantaneous according to the Newtonian theory. This contradicts to the special relativity, according to which the signal can propagate from the source to the absorbers and back with the speed c of light at maximum. Consequently, in one period t, the source must give to the gravitational field its energy of the order of

$$\frac{dE}{dt} \simeq \frac{G}{c^5}\left(\frac{mx^2}{t^3}\right)^2 \tag{3.61}$$

for possible absorbers at distances $r \simeq ct$ and behind, without knowing if they are present and in what phase. This energy is radiated away from the source in the form of perturbations of the gravitational field, even if there are no absorbers in distances much larger than ct. The exact amount of this energy is given by the dynamics of the space-time variability of the gravitational field as well as on the number of degrees of its freedom, which is different in the scalar Newtonian theory and in the tensor general relativity or in some alternative theories.[12]

[11] Such processes are observed as different effects of gravitational resonances e.g. in Solar system.

[12] The Newtonian analog of general-relativistic gravitational waves gives a warning concerning the complexity of the problem of interpretation of possibly observed gravitational waves. The observed loose of source energy can be

3.6.2 Weak-field solutions in vacuum

The general relativistic equation (3.52) for weak gravitational field reduces in vacuum to

$$\Box \bar{\gamma}_{\iota\kappa} = 0 \;, \tag{3.62}$$

with the gauge condition (3.47). General solution of this homogeneous equation can be written as a superposition of Fourier modes

$$\bar{\gamma}_{\iota\kappa}(x) = h_{\iota\kappa} \exp(i k_\lambda x^\lambda) \;, \tag{3.63}$$

satisfying, according to (3.62), the condition

$$k_\lambda k^\lambda = 0 \;, \tag{3.64}$$

i.e., the wave-vector k must be on the light-cone, and according to the gauge condition (3.47)

$$h^{\iota\kappa} k_\iota = 0 \;, \tag{3.65}$$

i.e., the tensor h of the wave amplitude must be orthogonal to the wave-vector. This equation imposes four restrictions on otherwise 10 algebraically independent components of h. However, the gauge transformation (3.42) is not defined uniquely by the condition of vanishing of the expression (3.45), because any additional transformation generated by vector ξ which satisfies condition

$$\Box \xi_\kappa = 0 \;, \tag{3.66}$$

will not change the divergence of $\bar{\gamma}$ (i.e. the orthogonality of h with respect to k). In particular, for a planar wave (3.63) we can choose ξ satisfying

$$\xi^\iota{}_{,\iota} = -\frac{1}{2} \bar{\gamma}^\iota{}_\iota \;, \tag{3.67}$$

so that following (3.45) the new tensor $\bar{\gamma}$ will be traceless,

$$\bar{\gamma}^\iota{}_\iota = -\gamma^\iota{}_\iota = 0 \;, \tag{3.68}$$

and hence $\bar{\gamma} \equiv \gamma$. Obviously, ξ must be chosen also as a planar wave with the same wave-vector

$$\xi_\kappa = X_\kappa \exp(i k_\lambda x^\lambda) \;, \tag{3.69}$$

and with amplitude satisfying

$$X^\iota k_\iota = \frac{i}{2} h^\iota{}_\iota \;. \tag{3.70}$$

The other three still free components of X can be used to suppress all but the two transverse components of h.

---

influenced by its interaction with some absorbers in the near zone or with some inciding gravitational radiation and a detectable intensity of gravitational waves can be dumped by some absorbers between the source and the detector.

Appendix A

Differential geometry

A.1 Tensor algebra

In this chapter will be summarized basic properties of vectors, tensors and other geometrical objects in flat spaces like momentum space or the Minkowskian spacetime in special theory of relativity.

Definition 1 *Let T be an n-dimensional vector space over a field*[1] $\mathcal{R}$. *The* **dual vector space** T^* *is the space of linear mappings*

$$\xi : T \to \mathcal{R} \ .$$

$\{\theta^k|_{k=1}^n \in T^*\} = \{e_k|_{k=1}^n \in T\}^*$ *are mutually* **dual bases** *of T and T^*, if*

$$\theta^i(e_j) = \delta^i_j \ . \tag{A.1}$$

Obviously, T^* is also n-dimensional. It is common to identify $T^{**} \equiv T$ and to write

$$\xi(X) = \langle \xi X \rangle = X^i \xi_i \ ,$$

where $X = X^i e_i \in T$ and $\xi = \xi_i \theta^i \in T^*$ (following the Einstein notation, the summation is over identical upper and lower indices).

Definition 2 *The space T^p_q of* **tensors** *of type (p, q) is the space of multilinear mappings*

$$A : \underbrace{T^* \times ... \times T^*}_{p} \times \underbrace{T \times ... \times T}_{q} \to \mathcal{R} \ .$$

In the set of tensors of all types, there is defined the operation **tensor product**

$$\otimes : T^p_q \times T^r_s \to T^{p+r}_{q+s}$$

in such a way that

$$(A \otimes B)(\underbrace{\xi, ...,}_{p} \underbrace{\eta, ...,}_{r} \underbrace{X, ...,}_{q} \underbrace{Y, ...}_{s}) = A(\underbrace{\xi, ...,}_{p} \underbrace{X, ...}_{q}) B(\underbrace{\eta, ...,}_{r} \underbrace{Y, ...}_{s}) \ , \tag{A.2}$$

[1] A field is a commutative and associative ring with a unit, i.e., a set with the operations addition (with respect to which the set is a group) and multiplication. In general relativity we deal mostly with the field of real numbers, sometimes also complex numbers.

and the mapping **contraction** *in i-th and j-th index*

$$\langle\quad\rangle : T^p_q \to T^{p-1}_{q-1} \ ,$$

such that

$$\langle A\rangle(\underbrace{\xi,...,}_{i-1}\underbrace{\eta,...,}_{p-i}\underbrace{X,...,}_{j-1}\underbrace{Y,...}_{q-j}) = \sum_{k=1}^{n} A(\underbrace{\xi,...,}_{i-1}\theta^k,\underbrace{\eta,...,}_{p-i}\underbrace{X,...,}_{j-1}e_k,\underbrace{Y,...}_{q-j}) \ , \tag{A.3}$$

where $\{e_k|^n_{k=1}\} = \{\theta^k|^n_{k=1}\}^*$ *are any dual bases.*

Obviously $\dim(T^p_q) = n^{p+q}$. The tensor product forms an algebra in the space $\oplus^\infty_{p,q=0}T^p_q$ of all tensors. Every (not necessarily dual) couple of bases in T and T^* induces a basis in T^p_q (formed by tensor products of their elements), in which any tensor $A \in T^p_q$ can be expressed by the components

$$A = A^{i_1,...,i_p}_{j_1,...,j_q} e_{i_1} \otimes ... \otimes e_{i_p} \otimes \theta^{j_1} \otimes ... \otimes \theta^{j_q} \ .$$

It can be verified that the definition of contraction is independent of the choice of the couple of dual bases.

