Continuous interacting populations

General idea:
Important solutions are near steady-states (or at least accessible)

Model near equilibrium
Do this to first order

$$\dot{u} = u(1-v)$$
$$\dot{v} = \alpha v(u-1)$$

$u=v=0$: $\begin{pmatrix} 1 & \\ & -\alpha \end{pmatrix}$

$u=v=1$: $\begin{pmatrix} & -1 \\ \alpha & \end{pmatrix}$

$$\frac{dx}{dt} = f_1(x,y)$$
$$\frac{dy}{dt} = f_2(x,y)$$

$$\left.\begin{aligned}\frac{dx}{dt} &= ax+by \\ \frac{dy}{dt} &= cx+dy\end{aligned}\right\} = A\begin{pmatrix} x \\ y \end{pmatrix}$$

Fact about 2D systems:
local $\Rightarrow$ global
- curves only intersect at steady-state
- closed loops go around steady-state

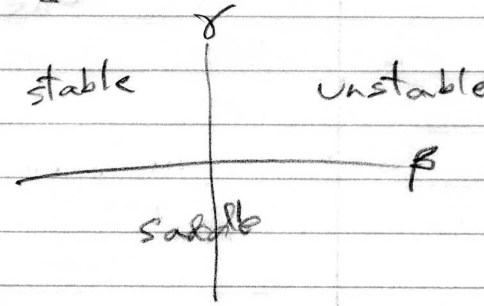
$$\Rightarrow \frac{d^2x}{dt^2} - \underset{\text{tr}A}{\underset{\shortparallel}{\beta}} \frac{dx}{dt} + \underset{\det A}{\underset{\shortparallel}{\gamma}} x = 0$$

$$\ddot{x} = a\dot{x} + b\dot{y}$$
$$d\dot{x} - b\dot{y} = (ad-bc)x$$

Solutions: $x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

$$\lambda_{1,2} = \frac{\beta \pm \sqrt{\beta^2 - 4\gamma}}{2} = \frac{\beta \pm \sqrt{\delta}}{2}$$

go to table
- β negative $\Rightarrow$ saddle



Higher dimensional

$Re[\lambda_i] > 0$ unstable

$Re[\lambda_i] < 0$ stable

Generalized Lotka-Volterra

$$\frac{dN_i}{dt} = N_i\left(a_i - \sum_{j=1}^{k} b_{ij} P_j\right)$$

$$\frac{dP_i}{dt} = P_i\left(\sum_{j=1}^{K} c_{ij} N_j - d_i\right)$$

trivial $N_i = P_i = 0$:
unstable

$$\left(\begin{array}{ccc|ccc} a_1 & & & & & \\ & \ddots & & & & \\ & & a_k & & & \\ \hline & & & -d_1 & & \\ & & & & \ddots & \\ & & & & & -d_k \end{array}\right)$$

$B\vec{P} = \vec{a}$, $C\vec{N} = \vec{d}$

Non-trivial:

$$A = \left(\begin{array}{c|c} 0 & -N_0^T \cdot B \\ \hline P_0^T \cdot C & 0 \end{array}\right)$$

solutions: $|A - \lambda I| = 0$ real poly

$\mathrm{tr} A = 0$

Elements real, or complex as complex conjugates

One λ_i with $Re\,\lambda_i \neq 0 \Rightarrow Re\,\lambda_j > 0$ exists

thus unstable