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In this problem we have two atoms with $J_1 = 1$ and $J_2 = 2$ that are coupled. The energy is described by $\hat{H} = \epsilon\vec{J}_1 \cdot \vec{J}_2$, where $\epsilon > 0$. We are supposed to find all of the energies and degeneracies for the coupled system.

To do this, we remember that we have $\vec{J} = \vec{J}_1 + \vec{J}_2$. Then, we have that $J^2 = J_1^2 + J_2^2 + 2\vec{J}_1 \cdot \vec{J}_2$. Thus, $\vec{J}_1 \cdot \vec{J}_2 = J^2 - J_1^2 - J_2^2$. This means that the hamiltonian becomes $\hat{H} = \frac{\epsilon}{2}(J^2 - J_1^2 - J_2^2) = \frac{\hbar^2\epsilon}{2}(J(J+1) - J_1(J_1+1) - J_2(J_2+1))$. Then, we know the values for J_1 and J_2, so we plug these in and get the hamiltonian of

$$\hat{H} = \frac{\hbar^2\epsilon}{2}(J(J+1) - 8).$$

Then, the possible values for$J$, are 1, 2, and 3. Plugging these in, we have $\hat{H} = \frac{-5}{2}\epsilon\hbar^2, -\epsilon\hbar^2$, and $2\epsilon\hbar^2$. The first value is 3 times degenerate, the second is 5 times degenerate, and the third is 7 times degenerate.

a

We can quickly see that

$$\hat{S}_+ |1,1\rangle = 0$$
$$\hat{S}_+ |1,0\rangle = \sqrt{2}\hbar |1,1\rangle$$
$$\hat{S}_+ |1,-1\rangle = \sqrt{2}\hbar |1,0\rangle$$
$$\hat{S}_- |1,1\rangle = \sqrt{2}\hbar |1,0\rangle$$
$$\hat{S}_- |1,0\rangle = \sqrt{2}\hbar |1,-1\rangle$$
$$\hat{S}_- |1,-1\rangle = 0$$

so

$$S_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$
$$S_- = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

Then we can find

$$S_x = \frac{\hbar}{2}(S_+ + S_-) = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$S_y = \frac{\hbar}{2i}(S_+ - S_-) = \frac{\hbar}{i\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Then we can just write that

$$S^2 = \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

since we are in the S_z basis with total spin 1.

b

The eigenstates of $\hat{S}_x$ can be found from Mathematica to be

$$\vec{v}_1 = \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

$$\vec{v}_2 = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$$\vec{v}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

for eigenvalues of $-\hbar$, $\hbar$, and 0 respectively.

c

1

The probabilities are

$$\left| \left\langle \hat{S}_z = \hbar \middle| \psi \right\rangle \right|^2 = \frac{1}{14}$$

$$\left| \left\langle \hat{S}_z = -\hbar \middle| \psi \right\rangle \right|^2 = \frac{9}{14}$$

$$\left| \left\langle \hat{S}_z = 0 \middle| \psi \right\rangle \right|^2 = \frac{2}{7}$$

so

$$\langle \hat{S}_z \rangle = \frac{1}{14}\hbar - \frac{9}{14}\hbar = -\frac{4}{7}\hbar$$

2

We find the density operator W, which is given by

$$W = |\psi\rangle \langle\psi| = \frac{1}{14}\begin{pmatrix} 1 & 2 & 3i \\ 2 & 4 & 6i \\ -3i & -6i & 9 \end{pmatrix}$$

so we have that

$$WS_x = \hbar\frac{1}{14\sqrt{2}}\begin{pmatrix} 2 & 1+3i & 2 \\ 4 & 2+6i & 4 \\ -6i & 9-3i & -6i \end{pmatrix}$$

Thus, we have that

$$\langle \hat{S}_x \rangle = Tr(WS_x) = \hbar\frac{4}{14\sqrt{2}} = 7\sqrt{2}\hbar$$

3

We find the probability as

$$\left|\left\langle \hat{S}_x = \hbar \middle| \, \psi \right\rangle\right|^2 = |\langle \vec{v}_2| \, \psi\rangle|^2 = \frac{1}{4}\frac{1}{14}(1 + 2\sqrt{2} + 3i)^2 = \frac{9 + 2\sqrt{2}}{28}$$

d

1

Using the matrices from above we can find the Hamiltonian to be

$$H = \hbar \begin{pmatrix} A + \frac{B}{2} & 0 & \frac{B}{2} \\ 0 & B & 0 \\ \frac{B}{2} & 0 & -A + \frac{B}{2} \end{pmatrix}$$