A.2 Exterior algebra

Definition 3 *The spaces of* p-**vectors** $\wedge^p T \subset T^p_0$ *or* p-**forms** $\wedge^p T^* \subset T^0_p$ *($p \le n$) are the subspaces of tensors of types $(p,0)$ or $(0,q)$ resp., which are antisymmetric in all arguments. The* **exterior product** *is the mapping*

$$\wedge : \wedge^p T \times \wedge^q T \to \wedge^{p+q} T \ ,$$

such that

$$(X \wedge Y)(\xi_1, ..., \xi_{p+q}) = \sum_\sigma \frac{(-1)^{|\sigma|}}{p!q!}(X \otimes Y)(\xi_{\sigma(1)}, ..., \xi_{\sigma(p+q)}) \ , \tag{A.4}$$

where $|\sigma|$ is the parity of permutation σ of numbers $1,...,p+q$ (i.e., $+1$ for even or -1 for odd permutation). The **interior product** *of a p-form ξ and q-vector X is the $|p-q|$-form (or -vector) formed by contraction of their tensor product in the first $\min(p,q)$ indices,*

$$\langle \xi X\rangle = \langle \xi \otimes X\rangle \ .$$

Obviously, $\dim(\wedge^p T) = \binom{n}{p}$, $\wedge^0 T = \mathcal{R}$, $\wedge^1 T = T^1_0 = T$, $\wedge^1 T^* = T^0_1 = T^*$. An exterior product is antisymmetric, i.e.,

$$(X \wedge Y) = (-1)^{qp}(Y \wedge X) \ , \tag{A.5}$$

linear and asociative. It thus forms (the so-called Grassman's) algebra in 2^n-dimensional space of direct products $\oplus^n_{p=0}(\wedge^p T)$.

Definition 4 *Let us choose a* **unit** n-**form** $\epsilon \in \wedge^n T^*$ *on an n-dimensional space T. The n-form ϵ then defines the* **dual mapping**

$$* : \wedge^p T \leftrightarrow \wedge^{n-p} T^*; ,$$

such that for $A \in \wedge^p T$, $\omega \in \wedge^p T^*$,

$$*A = \frac{1}{p!}\langle \epsilon A\rangle \ , \tag{A.6}$$

$$*\omega = \frac{1}{p!}\langle \omega E\rangle \ , \tag{A.7}$$

where $E \in \wedge^n T$ is the **unit** n-**vector**, *for which*

$$*E = 1 \ . \tag{A.8}$$

The following relation is valid

$$* * \omega = (-1)^{p(n-p)} \omega \ . \tag{A.9}$$

The exterior product of p-vectors could be confused with the vector product of vectors, which is also an antisymmetric bilinear operation (sometimes denoted by the same symbol $\wedge$) but defined only on a 3-dimensional vector space. In such a space with a metric the vector product Z of vectors X, Y can be constructed by raising the indices of their contraction with a unit 3-form, $Z \equiv X \times Y = g^{-1}(\langle \epsilon, X \otimes Y \rangle)$.

A.3 Scalar product

Definition 5 *The so-called* **covariant metric tensor** $g \in T^0_2$*, which is symmetric, and with components* g_{ij} *in any basis* T^* *which satisfy*

$$\det(g_{ij}) \neq 0 \ ,$$

defines on T *the* **scalar product**

$$(\quad . \quad) : T \times T \to \mathcal{F} \ ,$$

in such a way that $\forall X, Y \in T$

$$(X.Y) = (Y.X) = g(X, Y) = g_{ij} X^i Y^j \tag{A.10}$$

(where X^i *and* Y^j *are components in any dual basis).* g *also defines the* **lowering of indices***, i.e., the mapping*

$$g : T \to T^* \ ,$$

such that

$$\langle g(X) Y \rangle = g(X, Y) \ , \tag{A.11}$$

*and its inverse mapping (***raising of indices***)*

$$g^{-1} : T^* \to T \ ,$$

which defines the scalar product on T^*

$$(\xi . \eta) = (\eta . \xi) = g(g^{-1}(\xi), g^{-1}(\eta)) = g^{ij} \xi_i \eta_j \ , \tag{A.12}$$

where g^{ij} *are components of the* **contravariant metric tensor** $g^{-1} \in T^2_0$*. A basis* $\{e_i\}$ *is* **orthogonal***, if*

$$(e_i . e_j) = 0 \qquad \text{for} \qquad i \neq j \ ,$$

and it is **orthonormal***, if*

$$(e_i . e_j) = \pm \delta_{ij} \ .$$

A.4 Tensor analysis

In the present chapter will be summarised the basic properties of tangent vectors, vector and tensor fields, and their derivatives on curved spaces (manifolds) as is, for example, the case of general-relativistic spacetime.

Definition 6 *A topologic space M is a* **differentiable manifold** *of dimension n, if there is given a so-called complete atlas of coordinate maps, i.e., of the locally defined mappings*

$$\mu, \nu : M \to \mathcal{R}^n ,$$

covering the whole M such that (in the region where both are defined) $\mu \circ \nu^{-1}$ is infinitely differentiable with a non-zero jacobian ($\circ$ denotes the composition of the mappings).

Definition 7 *Let M be a differentiable manifold and $\mathcal{F}$ the space of real functions ("scalars") on M. The* **tangent vector** *at the point $A \in M$ is the mapping*

$$X_A : \mathcal{F} \to \mathcal{R} ,$$

which is linear, i.e., it satisfies

$$X_A(f + c.g) = X_A(f) + c.X_A(g) , \tag{A.13}$$

for any $f, g \in \mathcal{F}$, $c \in \mathcal{R}$, and which satisfies

$$X_A(f.g) = X_A(f).g(A) + f(A).X_A(g) . \tag{A.14}$$

The set T_A of all tangent vectors in a fixed point A is called **fibre**, *and their unification $B = \cup_{A \in M} T_A$ a* **tangent bundle**.

