a

The expectation value for the energy is given by

$$\langle E\rangle = \frac{-e^2}{2a_o}\left(\frac{16}{25}\frac{1}{1^2}+\frac{9}{25}\frac{1}{2^2}\right) = \frac{-e^2}{a_o}\frac{73}{200}$$

The expectation values for the momentum operators are given by

$$\langle \hat{L}^2\rangle = \hbar^2\frac{9}{25}(1) = \hbar^2\frac{9}{25}$$
$$\langle \hat{L}_z\rangle = \hbar\frac{9}{25}(1) = \hbar\frac{9}{25}$$
$$\langle \hat{L}_x\rangle = \frac{1}{2}\langle \hat{L}_+ + \hat{L}_-\rangle = 0$$

b

Let

$$\alpha = \frac{e^2}{2a_o}$$

We have that

$$|\phi(t)\rangle = e^{-i\hat{H}t/\hbar}|\phi\rangle = \frac{4}{5}e^{i\alpha t/\hbar}|1,0,0\rangle + \frac{3i}{5}e^{i\frac{\alpha}{4}t/\hbar}|2,1,1\rangle$$

It is easy to see that none of the expectation values vary with time. This is not surprising because, if $\hat{Q}$ is an operator, then

$$\frac{d\langle\hat{Q}\rangle}{dt} = c[\hat{H},\hat{Q}]$$

for some constant c. Here, we have that $\hat{L}^2$, $\hat{L}_z$, and $\hat{L}_x$ commute with the Hamiltonian, so

$$[\hat{H},\hat{L}^2] = [\hat{H},\hat{L}_z] = [\hat{H},\hat{L}_x] = 0$$

which implies that $\langle\hat{L}^2\rangle$, $\langle\hat{L}_z\rangle$, and $\langle\hat{L}_x\rangle$ are constant with time.

a

The expectation value for the energy is given by

$$\langle E\rangle = \frac{-e^2}{2a_o}(\frac{2}{5}\frac{1}{1^2} + \frac{3}{5}\frac{1}{2^2}) = \frac{-e^2}{a_o}\frac{11}{40}$$

b

Let

$$\alpha = \frac{e^2}{2a_o}$$

The probability of being in the state $|\phi_{211}\rangle$ at time t is

$$|\langle\phi(t)|\ \phi_{211}\rangle|^2 = \frac{1}{5}\left|e^{i\frac{\alpha}{4}t/\hbar}\right|^2 = \frac{1}{5}$$

c

The probability is given by

$$\int_0^{r_o} |\phi|^2\, r^2 dr = \int_0^{r_o} (\frac{2}{5}(R_{10}^2) + \frac{3}{5}(R_{21}^2))r^2 dr$$
$$= \frac{2}{5}\frac{4}{a_o^3}\frac{1}{4}a_o(a_o^2 - (2r_o^2 + 2a_o r_o + a_o^2)e^{-2\frac{r_o}{a_o}})$$
$$+\frac{3}{5}\frac{1}{8a_o^3}\frac{1}{3a_o^2}a_o(24a_o^4 - (r_o^r + 4r_o^3 a_o + 12r_o^2 a_o^2 + 24r_o a_o^3 + 24a_o^4)e^{-\frac{r_o}{a_o}})$$

From the values given, we have that

$$\frac{r_o}{a_o} = \frac{10^{-12}m}{5.29\times 10^{-11}m} = 0.019$$

Plugging this in to the above equation gives that the probabilty is approximately 3.6×10^{-6}.

d

We start in the state

$$|\phi\rangle = \frac{1}{\sqrt{10}}(2\,|100\rangle + |210\rangle + \sqrt{2}\,|211\rangle + \sqrt{3}\,|21-1\rangle)$$

We have after the measurement that $L = 1$ and $L_x = +1$. This implies the final state $|\phi'\rangle$ must be the following linear combination.