The eigenvalues, which are the allowed energies, can be found from Mathematica to be

$$\hbar B, \frac{\hbar}{2}(B + \sqrt{4A^2 + B^2}), \frac{\hbar}{2}(B - \sqrt{4A^2 + B^2})$$

2

We are in the state

$$|\psi\rangle = \vec{v}_2 = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

so the density operator is

$$W = |\psi\rangle \langle\psi| = \frac{1}{4} \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix}$$

So the expectation value is

$$\langle \hat{S}_z \rangle = Tr(WS_z) = \frac{\hbar}{4}(1 - 1) = 0$$

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In this problem we are considering a particle in a state described by $\psi = N(x+y+2z)e^{-\alpha r}$, where N is a normalization factor.

For part (a) we want to rewrite the $Y_1^{\pm 1,0}$ functions in terms of x, y, z, and r. We do this by using the spherical coordinate to cartesian coordinate transformations. These are

$$x = r\sin\theta\cos\phi \ y = r\sin\theta\sin\phi \ z = r\cos\theta.$$

We have that the sperical harmonics are given by

$$Y_1^0 = \tfrac{1}{2}\sqrt{\tfrac{3}{\pi}}\cos\theta \ Y_1^{\pm 1} = \mp\tfrac{1}{2}\sqrt{\tfrac{3}{2\pi}}\sin\theta(\cos\phi \pm i\sin\phi).$$

Using the sperical coordinate to cartesian coordinate transformations stated above, we are able to obtain the sperical harmonics in terms of x, y, z, and r to be

$$Y_1^0 = \sqrt{\tfrac{3}{4\pi}}\tfrac{z}{r} \ Y_1^{\pm 1} = \mp\sqrt{\tfrac{3}{4\pi}}\tfrac{x \pm iy}{\sqrt{2}r}.$$

For part (b) we want to show that for a particle described in the state ψ above, we have that $P(L_z = 0) = \frac{2}{3}$, $P(L_z = \hbar) = \frac{1}{6}$, and $P(L_z = -\hbar) = \frac{1}{6}$.

We do this by writing the state ψ in terms of the spherical harmonics. Therefore, $\psi = Nr\sqrt{\frac{2\pi}{3}}[(-i-1)Y_1^1 + (1-i)Y_1^{-1} + 2\sqrt{2}Y_1^0]e^{-\alpha r}$. This equaiton needs to be normalized, however. Doing this we obtain $\psi = Nr\sqrt{\frac{2\pi}{3}}[\frac{(-i-1)}{\sqrt{12}}Y_1^1 + \frac{(1-i)}{\sqrt{12}}Y_1^{-1} + \frac{2\sqrt{2}}{\sqrt{12}}Y_1^0]e^{-\alpha r}$.

To get the probabilities for each L_z state, we just square each coefficient of the spherical harmonics. Thus, we get the probabilities of $P(L_z = 0) = \frac{2}{3}$, $P(L_z = \hbar) = \frac{1}{6}$, and $P(L_z = -\hbar) = \frac{1}{6}$, which is what we wanted to achieve.

Suppose that we have a spin-$\frac{1}{2}$ particle interacting with a magnetic field via the Hamiltonian

$$\hat{H} = \begin{cases} -\vec{\mu}\cdot\vec{B}, \vec{B} = B\hat{e}_z & 0 \le t < T \\ -\vec{\mu}\cdot\vec{B}, \vec{B} = B\hat{e}_y & 0 \le t < 2T \end{cases}$$

where $\vec{\mu} = \mu_B\vec{\sigma}$ and the system initially ($t = 0$) in the state

$$|\psi(0)\rangle = |x+\rangle = \frac{1}{\sqrt{2}}(|z+\rangle + |z-\rangle)$$

We can determine the probability that the state of the system at $t = 2T$ is

$$|\psi(2T)\rangle = |x+\rangle$$

using the time development operator.

$$|\psi(t)\rangle = (\cos kt_i(\vec{\sigma}\cdot\vec{n})\sin kt)\,|\psi(0)\rangle$$

$$\vec{\sigma}\cdot\vec{n} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|\psi(T)\rangle = \frac{1}{\sqrt{2}}(\cos kT\begin{pmatrix} 1 \\ 1 \end{pmatrix} + i\sin kT\begin{pmatrix} 1 \\ -1 \end{pmatrix}) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{ikT} \\ e^{-ikT} \end{pmatrix}$$

Now, we can use the state we just found at $t = T$ to be the intial state when we change into the y basis.