It can be shown that in a coordinate map μ ($\mu(A) = (x^1_A, ..., x^n_A)$), the tangent vector X_A is given by

$$X_A(f) = X^i_A \frac{\partial}{\partial x^i}(f \circ \mu^{-1})|_{\mu(A)} . \tag{A.15}$$

This definition corresponds to the intuitive understanding of a vector X_A as an arrow from the point $A \in M$ to the (infinitesimally distant) point $A + \Delta$ in such a way that $\forall f \in \mathcal{F}$ (and especially for f as one component of the coordinate map) $X_A(f)$ gives the linear term of the Taylor expansion

$$f(A + \Delta) \simeq f(A) + X_A(f) .$$

On the fibre T_A there is defined an addition and a multiplication by a number, with respect to which T_A is an n-dimensional vector space and $\{\frac{\partial}{\partial x^i_A}|^n_{i=1}\}$ is its coordinate basis induced by the map μ. The tangent bundle B forms a $2n$-dimensional manifold, the atlas of which is formed by all maps compatible with the map which attributes $\forall X_A \in T_A$ (i.e., to the point A and the vector X_A in it) the coordinates $(x^1_A, ..., x^n_A, X^1_A, ..., X^n_A) \in \mathcal{R}^{2n}$.

Definition 8 *Let M be a differentiable manifold and $\mathcal{F}$ the space of real functions ("scalars") on M. The tangent vector space T on M is the space of* **tangent vectors** *X (or "contravariant vector fields") on M, i.e., the space of mappings*

$$X : \mathcal{F} \to \mathcal{F} ,$$

which are linear and differentiable, i.e. satisfying

$$X(f + c.g) = X(f) + c.X(g) , \tag{A.16}$$

$$X(f.g) = X(f).g + f.X(g) , \tag{A.17}$$

for any $f, g \in \mathcal{F}$, $c \in \mathcal{R}$.

The definition region $\{A \in M\}$ of a vector field can be restricted to some $N \subset M$ ($X : \mathcal{F}(M) \to \mathcal{F}(N)$), especially also to a single point. If N is a submanifold of M, $T(N)$ is then a subspace of $T(M)$ called the space of vectors tangent to N. Obviously, any function on M is simultaneously a function on N, and thus each tangent vector on N is also a vector on M, defined only in points of N and 'insensitive' to a different behaviour of a function apart from N. Each coordinate map $\mu(A) = \{x^i(A)|_{i=1}^n\}$ induces in its definition region a coordinate basis $\{\partial_{x^i}|_{i=1}^n\}$ (here it is understood that $\partial_{x^i} f = \frac{\partial}{\partial x^i}(f \circ \mu^{-1})$). For the change to coordinates $\{y^i\}$ the elements of the coordinate basis are transformed according to

$$\partial_{x^i} = \frac{\partial y^j}{\partial x^i} \partial_{y^j} \ . \tag{A.18}$$

If the tangent vectors are defined as a kind of mappings of the space of scalars into the same space, there is a natural question whether their composition is also a tangent vector. It is obviously linear, however, it does not satisfy the condition (A.17), because

$$X \circ Y(f.g) = X(f.Y(g) + Y(f).g) = f.X \circ Y(g) + X \circ Y(f).g + X(f).Y(g) + Y(f).X(g) \ , \tag{A.19}$$

where the last two terms violate the condition. Subtracting from this equation the same with interchanged X and Y, these additional terms cancel. We have thus proved the following theorem (and definition at the same time):

Theorem 1 *The commutator*

$$[X,Y] = X \circ Y - Y \circ X \tag{A.20}$$

of vectors X, Y is also a vector.

The elements of a coordinate basis commute (they are 'holonomic'). In a general basis $\{e_i\} = \{\theta^i\}^*$, the action of the commutator on vectors expressed by their components can be written as

$$[X,Y] = (X(Y^k) - Y(X^k) + X^i Y^j c^k{}_{ij}) e_k \ , \tag{A.21}$$

where

$$c^k{}_{ij} = -c^k{}_{ji} = \theta^k([e_i, e_j]) \tag{A.22}$$

are the so-called **coefficients of structure** of the given basis.

Definition 9 *A cotangent vector space T^* is the dual vector space of* **1–forms** *or "covariant vector fields", i.e., the space of mappings*

$$\xi : T \to \mathcal{F} \ .$$

By analogy, the space T^p_q of **tensors** *of type p, q is the space of multilinear mappings*

$$A : \underbrace{T^* \times \ldots \times T^*}_{p} \times \underbrace{T \times \ldots \times T}_{q} \to \mathcal{F} \ ,$$

*and spaces of p-***vectors** *or p-***forms** *are the antisymmetric subspaces T^p_0 or T^0_p, resp. In the spaces of all tensors or p-vectors and p-forms there are defined the tensor product and the contraction or the external and inner product, resp. in analogy to Definitions 2 and 3.*

A tensor of type T^0_p on M is simultaneously a tensor of the same type on a submanifold $N \subset M$. Especially, a p-form on M is a p-form on N, which can be non-zero only if $p \leq \dim N$.

Definition 10 *An* **exterior derivative** *is the linear mapping*

$$d : \wedge^p T^* \to \wedge^{p+1} T^* \ ,$$

such that $\forall f \in \wedge^0 T^* = \mathcal{F}$

$$df(X) = X(f) \ , \tag{A.23}$$

$\forall \omega \in \wedge^p T^*$

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^p \omega \wedge (d\eta) \ , \tag{A.24}$$

and

$$dd\omega = 0 \ . \tag{A.25}$$

For any coordinate map $\{x^i|_{i=1}^n\}$ the exterior derivatives $\{dx^i|_{i=1}^n\}$ form a basis of T^*, which transforms according to

$$dx^i = \frac{\partial x^i}{\partial y^j} dy^j \ , \tag{A.26}$$

and is dual with respect to the coordinate basis $\{\partial_{x^i}|_{i=1}^n\}$ of T.

Theorem 2 *For a 1-form* ω*, its exterior derivative satisfies*

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) \ . \tag{A.27}$$

Proof: According to (A.21), in any coordinate basis (where ${c^k}_{ij}$ vanishes and $\omega = \omega_j dx^j$), it is valid that

$$\begin{aligned} X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) &= X(\omega_j Y^j) - Y(\omega_j X^j) - \omega_j(X(Y^j) - Y(X^j)) = \\ &= X(\omega_j)Y^j - Y(\omega_j)X^j = (d\omega_j \wedge dx^j)(X,Y) = d\omega(X,Y) \ . \end{aligned}$$

Q.E.D.

