Solutions of Einstein's Equations:
The Field Outside Of A Massive Charged Spherical Body

Below, we list the non–zero components of all of the relevant tensors and other objects which are involved in showing that the Reissner–Nordstrom solution ($Q{\neq}0$) and the Schwarzschild solution ($Q{=}0$) are solutions to Einstein's equations.

The (covariant) Metric

$$
\begin{aligned}
g_{tt} &= -\left(1-2\,\frac{m}{r}+\frac{Q^2}{r^2}\right)\\
g_{rr} &= \left(1-2\frac{m}{r}+\frac{Q^2}{r^2}\right)^{-1}\\
g_{\theta\theta} &= r^2\\
g_{\phi\phi} &= r^2\sin^2\theta
\end{aligned}
$$

The (contravariant) Inverse Metric

$$
\begin{aligned}
g^{tt} &= -\left(1-2\frac{m}{r}+\frac{Q^2}{r^2}\right)^{-1}\\
g^{rr} &= \left(1-2\,\frac{m}{r}+\frac{Q^2}{r^2}\right)\\
g^{\theta\theta} &= \frac{1}{r^2}\\
g^{\phi\phi} &= \frac{1}{r^2\sin^2\theta}
\end{aligned}
$$

The Affine (metric) Connection

$$
\begin{aligned}
\Gamma^{t}_{tr} &= \frac{m\,r - Q^2}{r\,(r^2 - 2\,m\,r + Q^2)} \\
\Gamma^{r}_{tt} &= \frac{(r^2 - 2\,m\,r + Q^2)\,(m\,r - Q^2)}{r^5} \\
\Gamma^{r}_{rr} &= -\frac{m\,r - Q^2}{r\,(r^2 - 2\,m\,r + Q^2)} \\
\Gamma^{r}_{\theta\theta} &= -\frac{r^2 - 2\,m\,r + Q^2}{r} \\
\Gamma^{r}_{\phi\phi} &= -\frac{(r^2 - 2\,m\,r + Q^2)\sin^2\theta}{r} \\
\Gamma^{\theta}_{\phi\theta} &= \frac{1}{r} \\
\Gamma^{\theta}_{\phi\phi} &= -\sin\theta\cos\theta \\
\Gamma^{\phi}_{r\phi} &= \frac{1}{r} \\
\Gamma^{\phi}_{\theta\phi} &= \frac{\cos\theta}{\sin\theta}
\end{aligned}
$$

The Riemann–Christoffel Tensor

$$
\begin{aligned}
R_{trtr} &= \frac{2\,m\,r - 3\,Q^2}{r^4} \\
R_{t\theta t\theta} &= -\frac{(r^2 - 2\,m\,r + Q^2)\,(m\,r - Q^2)}{r^4} \\
R_{t\phi t\phi} &= -\frac{(r^2 - 2\,m\,r + Q^2)\,(m\,r - Q^2)\sin(\theta)^2}{r^4} \\
R_{r\theta r\theta} &= \frac{m\,r - Q^2}{r^2 - 2\,m\,r + Q^2} \\
R_{r\phi r\phi} &= \frac{(m\,r - Q^2)\sin^2\theta}{r^2 - 2\,m\,r + Q^2} \\
R_{\theta\phi\theta\phi} &= -2\sin^2\theta\,m\,r + \sin^2\theta\,Q^2
\end{aligned}
$$

The Ricci tensor

$$
\begin{aligned}
R_{tt} &= \frac{\left(r^2 - 2\,m\,r + Q^2\right) Q^2}{r^6} \\
R_{rr} &= -\frac{Q^2}{r^2\left(r^2 - 2\,m\,r + Q^2\right)} \\
R_{\theta\theta} &= \frac{Q^2}{r^2} \\
R_{\phi\phi} &= \frac{\sin^2\theta\, Q^2}{r^2}
\end{aligned}
$$

The Ricci Scalar

$$R = 0$$

The Einstein Tensor: $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R.$

$$
\begin{aligned}
G_{tt} &= \frac{\left(r^2 - 2\,m\,r + Q^2\right) Q^2}{r^6} \\
G_{rr} &= -\frac{Q^2}{r^2\left(r^2 - 2\,m\,r + Q^2\right)} \\
G_{\theta\theta} &= \frac{Q^2}{r^2} \\
G_{\phi\phi} &= \frac{\sin^2\theta\, Q^2}{r^2}
\end{aligned}
$$

The Maxwell Tensor

$$F_{tr} = E_r(r) = -F_{rt}.$$

The Energy–Momentum Tensor: $4\pi T_{\mu\nu} = -g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} + \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$

$$\begin{aligned}
T_{tt} &= -\frac{E_r(r)^2(r^2 - 2\,m\,r + Q^2)}{2r^2} \\
T_{rr} &= \frac{E_r(r)^2 r^2}{2\,(r^2 - 2\,m\,r + Q^2)} \\
T_{\theta\theta} &= -\frac{E_r(r)^2 r^2}{2} \\
T_{\phi\phi} &= -\frac{E_r(r)^2 r^2 \sin^2\theta}{2}
\end{aligned}$$

So we see that Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi T_{\mu\nu}$$

are satisfied with the field for a point charge $E_r(r) = Q/r^2$.

For Q=0 we have the field/geometry outside a neutral spherical body of mass m. The source terms vanish identically. This is an 'empty–space' solution of Einstein's equations. For the $Q\neq 0$ case, we have the field/geometry outside a spherical body of electric charge Q and mass m. It is not an empty–space solution. Space is filled with the 'field lines' due to the point source of charge at $r = 0$.

You have studied many properties of these solutions in class, exercises and in the midterm exam.

–Clifford V. Johnson