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}(\cos kt\begin{pmatrix} e^{ikT} \\ e^{-ikT} \end{pmatrix} - \sin kt\begin{pmatrix} -e^{ikT} \\ e^{-ikT} \end{pmatrix}$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{ikT}\cos kt + e^{-ikT}\sin kt \\ e^{-ikT}\cos kt - e^{ikT}\sin kt \end{pmatrix}$$

So the probability of the system at $t = 2T$ (or for the wave function calculated from $T \le t < 2T$, $t = T$) in state $|x+\rangle$ is

$$|\langle x + |\ \psi(2T)\rangle|^2 = \frac{1}{4}|e^{ikT}\cos kT + e^{-ikT}\sin kT + e^{-ikT}\cos kT - e^{ikT}\sin kT|^2$$

$$= \frac{1}{4}|2\cos^2 kT - 2i\sin^2 kT|^2$$

$$= \cos^4 kT + \sin^4 kT$$

a

We have

$$\vec{B}_o = B_o \vec{e}_z$$
$$\vec{B}_1 = B_1 \vec{e}_x$$
$$\vec{\mu} = \mu_o \vec{\sigma}$$

Define

$$B = \sqrt{B_o^2 + B_1^2}$$
$$\theta = \tan^{-1} \frac{B_1}{B_0}$$

Then

$$\vec{B} = \vec{B}_o + \vec{B}_1 = B\hat{\theta}$$

The initial state, in the z-axis basis, is

$$|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We can find the state in z'-axis by applying the rotation matrix in two dimensions. So,

$$|\psi'(0)\rangle = R(\frac{\theta}{2})\,|\psi(0)\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \end{pmatrix}$$

so the probability is given by

$$\left|\left\langle m' = +\frac{1}{2} \middle| \phi'(0) \right\rangle\right|^2 = \cos^2 \frac{\theta}{2}$$

b

The time evolution operator in the z'-axis is

$$U' = e^{-i\hat{H}t/\hbar}$$

so

$$U' |\phi'(0)\rangle = e^{-i\hat{H}t/\hbar} |\phi'(0)\rangle = e^{i\mu_o B t\sigma'_z/\hbar} |\phi(0)'\rangle$$

Let

$$\alpha = \frac{\mu_o B}{\hbar}$$

Then

$$|\psi'(t)\rangle = U' |\psi'(0)\rangle = e^{i\mu_o B t\sigma'_z/\hbar} |\phi(0)'\rangle = (\cos(\alpha t) + i\sigma_z \sin(\alpha t)) |\psi'(0)\rangle$$
$$= \begin{pmatrix} (\cos(\alpha t) + i\sin(\alpha t)) \cos\frac{\theta}{2} \\ -(\cos(\alpha t) - i\sin(\alpha t) \sin\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} e^{i\alpha t} \\ -\sin\frac{\theta}{2} e^{-i\alpha t} \end{pmatrix}$$

c

We want to find the probability of being in the state

$$|\phi\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So, we begin by rotating $|\psi'(t)\rangle$ back into the z-axis. This gives

$$|\psi(t)\rangle = R(-\frac{\theta}{2}) |\psi'(t)\rangle = \begin{pmatrix} \cos^2\frac{\theta}{2} e^{i\alpha t} + \sin^2\frac{\theta}{2} e^{-i\alpha t} \\ \cos\frac{\theta}{2} \sin\frac{\theta}{2} (e^{i\alpha t} - e^{-i\alpha t}) \end{pmatrix} = \begin{pmatrix} \cos^2\frac{\theta}{2} e^{i\alpha t} + \sin^2\frac{\theta}{2} e^{-i\alpha t} \\ i\sin\theta \sin(\alpha t) \end{pmatrix}$$