Consequently, alternatively to (A.22) the coefficients of structure can be expressed as

$${c^k}_{ij} = -d\theta^k(e_i, e_j) \ . \tag{A.28}$$

Definition 11 $\omega \in \wedge^p T^*$ *defines the* **measure** *(i.e., the integration) on a p-dimensional submanifold* $N \subset M$

$$\int \omega = \int \omega(\partial_{x^1}, ..., \partial_{x^p}) dx^1 ... dx^p \ , \tag{A.29}$$

where $\{x^i|_{i=1}^p\}$ *is an arbitrary coordinate map on* N.

Theorem 3 Stokes theorem *For a closed p-dimensional boundary* ∂V *of a* $(p+1)$*-dimensional region* V*, it is valid that*

$$\int_V d\omega = \int_{\partial V} \omega \ . \tag{A.30}$$

A.5 Affine connection

Definition 12 *An* **affine connection** *is the mapping*

$$\nabla : T \times T \to T \ ,$$

such that $\forall X, Y, Z \in T, f \in \mathcal{F}$

$$\nabla_{f.Y+Z} X = f.\nabla_Y X + \nabla_Z X; , \tag{A.31}$$

$$\nabla_X (f.Y + Z) = X(f).Y + f.\nabla_X Y + \nabla_X Z \ . \tag{A.32}$$

An affine connection $\nabla_X Y$ thus has a tensor character with respect to the index X and the differential behaviour with respect to the argument Y, i.e., it describes the change of the field Y along the field X. In any dual basis $\{e_i\} = \{\theta^i\}^*$,

$$\nabla_X Y = \left(X(Y^k) + Y^i \omega^k{}_i(X)\right) e_k \ , \tag{A.33}$$

where

$$\omega^k{}_i = \omega^k{}_{ij} \theta^j = \theta^k(\nabla_{e_j} e_i) \theta^j \tag{A.34}$$

are the **1-forms of connection** in the corresponding basis (their components $\omega^k{}_{ij}$ are the so-called **Ricci rotation coefficients**).

By the requirement

$$X(\langle \xi, Y \rangle) = \langle \nabla_X \xi, Y \rangle + \langle \xi, \nabla_X Y \rangle \ , \tag{A.35}$$

or by the analogous requirement on the derivative of the tensor product and its contraction, there is also defined the affine connection for 1-forms by the relation

$$\nabla_X \xi = \left(X(\xi_k) - \xi_i \omega^i{}_k(X)\right) \theta^k \ , \tag{A.36}$$

or for tensors T^p_q as the mapping

$$\nabla : T \times T^p_q \to T^p_q \ ,$$

such that $\forall A \in T^p_q, X, Y, ... \in T, \xi, ... \in T^*$

$$X(A(\xi, ..., Y, ...)) = \nabla_X A(\xi, ..., Y, ...) + A(\nabla_X \xi, ..., Y, ...) + ... + A(\xi, ..., \nabla_X Y, ...) + ... \ . \tag{A.37}$$

Definition 13 *The* **covariant derivative** *(corresponding to the affine connection* ∇*) is the mapping*

$$D : T^p_q \to T^p_{q+1} \ ,$$

such that $\forall A \in T^p_q, X \in T$

$$\langle DA, X \rangle = \nabla_X A \ . \tag{A.38}$$

In any dual basis $\{e_i\} = \{\theta^i\}^*$ it is useful to denote

$$DA = \nabla_{e_k} A \otimes \theta^k = A^{i_1,...,i_p}_{j_1,...,j_q;k} e_{i_1} \otimes ... \otimes e_{i_p} \otimes \theta^{j_1} \otimes ... \otimes \theta^{j_q} \otimes \theta^k \ , \tag{A.39}$$

and to write

$$A^{i_1,...,i_p}_{j_1,...,j_q;k} = A^{i_1,...,i_p}_{j_1,...,j_q,k} + A^{i,...,i_p}_{j_1,...,j_q} \omega^{i_1}{}_{ik} + ... - A^{i_1,...,i_p}_{j,...,j_q} \omega^j{}_{j_1 k} - ... \ , \tag{A.40}$$

where

$$A^{i_1,...,i_p}_{j_1,...,j_q,k} = e_k(A^{i_1,...,i_p}_{j_1,...,j_q}) \ .$$

The matrix of 1-forms of connection (see (A.34)) does not behave as a tensor in the indices k and i for a transition to another basis, however, it is possible to produce a tensor from it.

Theorem 4 *If* ∇ *is an affine connection, then there exists a* **tensor of torsion** *(determined by this connection)* $Q \in (T \otimes \wedge^2 T^*) \subset T^1_2$ *such, that* $\forall X, Y \in T$

$$\nabla_X Y - \nabla_Y X - [X,Y] = \langle Q, X \otimes Y \rangle \tag{A.41}$$

and a **tensor of curvature** $R \in (T \otimes T^* \otimes \wedge^2 T^*) \subset T^1_3$*, such that* $\forall X, Y, Z \in T$

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = \langle R, Z \otimes X \otimes Y \rangle \ . \tag{A.42}$$

To proove this theorem, we must check that there mutually cancel additional terms like $X(f)$, arising from both the covariant derivatives and the commutator if $Y \to fY$ is substituted. The linearity with respect to the addition is obvious.

Both these tensors are antisymmetric in the last two arguments (X and Y). It is thus possible to represent them in a chosen basis by 2-forms of torsion τ^k or curvature $\Omega^k{}_i$ resp., according to the relations

$$Q = e_k \otimes \tau^k \ , \tag{A.43}$$
$$R = e_k \otimes \theta^i \otimes \Omega^k{}_i \ . \tag{A.44}$$

Comparing (A.33) and (A.21), it is obvious that

$$\nabla_X Y - \nabla_Y X - [X,Y] = \left(Y^i \omega^k{}_i(X) - X^i \omega^k{}_i(Y) - X^i Y^j c^k{}_{ij} \right) .e_k \ , \tag{A.45}$$

so that

$$Q^k{}_{ij} = \omega^k{}_{ji} - \omega^k{}_{ij} - c^k{}_{ij} \ . \tag{A.46}$$

Theorem 5 *The 2-forms of torsion are given by the first* **Cartan's equations of structure***, which read*

$$\tau^k = d\theta^k + \omega^k{}_i \wedge \theta^i \ , \tag{A.47}$$

and the 2-forms of curvature by the second Cartan's eqs. of structure

$$\Omega^k{}_i = d\omega^k{}_i + \omega^k{}_j \wedge \omega^j{}_i \ . \tag{A.48}$$

Specially, for a connection without torsion, the right-hand side of (A.47) is identically equal to zero. The proof of Eq. (A.47) follows from the substitution of (A.28) into (A.46). Similarly (A.48) follows from (A.42) by substituting (A.33) and using (A.27).