$$|\phi'\rangle = a\,|210\rangle + b\,|211\rangle + c\,|21-1\rangle$$

with

$$a^2 + b^2 + c^2 = 1$$

Since we are now in the eigenstate +1 of $\hat{L}_x$, we apply $\hat{L}_x$ using the relation

$$\hat{L}_x\,|\phi'\rangle = |\phi'\rangle$$

this gives

$$\begin{aligned}\hat{L}_x\,|\phi'\rangle = \frac{1}{2}(\hat{L}_+ + \hat{L}_-) &= \frac{1}{2}(a(\sqrt{2}\,|211\rangle + \sqrt{2}\,|21-1\rangle) + b\sqrt{2}\,|210\rangle + c\sqrt{2}\,|210\rangle) \\ &= a\,|210\rangle + b\,|211\rangle + c\,|21-1\rangle\end{aligned}$$

This implies that

$$\begin{aligned}a &= \frac{b}{\sqrt{2}} + \frac{c}{\sqrt{2}} \\ b &= \frac{a}{\sqrt{2}} \\ c &= \frac{a}{\sqrt{2}} = b\end{aligned}$$

So, we have

$$a^2 + b^2 + c^2 = 2a^2 = 1$$

so

$$a = \frac{1}{\sqrt{2}}$$
$$b = \frac{1}{2}$$
$$c = \frac{1}{2}$$

This gives

$$|\phi'\rangle = \frac{1}{\sqrt{2}}\,|210\rangle + \frac{1}{2}\,|211\rangle + \frac{1}{2}\,|21-1\rangle$$

Given

$$V(r) = -\frac{e^2}{r}(1 + \frac{b}{r})$$

Then

$$ER = -\frac{\hbar}{2\mu r^2}\frac{d}{dr}r^2\frac{dR}{dr} + (\frac{l(l+1)\hbar}{2\mu r^2} - \frac{e^2}{r} - b\frac{e^2}{r^2})R$$

We can see this is very close to an energy we can solve for except for the extra $b\frac{e^2}{r^2}$ term. In order to change this equation into an equation we can solve for let us redefine $l \to l'$ which will absorb the extra term.

$$\frac{l(l+1)\hbar}{2\mu r^2} - \frac{e^2}{r} - b\frac{e^2}{r^2} = \frac{l'(l'+1)\hbar}{2\mu r^2} - \frac{e^2}{r}$$

$$l' = \sqrt{l^2 + l + \frac{2b\mu e^2}{\hbar^2} + \frac{1}{4}}$$

Therefore, we have

$$ER = -\frac{\hbar}{2\mu r^2}\frac{d}{dr}r^2\frac{dR}{dr} + (\frac{l'(l'+1)\hbar}{2\mu r^2} - \frac{e^2}{r})R$$

$$E_n = -\frac{\mu e^4}{2\hbar^2(x+l')^2}$$

$$R_{nl} = -[(\frac{2}{(n+l')a_0})^3\frac{(n-1)!}{2(n+1)(n+2l')!^3}]^{\frac{1}{2}}e^{-\frac{1}{2}\rho}e^{l'}L^{2l+1}_{n+l}(l)$$

a

The potential $V = 0$, which gives in 2D the time-independent Schrodinger equation

$$\frac{-\hbar^2}{2m}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\psi = E\psi$$

b

Let

$$k^2 = \frac{2mE}{\hbar^2}$$

where $E > 0$. This implies, with separation of variables, solutions of the form

$$X(x) = A\sin kx + B\cos kx$$
$$Y(y) = C\sin(ky + \phi)$$

c

Applying the boundary conditions we have

$$X(0) = 0 = X(a)$$

which implies that

$$X(x) = A\sin(\frac{n_x\pi}{a}x)$$

since

$$k_x = \frac{\pi n_x}{a}$$

and

$$Y(c) = Y(c + 2\pi R)$$

which implies that

$$Y(y) = C\sin(n_y \frac{y}{R} + \phi)$$

since

$$k_y = \frac{n_y}{R}$$

d

The energy is given by

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m}(\frac{\pi^2 n_x^2}{a^2} + \frac{n_y^2}{R^2})$$

where

$$n_x + n_y = n$$

The degeneracy depends on whether the ratio $\frac{\frac{\pi^2}{a^2}}{R^2}$ is close to unity.

e

Fix n_y. Then the new E_n contains

$$E_{\text{old}} = \frac{\hbar^2 \pi^2 n_x^2}{2ma^2}$$

Varying n_y introduces new states.

f

When $R << a$ we have that

$$E_n = \frac{\hbar^2}{2m}\frac{1}{R^2}(\frac{R^2}{a^2}\pi^2 n_x^2 + n_y^2) = \frac{\hbar^2}{2m}\frac{n_y^2}{R^2}$$

These energies are very high for low n_y, so the experimental consequences are that they cannot observe these since the experiments are only conducted at low energies.