Now we find the probability at time T as

$$|\langle\phi|\ \psi(T)\rangle|^2 = \sin^2(\theta) \sin^2(\alpha T)$$

where

$$\alpha = \frac{\mu_o B}{\hbar} = \omega_o$$

Given a beam of spin-$\frac{1}{2}$ particles traveling in the y-direction. It is sent through a Stern-Gerlach apparatus, which is aligned in the z-direction, and which divides the incident beam into two beams with $m = -\frac{1}{2}$. The $m = \frac{1}{2}$ beam is then allowed to impinge on a second Stern-Gerlach apparatus aligned along the direction given by

$$\hat{e} = \sin\theta\hat{x} + \cos\theta\hat{z}$$

(a) First, we want to evaluate $\hat{S} = \frac{\hbar}{2}\hat{\sigma}\cdot\hat{e}$, where $\hat{\sigma}$ is represented by the Pauli matrices.

$$\hat{S} = \tfrac{\hbar}{2}\hat{\sigma}\cdot\hat{e} = \tfrac{\hbar}{2}\hat{\sigma}\cdot(\sin\theta\hat{x} + \cos\theta\hat{z})$$

$$\hat{S} = \tfrac{\hbar}{2}(\sin\theta\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \cos\theta\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix})$$

$$\hat{S} = \tfrac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}$$

Now, we can solve for the eigenvalues.

$$\lambda_1 = -\tfrac{\hbar}{2}, \quad \lambda_2 = \tfrac{\hbar}{2}$$

(b) We can calculate the normalized eigenvectors of $\hat{S}$.

$$v_1 = \tfrac{1}{\sqrt{2(1-\cos\theta)}}\begin{pmatrix}(-1+\cos\theta)\\ \sin\theta\end{pmatrix}$$

$$v_2 = \tfrac{1}{\sqrt{2(1+\cos\theta)}}\begin{pmatrix}(1+\cos\theta)\\ \sin\theta\end{pmatrix}$$

(c) If the original beam has an intensity, I_0, then after leaving the first Stern-Gerlach the beams will be evenly split with intensities of $\frac{1}{2}I_0$.The intensities of the two beams which emerge from the second Stern-Gerlach apparatus are therefore,

$$(\tfrac{1}{2}I_0)|\langle v_1|\uparrow\rangle|^2 = |\tfrac{\cos\theta-1}{\sqrt{2(1-\cos\theta)}}|^2 = \tfrac{1-\cos\theta}{4}I_0$$

$$(\tfrac{1}{2}I_0)|\langle v_2|\uparrow\rangle|^2 = |\tfrac{\cos\theta+1}{\sqrt{2(1+\cos\theta)}}|^2 = \tfrac{1+\cos\theta}{4}I_0$$

The ratio of the two intensities is

$$\tfrac{1+\cos\theta}{1-\cos\theta}$$

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In this probem we have a coupled spin system in a magnetic field. The Hamiltonian for this system is $H = A + J\frac{\vec{S}_1 \cdot \vec{S}_2}{\hbar^2} + B\frac{S_{1z}+S_{2z}}{\hbar}$. The factors of $\hbar$ are there to give the constants of A, B and J the units of energy.

We want to find the eigenvalues and eigenstates of the system when $S_1 = 1$ and $S_2 = \frac{1}{2}$. So, we can simplify the term multiplying J by realizing that $S^2 = S_1^2 + S_2^2 + 2\vec{S}_1 \cdot \vec{S}_2$ and $S_z = S_{1z} + S_{2z}$. Therefore, the Hamiltonian becomes $H = A + \frac{J}{\hbar^2}(S^2 - S_1^2 - S_2^2) + \frac{BS_z}{\hbar}$. Then, we can evaluate this using the values of $\frac{1}{2}$ and 1 for S_2 and S_1. Thus, we are left with the Hamiltonian of $H = A + J(S(S+1) - \frac{11}{4}) + \frac{BS_z}{\hbar}$.

Then, we can look at the case if $B = 0$. For $S = \frac{3}{2}$, $H = A + \frac{1}{2}J$ and for $S = \frac{1}{2}$, $H = A - J$. When $S = \frac{3}{2}$, the Hamiltonian is 4 times degenerate because there are 4 possible values of S_z. When $S = \frac{1}{2}$, the Hamiltonian is 2 times degenerate because there are only 2 possible values of S_z.