Definition 14 *An affine connection* ∇ *which preserves a scalar product, i.e., which satisfies*

$$Z(X,Y) = (\nabla_Z X, Y) + (X, \nabla_Z Y) \tag{A.49}$$

for all X, Y, Z*, is named a* **metric connection***.*

Inserting here the basis vectors, we arrive at the expression

$$dg_{ij} = \omega_{ij} + \omega_{ji} \ , \tag{A.50}$$

where the 1-forms $\omega_{ij} = g_{ik}\omega^k{}_j$. Rewriting this equation in components

$$e_k(g_{ij}) = \omega_{ijk} + \omega_{jik} \ , \tag{A.51}$$

and using Eq. (A.46) with lowered index k, we can solve for ${\omega^i}_j$, i.e.,

$$\begin{aligned}\omega_{ijk} &= \frac{1}{2}(e_k(g_{ij}) + e_j(g_{ik}) - e_i(g_{jk}) \\ &\quad -c_{ijk} + c_{jik} + c_{kij} \\ &\quad -Q_{ijk} + Q_{jik} + Q_{kij}\ ) .\end{aligned} \tag{A.52}$$

An affine connection is thus uniquely determined by requirements of being metric and torsion free (i.e., the third line of this expression being identically equal to zero). The first line of this expression, i.e., the components ${\omega^i}_{jk}$ in a coordinate basis, where also the second line of this expression vanishes, is named Christoffel symbols (usually denoted as ${\Gamma^i}_{jk}$).

Theorem 6 *For a metric connection, the components ${R^i}_{klm} = \langle {\Omega^i}_k, e_l \otimes e_m \rangle$ of the tensor of curvature safisfy the antisymmetry*

$$R_{iklm} = -R_{kilm} . \tag{A.53}$$

For a torsion-free connection they satisfy the symmetry

$${R^i}_{(klm)} \equiv {R^i}_{klm} + {R^i}_{lmk} + {R^i}_{mkl} = 0 , \tag{A.54}$$

and the **Bianchi identities**

$${R^i}_{k(lm;n)} \equiv {R^i}_{klm;n} + {R^i}_{knl;m} + {R^i}_{kmn;l} = 0 . \tag{A.55}$$

In an n-dimensional manifold the **Riemann tensor***, i.e., the tensor of curvature determined by the torsion-free metric connection, thus has $\frac{1}{12}n^2(n^2-1)$ algebraically independent components.*

The **Ricci tensor** $R_{km} = {R^i}_{kim}$ *is then symmetric.*

Theorem 7 *The tangent vector $X = \frac{d}{dt}$ of a line $x = x(t)$ which is an extremal of the length S (i.e., the line is a geodesic),*

$$0 = \delta S = \delta \int \sqrt{g(X,X)}dt , \tag{A.56}$$

is parallely transported along the line, i.e.,

$$\nabla_X X = 0 . \tag{A.57}$$

Proof: Equations for extremals of an action $S = \int L(x,\dot{x})dt$ are obtained via a variation δx of the path and its derivatives $\delta\dot{x}$. To get the Lagrangian of the geodesics in this form, it is thus advantageous to express its tangent vector X in a coordinate basis of any coordinate system, i.e.,

$$L(x,\dot{x}) = \sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j} . \tag{A.58}$$

Inserting this Lagrangian into the standard procedure

$$0 = \delta S = \int \delta L(x,\dot{x})dt = \int \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right]\delta x dt , \tag{A.59}$$

we arrive at the equations

$$0 = \frac{1}{2\sqrt{g(\dot{x},\dot{x})}} g_{ij,k}(x)\dot{x}^i\dot{x}^j - \frac{d}{dt}\left(\frac{g_{ik}\dot{x}^i + g_{kj}\dot{x}^j}{2\sqrt{g(\dot{x},\dot{x})}}\right) , \tag{A.60}$$

i.e.,

$$0 = \frac{1}{2}(g_{ij,k} - g_{ik,j} - g_{kj,i})\dot{x}^i\dot{x}^j - g_{ik}\ddot{x}^i - g_{ik}\dot{x}^i\sqrt{g(\dot{x},\dot{x})}\frac{d}{dt}\left(\frac{1}{\sqrt{g(\dot{x},\dot{x})}}\right) . \tag{A.61}$$

This is valid for any parameterization of the path $x(t)$. If we impose the condition of t being the affine parameter, i.e., $t = S$, then $g(\dot{x},\dot{x}) = 1$ and the last term vanishes. With this condition, the geodesic can thus be calculated equivalently from the variational principle

$$0 = \delta \int g(\dot{x},\dot{x})dt . \tag{A.62}$$

Comparing the first term with Eq. (A.52), we find (in view of $c_{ijk} = 0$ in coordinate basis), that it corresponds to $-\omega_{kij}\dot{x}^i\dot{x}^j$ of a torsion-free ($Q_{ijk} = 0$) metric connection, and the first two terms thus give the right-hand side of Eq. (A.57).