Maggie Regan
Physics 113
Boccio 9.7.40.

In this problem a particle of spin-$\frac{1}{2}$ is in a D-state of orbital angular momentum. This means that we are in a state with $l = 2$. The possible values of total angular momentum are $J = 2 \pm \frac{1}{2} = \frac{3}{2}, \frac{5}{2}$. For these possible values of J, the states of total angular momentum are those for which the $J_z = m$ values are considered. These values range from $-J$ to $+J$. Therefore, the possible states are:

$$|\tfrac{3}{2}, -\tfrac{3}{2}\rangle, |\tfrac{3}{2}, -\tfrac{1}{2}\rangle, |\tfrac{3}{2}, \tfrac{1}{2}\rangle, |\tfrac{3}{2}, \tfrac{3}{2}\rangle \ |\tfrac{5}{2}, -\tfrac{5}{2}\rangle, |\tfrac{5}{2}, -\tfrac{3}{2}\rangle, |\tfrac{5}{2}, -\tfrac{1}{2}\rangle, |\tfrac{5}{2}, \tfrac{1}{2}\rangle, |\tfrac{5}{2}, \tfrac{3}{2}\rangle, |\tfrac{5}{2}, \tfrac{5}{2}\rangle.$$

Then, we are given a single particle Hamiltonian of $H = A + B\vec{L} \cdot \vec{S} + C\vec{L} \cdot \vec{L}$. We know that $\vec{J} = \vec{L} + \vec{S}$. Therefore, $\vec{L} \cdot \vec{S} = \frac{1}{2}(J^2 - L^2 - S^2)4$.

This changes the Hamiltonian to the equation

$$\begin{aligned} H &= A + \frac{B}{2}(J^2 - L^2 - S^2) + CL^2 \\ &= A + \frac{B\hbar^2}{2}[(J(J+1) - L(L+1) - S(S+1)] + \hbar^2 C[L(L+1)]. \end{aligned}$$

We know the value of S to be $\frac{1}{2}$ and the value of L to be 2. So, the Hamiltonian is then $H = A + \frac{B\hbar^2}{2}[J(J+1) - \frac{27}{4}] + 6\hbar^2 C$.

Let us consider the $l = 2$ states (for some given principal quantum number n, which is irrelevant) of the H atom, taking into account the electron spin$= \frac{1}{2}$.

(a) We can enumerate all the states in the J, M representation arising from the $l = 2$, $s = \frac{1}{2}$ states.

$$\left|\tfrac{3}{2}, -\tfrac{3}{2}\right\rangle, \left|\tfrac{3}{2}, -\tfrac{1}{2}\right\rangle, \left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle, \left|\tfrac{3}{2}, \tfrac{3}{2}\right\rangle$$
$$\left|\tfrac{5}{2}, -\tfrac{5}{2}\right\rangle, \left|\tfrac{5}{2}, -\tfrac{3}{2}\right\rangle, \left|\tfrac{5}{2}, -\tfrac{1}{2}\right\rangle, \left|\tfrac{5}{2}, \tfrac{1}{2}\right\rangle, \left|\tfrac{5}{2}, \tfrac{3}{2}\right\rangle, \left|\tfrac{5}{2}, \tfrac{5}{2}\right\rangle$$

(b) The two states that have $M = \frac{1}{2}$ are $\left|\frac{3}{2}, \frac{1}{2}\right\rangle$ and $\left|\frac{5}{2}, \frac{1}{2}\right\rangle$. We can rewrite them in terms of the product space kets because we know that the product space kets must be a linear combination of the J, M representation.

$$\left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle = a\left|2, 1, \tfrac{1}{2}, -\tfrac{1}{2}\right\rangle + b\left|2, 1, \tfrac{1}{2}, \tfrac{1}{2}\right\rangle$$
$$\left|\tfrac{5}{2}, \tfrac{1}{2}\right\rangle = c\left|2, 1, \tfrac{1}{2}, -\tfrac{1}{2}\right\rangle + d\left|2, 1, \tfrac{1}{2}, \tfrac{1}{2}\right\rangle$$