Now, we need to look at the case where $B \neq 0$, which gets rid of the degeneracies. If $S_z = \pm\frac{1}{2}$, then the only possible value of S is $\frac{1}{2}$. For these possible values, the Hamiltonian becomes $H = A - J \pm \frac{B}{2\hbar}$. If $S_z = \pm\frac{3}{2}$, then the possible values of S are $\frac{3}{2}$ and $\frac{1}{2}$. For $S = \frac{3}{2}$, the Hamiltonian is $H = A + \frac{1}{2}J \pm \frac{3B}{2\hbar}$. For $S = \frac{1}{2}$, the Hamiltonian is $H = A - J \pm \frac{3B}{2\hbar}$. For each set of levels, all of the ordering is the same as you would order the S_z values from largest negative value to the largest positive value.

We are given two nonidentical spin-1/2 particles. So, we have

$$s_1 = \frac{1}{2}$$
$$s_2 = \frac{1}{2}$$
$$s = 0, 1$$

For this system, we have that

$$\hat{S}_{1_z} + \hat{S}_{2_z} = \hat{S}_z$$

so we can write the Hamiltonian as

$$\hat{H} = \frac{\epsilon_o}{\hbar^2}(\hat{S}_1^2 + \hat{S}_2^2) - \frac{\epsilon_o}{\hbar}(\hat{S}_{1_z} + \hat{S}_{2_z}) = \frac{\epsilon_o}{\hbar^2}(\hat{S}_1^2 + \hat{S}_2^2) - \frac{\epsilon_o}{\hbar}(\hat{S}_z)$$
$$= \epsilon_o((\frac{1}{2})(1 + \frac{1}{2}) + (\frac{1}{2})(1 + \frac{1}{2}) - \frac{1}{\hbar}\hat{S}_z) = \epsilon_o(\frac{3}{2} - \frac{1}{\hbar}\hat{S}_z)$$

The possible states are

$$(s, m_s) = (1, 1), (1, 0), (1, -1), (0, 0)$$

so applying the Hamiltonian gives the respective energies as

$$E = \frac{5}{2}\epsilon_o, \frac{3}{2}\epsilon_o, \frac{1}{2}\epsilon_o, \frac{3}{2}\epsilon_o$$

so $E = \frac{3}{2}\epsilon_o$ is double degenerate.

We are given the Hamiltonian

$$\hat{H} = \frac{\epsilon_1}{\hbar^2}(\hat{S}_1 + \hat{S}_3)\dot{\hat{S}}_2 + \frac{\epsilon_2}{\hbar^2}(\hat{S}_{1z} + \hat{S}_{2z} + \hat{S}_{3z})^2$$

First, $\hat{S}_3$ and $\hat{S}_1$ are coupled therefore, $\hat{S}_1 + \hat{S}_3 = \hat{S}_{13}$. Next, we can multiply $\hat{S}_2$ by $\hat{S}_{13}$.

$$\hat{H} = \frac{\epsilon_1}{\hbar^2}(\hat{S}_{13}\dot{\hat{S}}_2) + \frac{\epsilon_2}{\hbar^2}(\hat{S}_{1z} + \hat{S}_{2z} + \hat{S}_{3z})^2$$

$$\hat{H} = \frac{\epsilon_1}{\hbar^2}(2\hat{S}^2 - 2\hat{S}_{13} - 2\hat{S}_2) + \frac{\epsilon_2}{\hbar^2}\hat{S}_z^2$$

$$E_{s,s_{13},s_2,m} = \frac{\epsilon_1}{4}(2s(s+1) - 2s_{13}(s_{13}+1) - 2s_2(s_2+1)) + \epsilon_2 m^2$$

$$E_{\frac{1}{2},0,\frac{1}{2},\pm\frac{1}{2}} = \frac{\epsilon_2}{4}$$

$$E_{\frac{1}{2},1,\frac{1}{2},\pm\frac{1}{2}} = -\epsilon_1 + \frac{\epsilon_2}{4}$$

$$E_{\frac{3}{2},1,\frac{3}{2},\pm\frac{1}{2}} = \frac{\epsilon_1}{2} + \frac{\epsilon_2}{4}$$

$$E_{\frac{3}{2},1,\frac{3}{2},\pm\frac{3}{2}} = -\frac{\epsilon_1}{2} + \frac{9\epsilon_2}{4}$$