A.6 Lie derivatives and Killing vectors

Definition 15 *Let the* **Lie derivative** $\pounds_X$ *with respect to a vector field* X *be a mapping* $\pounds_X : T^p_q \to T^p_q$ *such that*

$$\pounds_X(A \otimes B) = (\pounds_X A) \otimes B + A \otimes (\pounds_X B) , \tag{A.63}$$

$$\pounds_X\langle A\rangle = \langle \pounds_X A\rangle , \tag{A.64}$$

$$\pounds_X f = X(f) , \qquad \text{for } f \in T^0_0 , \tag{A.65}$$

and

$$\pounds_X Y = [X,Y] , \qquad \text{for } Y \in T^1_0 . \tag{A.66}$$

Obviously, comparing with Eq. (A.31), there is an additional third term in the expression

$$\pounds_{fX+Z}Y = f\pounds_X Y + \pounds_Z Y - Y(f)X . \tag{A.67}$$

Consequently, in a basis $\{e_i\}$

$$\pounds_X Y = X^i\pounds_{e_i}Y - Y(X^i)e_i = X^i e_i(Y^k)e_k - Y^k e_k(X^i)e_i + X^iY^k[e_i,e_k] , \tag{A.68}$$

specially in a coordinate basis

$$(\pounds_X Y)^i = X^k Y^i{}_{,k} - Y^k X^i{}_{,k} . \tag{A.69}$$

Because for $\sigma \in T^0_1$

$$\pounds_X\langle\sigma, Y\rangle = \langle\pounds_X\sigma, Y\rangle + \langle\sigma, \pounds_X Y\rangle , \tag{A.70}$$

for the components of a covector Lie-derivative

$$\begin{aligned}(\pounds_X\sigma)_i &= \langle\pounds_X\sigma, e_i\rangle = \pounds_X\langle\sigma, e_i\rangle - \langle\sigma, \pounds_X e_i\rangle = X(\sigma_i) - \langle\sigma, [X, e_i]\rangle \\ &= X^k e_k(\sigma_i) + e_i(X^k)\sigma_k - X^k\langle\sigma, [e_k, e_i]\rangle ,\end{aligned} \tag{A.71}$$

and in a coordinate basis

$$(\pounds_X\sigma)_i = X^k\sigma_{i,k} + \sigma_k X^k{}_{,i} . \tag{A.72}$$

For Lie-derivatives of higher tensors, in addition to the derivative of the component, there appears for each contravariant or covariant index a term with derivatives of X analogous the second term of Eq. (A.69) or Eq. (A.72), resp. If the manifold is Riemannian, the partial derivatives in these equations can also be replaced by the contravariant derivatives, because the additional terms with connections mutually cancel due to their symmetry $\omega_{i(jk)}$.

Definition 16 *In a Riemannian manifold with a metric g, a vector ξ is named a* **Killing vector** *if*

$$\pounds_\xi g = 0 \ . \tag{A.73}$$

Expressing this condition in coordinate components

$$0 = (\pounds_\xi g)_{ik} = \xi^l g_{ik;l} + g_{lk}\xi^l{}_{;i} + g_{il}\xi^l{}_{;k} \ , \tag{A.74}$$

we arrive at the condition

$$0 = \xi_{k;i} + \xi_{i;k} \ , \tag{A.75}$$

because the covariant derivatives of the metric components vanish identically for the metric connection.

Theorem 8 *If $X = \frac{d}{dt}$ is a tangent vector to a geodesic, than for any Killing vector ξ the scalar product $g(\xi, X)$ is constant along the geodesic.*

Proof:

$$\frac{d}{dt} g(\xi, X) = (\xi_k X^k)_{;i} X^i = \xi_{k;i} X^i X^k + \xi_k X^k{}_{;i} X^i = 0 \ , \tag{A.76}$$

because the first term vanishes due to the antisymmetry $\xi_{[k;i]}$ according to Eq. (A.75), and the second one due to the equation (A.57) of the geodesic.

Appendix B

Variational principles

B.1 Systems with finite degrees of freedom

In classical mechanics the motion of a system with n degrees of freedom $x^i|_{i=1}^n$ (e.g., a motion of point mass) can be expressed by the variatioal principle

$$0 = \delta S \ , \tag{B.1}$$

where the action S is a function of the motion $x = x(t)$ of the system given by the lagrangian $L = L(t, x, \dot{x})$, i.e.,

$$S = \int L(t, x, \dot{x}) dt \ , \tag{B.2}$$

and $\dot{} \equiv \frac{d}{dt}$. From the principle (B.1) then follow equations of motion in the form of the Euler – Lagrange equations

$$0 = \frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) \ . \tag{B.3}$$

If we define the generalized momentum $p_i = \frac{\partial L}{\partial \dot{x}^i}$, and we solve from here $\dot{x} = \dot{x}(t, x, p)$,[1] then in the phase space $\{x, p\}$ can be written equations equivalent to (B.3) in the form of the Hamilton canonical equations

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} \ , \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} \ , \tag{B.4}$$

where the Hamiltonian

$$H(t, x, p) = p_i \dot{x}^i - L(t, x, \dot{x}) \ . \tag{B.5}$$

Equations (B.4) can also be obtained as the Euler – Lagrange equations from the Hamilton – Jacobi variational principle

$$0 = \delta S = \delta \int (p_i \dot{x}^i - H(t, x, p)) dt \ , \tag{B.6}$$

where x and p are taken as independent variables. From the equations (B.4) there follows

$$\frac{d}{dt} H(t, x, p) = \frac{\partial H}{\partial t} \ , \tag{B.7}$$

[1] Equations $p_i = p_i(t, x, \dot{x})$ can be solved with respect to $\dot{x}$, if the matrix $\frac{\partial p_i}{\partial \dot{x}^j} = \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}$ is regular.

hence the Hamiltonian is an invariant of motion just if it is not explicitely dependent on time.

From the point of view of quantum mechanics, the principle of extremal (least) action is valid because in fact the system is spreading out from the initial event $(t_{\rm i}, x_{\rm i})$ and it can get to the end one $(t_{\rm f}, x_{\rm f})$ on all trajectories $x(t)$ with these terminal points, and each of these trajectories contributes a term $\exp(\frac{i}{\hbar}S)$ to the wave function $\psi(t_{\rm f}, x_{\rm f})$. The contributions of trajectories close to the extremal of the action add with the same (or close) phase, while the contributions of more different trajectories mutually cancel – see [1]. If the ending event is changed $(t_{\rm f}, x_{\rm f}) \equiv (t, x) \rightarrow (t + \Delta t, x + \Delta x)$, the action (B.2) varies, i.e.,

$$\Delta S(t, x) = L\Delta t + \frac{\partial L}{\partial \dot{x}^i}\delta x^i + \int \left[\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\right]\delta x^i dt \ , \tag{B.8}$$

where at the final point $\Delta x = \delta x + \dot{x}\Delta t$. The last term in (B.8) is zero along the extremal in consequence of (B.3), hence

$$\Delta S(t, x) = \frac{\partial L}{\partial \dot{x}^i}\Delta x^i - \left(\frac{\partial L}{\partial \dot{x}^i}\dot{x}^i - L\right)\Delta t = p_i \Delta x^i - H\Delta t \ . \tag{B.9}$$