We also know that $a^2 + b^2 = 1$, $c^2 + d^2 = 1$, and $ac + bd = 0$. We can therefore solve for the coeficients and see that

$$\left|\tfrac{3}{2}, \tfrac{1}{2}\right\rangle = \sqrt{\tfrac{3}{5}}\left|2, 1, \tfrac{1}{2}, -\tfrac{1}{2}\right\rangle - \sqrt{\tfrac{2}{5}}\left|2, 1, \tfrac{1}{2}, \tfrac{1}{2}\right\rangle$$
$$\left|\tfrac{5}{2}, \tfrac{1}{2}\right\rangle = \sqrt{\tfrac{2}{5}}\left|2, 1, \tfrac{1}{2}, -\tfrac{1}{2}\right\rangle + \sqrt{\tfrac{3}{5}}\left|2, 1, \tfrac{1}{2}, \tfrac{1}{2}\right\rangle$$

Maggie Regan
Physics 113
Zettili 6-05

In this problem we consider a spinless particle of mass m which is confined to move under the influence of a three-dimensional potential

$$V(x) = \begin{cases} 0 & \text{for} \quad 0 < x < a, 0 < y < a, 0 < z < b \\ +\infty & \text{elsewhere} \end{cases}$$

For part (a) we are asked to find the expression for the energy levels $E_{n_x n_y n_z}$ and their corresponding wave functions. These are given in the book for general rectangular box potentials. So, we have that $E_{n_x n_y n_z} = \frac{\hbar^2 \pi^2}{2m}\left(\frac{n_x^2}{a^2} + \frac{n_y^2}{a^2} + \frac{n_z^2}{b^2}\right)$ and the corresponding wave functions are $\psi_{n_x n_y n_z}(x, y, z) = \sqrt{\frac{8}{a^2 b}} \sin(\frac{n_x \pi}{a} x) \sin(\frac{n_y \pi}{a} y) \sin(\frac{n_z \pi}{b} z)$.

For part (b) we let $a = 2b$. Then, we want to find the energies of the five lowest states and their degeneracies. So, we have that the lowest states are $|1, 1, 1\rangle, |2, 1, 1\rangle, |1, 2, 1\rangle, |1, 1, 2\rangle$, and $|2, 2, 1\rangle$. Therefore, the energies of these states (in order and with their degeneracies) are

$$E_{111} = \frac{3\pi^2\hbar^2}{4mb^2} \; E_{211} = E_{121} = \frac{9\pi^2\hbar^2}{8mb^2} \; E_{221} = \frac{3\pi^2\hbar^2}{2mb^2} \; E_{112} = \frac{9\pi^2\hbar^2}{2mb^2}.$$

Given

$$V(x,y,z) = \begin{cases} \frac{1}{2}m\omega^2 z^2 \\ +\infty \end{cases}$$

(a) The Schrodinger equation is $\hat{H}\Psi(x,y,z) = E\Psi(x,y,z)$ where $\hat{H} = \frac{\hbar^2\Delta^2}{2m} + \frac{1}{2}m\omega^2 z^2$. The wave equation is the wave equation for a square well,

$$\Psi(x,y,z) = \tfrac{2}{a}\sin(\tfrac{\pi n_x}{a})\sin(\tfrac{\pi n_y}{a} Z_{n_z}$$

where Z_{n_z} are the Hermite polynomials. In this case, in order to satisfiy the boundary condition (the wave function equals zero at zero) n_z must be odd only.

(b) We know that the energy in terms of n_x, n_y, and n_z is the energy of an infinite potential well and a harmonic oscillator

$$E_{n_x n_y n_z} = \tfrac{\pi^2 n^2}{2ma^2}(n_x^2 + n_y^2) + \hbar\omega(n_z + \tfrac{1}{2})$$

(c) We can calculate the lowest energy levels in the x, y plane to be $(1,1)$, the degernerate energy level $(1,2),(2,1)$ also $(2,2) and (3,2)$.