

\*WARNING: Not Readable!\*

Ben Good  
week 11

\* we first want our solution to be stable  
when everything is well mixed, i.e.  $\nabla^2 S = 0$

so we want:

$$S_+ = \delta g \quad \text{to be stable.}$$

$$a_+ = \delta f$$

\* thus, we linearize

$$J = \delta \begin{pmatrix} g_s & g_a \\ f_s & f_a \end{pmatrix}, \quad \text{solutions are } e^{\lambda t}$$

where

$$|\delta J - \lambda I| = 0 \quad \text{or} \quad \begin{vmatrix} \delta g_s - \lambda & \delta g_a \\ \delta f_s & \delta f_a - \lambda \end{vmatrix} = 0$$

or

$$(\delta g_s - \lambda)(\delta f_a - \lambda) - \delta^2 f_s f_a = 0$$

$$\lambda^2 - \delta(g_s + f_a)\lambda + \delta^2(g_s f_a - f_s g_a) = 0$$

so if  $\lambda$  is always negative,

$$-\delta(g_s + f_a) > 0 \quad \text{or} \quad g_s + f_a < 0$$

$$\delta^2(g_s f_a - f_s g_a) > 0 \quad \text{or} \quad g_s f_a - f_s g_a > 0$$

Now we look at full problem near homogeneous eq.

$$s_0 = s^* + s', \quad w = \begin{bmatrix} s' \\ a' \end{bmatrix}$$

~~scribble~~

so guess  $w = \cancel{e^{kt}} e^{kt} w_k(\vec{r})$

which works if

$$\lambda w = \gamma J w - D k^2 w, \quad D = \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}$$

$$\text{or } |\gamma J - D k^2 - \lambda I| = 0$$

$$\text{or } \begin{vmatrix} (\gamma g_s - k^2) - \lambda & \gamma g_a \\ \gamma f_s & (\gamma f_a - d k^2) - \lambda \end{vmatrix} = 0$$

so conditions are either

$$(\gamma g_s - k^2) + (\gamma f_a - d k^2) \cancel{> 0} > 0$$

$$\gamma(g_s + f_a) - (1+d)k^2 > 0 \rightarrow \text{but always negative}$$

$$\text{or } (\gamma g_s - k^2)(\gamma f_a - d k^2) - \gamma^2 f_s g_a < 0$$

or

$$dk^4 - \gamma(dg_s + f_a)k^2 + \gamma^2(g_s f_a - f_s g_a) < 0$$

so  $dk^4 > 0$ , last term  $> 0$ ,

so at minimum,

$$dg_s + f_a > 0, \text{ but } g_s + f_a < 0,$$

so  $d \neq 1$ ,  $g_s, f_a$  diff. signs.

Final condition:

$$\cancel{dg_s + f_a} \quad dg_s + f_a > 2\sqrt{d(g_s f_a - f_s g_a)} > 0$$

so all conditions for diffusive instability:

$$g_s + f_a < 0$$

$$g_s f_a - f_s g_a > 0$$

$d \neq 1$ ,  $f_a, g_s$  diff. signs.

$$dg_s + f_a > 2\sqrt{d(g_s f_a - f_s g_a)} > 0$$

In 1 dimension,

$$\nabla^2 w_k = \frac{d^2 w_k}{dx^2}, \text{ so } w_k = A \sin kx + B \cos kx$$

and boundary conditions imply

$$A=0, \quad k = \frac{\pi n}{L}$$

In higher dimensions, tougher

thus, we substitute in  $w$ , and get

$$\lambda w = \gamma J w - D k^2 w$$

so  $w$  is a solution if

$$\det(\lambda - \gamma J + D k^2) = 0$$

$$\begin{vmatrix} \lambda - \gamma g_s + D k^2 & -\gamma g_a \\ -\gamma s_s & \lambda - \gamma s_a + D k^2 \end{vmatrix} = 0$$

But now we want  $\lambda > 0$  for some  $k^2$   
so that our disturbances can form  
patterns.

by multiplying out, we obtain

$$-\gamma^2 g_a f_s + \lambda^2 + \gamma^2 g_s f_a + dk^4 - \gamma \lambda (f_a + g_s) + \lambda (k^2 + dk^2)$$

~~$-\gamma g_s dk^2 - \gamma f_a k^2$~~

$$\lambda^2 + \lambda [k^2(1+d) - \gamma(f_a + g_s)] + h(k^2) = 0$$

$$h(k^2) = dk^4 - \gamma(dg_s + f_a)k^2 + \gamma^2(g_s f_a - g_a f_s)$$

so if  $\lambda$  is to be negative, either

$$[k^2(1+d) - \gamma(f_a + g_s)] > 0 \quad * \text{ different from book}$$

or

$$h(k^2) < 0$$

but  $(g_s f_a - g_a f_s)$  is  $> 0$ , so

we must have

$$dg_s + f_a > 0. \quad \text{But } g_s + f_a < 0, \text{ so}$$

we must have

$d \neq 1$  and  $g_s, f_a$  have opposite signs.

so first we want to make sure the system is stable without diffusion

so we want

$$s_+ = \gamma g(s, a) \quad \text{to be stable}$$

$$a_+ = \gamma f(s, a)$$

what do we do?

linear analysis.

$$J = \begin{pmatrix} \gamma g_s & \gamma g_a \\ \gamma f_s & \gamma f_a \end{pmatrix} \quad \text{solutions } e^{\lambda t} \text{ where } \lambda \text{ is an eigenvalue}$$

i.e.

$$(\gamma g_s - \lambda)(\gamma f_a - \lambda) - \gamma f_s \gamma g_a = 0$$

$$\text{or } \lambda^2 - \lambda \gamma g_s - \lambda \gamma f_a - \gamma f_s \gamma g_a + \gamma f_s \gamma g_a = 0$$

$$\text{or } \lambda^2 - (\gamma g_s + \gamma f_a) \lambda - \gamma f_s \gamma g_a = 0$$

$$A=1, \quad B=-(\gamma g_s + \gamma f_a) \quad C=-\gamma f_s \gamma g_a + \gamma f_s \gamma g_a$$

$$\text{so } \lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2}$$