In Newtonian mechanics where the universal time t exists, are both the action S and the Lagrangian L scalar quantities with respect to any transformation $x \rightarrow x' = x'(t, x)$. However, in relativity the time coordinate $x^0 \equiv ct$ is equivalent to the space coordinates, and each non-zero variation δx of the worldline has at least from the point of view of some observers a non-zero variation also of the time component. Unlike the action which has a Lorentz invariant physical meaning, the Lagrangian in the expression (B.2) is dependent on the chosen coordinate system. To get to the relativistic formulation of the variational principle, we must make the position of all spacetime coordinates equal using the parametric description of the motion $x^\iota = x^\iota(w)|_{\iota=0}^n$, where x^0 is an additional (the $n+1$-st) up to now nonspecified function of the parameter w. For any parameterization the action is then given by the expression

$$S = \int f(x, x')dw \ , \tag{B.10}$$

where $' \equiv \frac{d}{dw}$ and the parametric Lagrangian

$$f(x, x') = t'L(t, x, \dot{x}) \ , \tag{B.11}$$

are not explicitely dependent on w. Because $\dot{x} = \frac{x'}{t'}$, the generalized momentum canonically conjugated to the space coordinates

$$p_i \equiv \frac{\partial f}{\partial x'^i} = \frac{\partial L}{\partial \dot{x}^i} \tag{B.12}$$

is unchanged. If the time coordinate $t(w)$ is also taken as a dynamical variable, the generalized momentum conjugated to t will be

$$p_0 \equiv \frac{\partial f}{\partial t'} = L - \frac{\partial L}{\partial \dot{x}^i}\frac{x'^i}{t'} = -H(t, x, p) \ . \tag{B.13}$$

The space components of the Euler – Lagrange equations obtained from f are a multiple by t' of the equations (B.3), and their time- component is also their linear combination (the sum of multiples by x'). All $n+1$ Lagrange equations are thus consistent with the original time equations, but there is not a unique solution because they are satisfied by any parameterization. To get a uniqueness it is necessary to add to the variational principle (B.1) for the parametric formulation of the dependence $x(w)$ an additional condition fixing the parameterization. This can have the form of an algebraic equation

$$\omega(x, x') = 0 \ , \tag{B.14}$$

where ω is an arbitrary function. Spacetime variations δx^ι are then no more independent, but they must satisfy the linear equation

$$\delta\omega = \frac{\partial\omega}{\partial x}\delta x + \frac{\partial\omega}{\partial x'}\delta x' = 0 \ , \tag{B.15}$$

so that in (B.1) must be taken the variation of (B.10) with the condition (B.14). The Lagrangian variational principle for a parametric formulation of the motion thus reads

$$0 = \delta S = \delta \int f(x, x') + n(w)\omega(x, x')dw \ , \tag{B.16}$$

where n is the Lagrangian coefficient of the variational problem with a condition.

Similarly, the Hamiltonian formalism for a parametric description of the motion can be obtained from the action (B.10), the system of equations (B.12) and (B.13) for t' and x' does not have a unique solution (the condition from the footnote [1] on p. 50 is not satisfied for f given by (B.11)). The expression $p_\iota x'^\iota - f$ formally produced from f for x^ι analogously to the Hamiltonian (B.5) from L for x^i is identically equal to zero, which corresponds to the fact that the Hamiltonian made in this way, which can not be explicitely dependent on w, should be – in analogy with (B.7) – a trivial integral of motion. In view of (B.13) the integral in (B.6) can be expressed as $\int p_\iota x'^\iota dw$ (the Hamilton function for $x^\iota(w)$ would thus be identically equal to 0). However, p_ι are not independent because they must satisfy the condition (B.13). The Hamilton – Jacobi variational principle for parametric description of the motion must be thus in the form

$$0 = \delta \int (p_\iota x'^\iota - N(w)\mathcal{H}(x, p))dw \ , \tag{B.17}$$

where N again is the Lagrangian coefficient, and the so-called superhamiltonian

$$\mathcal{H}(x, p) = p_0 + H(t, x, p) \tag{B.18}$$

is the expression, which put equal to zero gives the condition (B.13). Taking the variation of (B.17) with respect to p_0 we get

$$0 = x'^0 - N(w) \ , \tag{B.19}$$

from where the physical meaning of $N(w)$ is obvious.

B.2 Fields

For a general spacetime field $\varphi_I = \varphi_I(x)$ (where the index "$_I$" denotes individual components of the field φ), the action S is the integral of the Lagrangian density $\mathcal{L} = \mathcal{L}(\varphi, \partial\varphi)$, i.e.,

$$S = \int \mathcal{L} d^4x \ , \tag{B.20}$$

hence its variation is

$$\begin{aligned} \delta S &= \int \delta\mathcal{L} d^4x = \int \sum_I \left(\frac{\partial\mathcal{L}}{\partial\varphi_I}\delta\varphi_I + \frac{\partial\mathcal{L}}{\partial\varphi_{I,\kappa}}\delta\varphi_{I,\kappa} \right) d^4x \\ &= \int \sum_I \left(\frac{\partial\mathcal{L}}{\partial\varphi_I} - \frac{\partial}{\partial x^\kappa}\frac{\partial\mathcal{L}}{\partial\varphi_{I,\kappa}} \right) \delta\varphi_I d^4x \ , \end{aligned} \tag{B.21}$$

and the field equations read

$$0 = \frac{\partial \mathcal{L}}{\partial \varphi_I} - \frac{\partial}{\partial x^\kappa} \left(\frac{\partial \mathcal{L}}{\partial \varphi_{I,\kappa}} \right) . \tag{B.22}$$

If the Lagrangian (density) does not depend explicitely on the coordinates, then its spacetime variation is given by the change of the field only

$$\frac{\partial \mathcal{L}}{\partial x^\iota} = \frac{\partial \mathcal{L}}{\partial \varphi} \varphi_{,\iota} + \frac{\partial \mathcal{L}}{\partial \varphi_{,\kappa}} \varphi_{,\kappa\iota} = \frac{\partial}{\partial x^\kappa} \left(\frac{\partial \mathcal{L}}{\partial \varphi_{,\kappa}} \varphi_{,\iota} \right) , \tag{B.23}$$

(The field index for which this expression should be summed is omitted for brevity). We can thus define the canonical stress-energy tensor

$$T_\iota{}^\kappa \equiv \frac{\partial \mathcal{L}}{\partial \varphi_{,\kappa}} \varphi_{,\iota} - \delta_\iota^\kappa \mathcal{L} + t_\iota{}^{[\kappa\lambda]}{}_{,\lambda} , \tag{B.24}$$

which satisfies the conservation law

$$T^{\iota\kappa}{}_{,\kappa} = 0 . \tag{B.25}$$

t is an arbitrary tensor with the marked antisymmetry due to which it cancels in the conservation law.

of the dependence $x(w)$ an additional condition fixing the parameterization. This can have the form of an algebraic equation

$$\omega(x, x') = 0 \ , \tag{B.14}$$

where ω is an arbitrary function. Spacetime variations δx^ι are then no more independent, but they must satisfy the linear equation

$$\delta\omega = \frac{\partial\omega}{\partial x}\delta x + \frac{\partial\omega}{\partial x'}\delta x' = 0 \ , \tag{B.15}$$

so that in (B.1) must be taken the variation of (B.10) with the condition (B.14). The Lagrangian variational principle for a parametric formulation of the motion thus reads

$$0 = \delta S = \delta \int f(x, x') + n(w)\omega(x, x')dw \ , \tag{B.16}$$

where n is the Lagrangian coefficient of the variational problem with a condition.

Similarly, the Hamiltonian formalism for a parametric description of the motion can be obtained from the action (B.10), the system of equations (B.12) and (B.13) for t' and x' does not have a unique solution (the condition from the footnote [1] on p. 50 is not satisfied for f given by (B.11)). The expression $p_\iota x'^\iota - f$ formally produced from f for x^ι analogously to the Hamiltonian (B.5) from L for x^i is identically equal to zero, which corresponds to the fact that the Hamiltonian made in this way, which can not be explicitely dependent on w, should be – in analogy with (B.7) – a trivial integral of motion. In view of (B.13) the integral in (B.6) can be expressed as $\int p_\iota x'^\iota dw$ (the Hamilton function for $x^\iota(w)$ would thus be identically equal to 0). However, p_ι are not independent because they must satisfy the condition (B.13). The Hamilton – Jacobi variational principle for parametric description of the motion must be thus in the form

$$0 = \delta \int (p_\iota x'^\iota - N(w)\mathcal{H}(x, p))dw \ , \tag{B.17}$$

where N again is the Lagrangian coefficient, and the so-called superhamiltonian

$$\mathcal{H}(x, p) = p_0 + H(t, x, p) \tag{B.18}$$

is the expression, which put equal to zero gives the condition (B.13). Taking the variation of (B.17) with respect to p_0 we get

$$0 = x'^0 - N(w) \ , \tag{B.19}$$

from where the physical meaning of $N(w)$ is obvious.

B.2 Fields

For a general spacetime field $\varphi_I = \varphi_I(x)$ (where the index "$_I$" denotes individual components of the field φ), the action S is the integral of the Lagrangian density $\mathcal{L} = \mathcal{L}(\varphi, \partial\varphi)$, i.e.,

$$S = \int \mathcal{L} d^4x \ , \tag{B.20}$$

hence its variation is

$$\begin{aligned} \delta S &= \int \delta\mathcal{L} d^4x = \int \sum_I \left(\frac{\partial\mathcal{L}}{\partial\varphi_I}\delta\varphi_I + \frac{\partial\mathcal{L}}{\partial\varphi_{I,\kappa}}\delta\varphi_{I,\kappa} \right) d^4x \\ &= \int \sum_I \left(\frac{\partial\mathcal{L}}{\partial\varphi_I} - \frac{\partial}{\partial x^\kappa}\frac{\partial\mathcal{L}}{\partial\varphi_{I,\kappa}} \right) \delta\varphi_I d^4x \ , \end{aligned} \tag{B.21}$$

and the field equations read

$$0 = \frac{\partial \mathcal{L}}{\partial \varphi_I} - \frac{\partial}{\partial x^\kappa}\left(\frac{\partial \mathcal{L}}{\partial \varphi_{I,\kappa}}\right) . \tag{B.22}$$

If the Lagrangian (density) does not depend explicitely on the coordinates, then its spacetime variation is given by the change of the field only

$$\frac{\partial \mathcal{L}}{\partial x^\iota} = \frac{\partial \mathcal{L}}{\partial \varphi}\varphi_{,\iota} + \frac{\partial \mathcal{L}}{\partial \varphi_{,\kappa}}\varphi_{,\kappa\iota} = \frac{\partial}{\partial x^\kappa}\left(\frac{\partial \mathcal{L}}{\partial \varphi_{,\kappa}}\varphi_{,\iota}\right) , \tag{B.23}$$

(The field index for which this expression should be summed is omitted for brevity). We can thus define the canonical stress-energy tensor

$$T_\iota{}^\kappa \equiv \frac{\partial \mathcal{L}}{\partial \varphi_{,\kappa}}\varphi_{,\iota} - \delta_\iota^\kappa \mathcal{L} + t_\iota{}^{[\kappa\lambda]}{}_{,\lambda} , \tag{B.24}$$

which satisfies the conservation law

$$T^{\iota\kappa}{}_{,\kappa} = 0 . \tag{B.25}$$

t is an arbitrary tensor with the marked antisymmetry due to which it cancels in the conservation law.

Index (English – Norwegian – Czech)

Bibliography

[1] Feynman R.P., Hybbs A.R.: Quantum Mechanics and Path Integrals, McGraw-Hill, New York 1965

[2] Kenyon I. R.: *General Relativity*, Oxford University Press, Oxford 1990

[3] Martin J. L.: *General Relativity: A Guide to its Consequences for Gravity and Cosmology*, Ellis Horwood Ltd., Chichester 1988

[4] Misner C. W., Thorne K. S., Wheeler J. A.: *Gravitation*, Freeman & Co., San Francisco 1973

[5] Stephani H.: *General relativity, An introduction to the theory of the gravitational field*, Cambridge University Press, Cambridge 1982, 1985

[6] Wald R. M.: *General Relativity*, The University of Chicago Press, Chicago and London 1984, ISBN 0-226-87033-2

[7] Weinberg S.: *Gravitation and Cosmology*, J. Wiley & Sons, New York... 1972