We begin by expressing the initial state in a basis of the spherical harmonics, which will allow us to apply the operators $\hat{L}^2$ and $\hat{L}_z$.

$$\langle \theta, \phi | \, \phi(0) \rangle = \sqrt{\frac{3}{4\pi}} \sin\theta \sin\phi = \frac{1}{2i}\sqrt{2}(Y_{1,-1} - Y_{1,1}) = \frac{i}{\sqrt{2}}(Y_{1,1} - Y_{1,-1})$$

Now we apply the Hamiltonian to $\langle \theta, \phi | \, \phi(0) \rangle$.

$$\langle \theta, \phi | \, \hat{H} \, |\phi(0)\rangle = (\frac{\hat{L}^2}{2I} + \omega_o \hat{L}_z) \, \langle \theta, \phi | \, \phi(0) \rangle = \frac{i}{\sqrt{2}}((\frac{\hbar^2}{2I} + \hbar\omega_o)Y_{11} - (\frac{\hbar^2}{2I} - \hbar\omega_o)Y_{1,-1})$$

Now we can use these results to find $\langle \theta, \phi | \, \phi(t) \rangle$.

$$\langle \theta, \phi | \, \phi(t) \rangle = \langle \theta, \phi | \, e^{-i\hat{H}t/\hbar} \, |\phi(0)\rangle = e^{-i(\frac{\hbar}{2I} + \omega_o)t} Y_{11} - e^{-i(\frac{\hbar}{2I} - \omega_o)t} Y_{1,-1}$$

Finally, we compute $\langle \hat{L}_x \rangle$.

$$\langle \hat{L}_x \rangle = \frac{1}{2} \langle \phi(t) | \, (\hat{L}_+ + \hat{L}_-) \, |\phi(t)\rangle = \frac{1}{2} \langle \phi(t) | \, ((...)Y_{10} + (...)Y_{10}) = 0$$

Maggie Regan
Physics 113
Boccio 9.7.7.

In this problem we are given the three operators on a 3-dimensional Hilbert space,

$$L_x = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$L_y = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix},$$

$$L_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

.

In part (a) we want to find the possible values that we can obtain if L_z is measured. Since L_z is already diagonalized, the possible values are the diagonal entries of $1, 0,$ and -1.

In part (b), a state of $L_z = 1$ is taken. Then, we are asked to find $\langle L_x \rangle, \langle L_x^2 \rangle$, and $\triangle L_x = \sqrt{\langle L_x^2 \rangle - \langle L_x \rangle^2}$. So,

$$|L_z = 1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

.

Also,

$$L_x^2 = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

Then, $\langle 1|L_x|1\rangle = 0$, $\langle 1|L_x^2|1\rangle = \frac{1}{2}$, and thus, $\triangle L_x = \sqrt{\frac{1}{2}}$.

For part (c), we want to find the normalized eigenstates and eigenvectors of L_x in the L_z basis. The eigenvalues of the L_x matrix are 1, 0, and -1. The corresponding normalized eigenvectors are

$$\vec{v_1} = \begin{pmatrix} \frac{1}{2} \\ -\sqrt{\frac{1}{2}} \\ \frac{1}{2} \end{pmatrix}$$

,

$$\vec{v_2} = \begin{pmatrix} -\sqrt{\frac{1}{2}} \\ 0 \\ \sqrt{\frac{1}{2}} \end{pmatrix}$$

, and

$$\vec{v_3} = \begin{pmatrix} \frac{1}{2} \\ \sqrt{\frac{1}{2}} \\ \frac{1}{2} \end{pmatrix}$$

.

These vectors and eigenvalues are already in the L_z basis, so no change of basis needs to be done.

For part (d) we are given that the particle is in the state with $L_z = -1$. Then, L_x is measured. We want to find the possible outcomes and their probabilities. To do this we find the projection operators for the eigenvalues of the L_x matrix. So, $|1\rangle\langle 1| = \frac{1}{4}, |0\rangle\langle | = \frac{1}{2}$, and $|-1\rangle\langle -1| = \frac{1}{4}$.

In part (e) we have the state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 1 \end{pmatrix}$$

,

which is in the L_z basis. We have that L_z^2 is measured and a result of 1 is obtained. So, for L_z^2, 1 is a degenerate eigenvalue, so the state will be a linear combination of two vectors. To find this linear combination we have that for $L_z|\psi\rangle$, prob(1) $= \frac{1}{3}$, prob(0) $= 0$, and prob(-1) $= \frac{2}{3}$. We get these values by renormalizing the vector. This renormalization factor is $\frac{3}{4}$ because from the original state, we have that prob(1) $= \frac{1}{4} + \frac{2}{4} = \frac{3}{4}$.

Thus, the state is $\frac{1}{3}\vec{v}_1 + \frac{2}{3}\vec{v}_3$, where

$$\vec{v_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\vec{v_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

.

Thus,

$$|\psi\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 0 \\ \sqrt{2} \end{pmatrix}$$

.

For part (f), we have that a particle is in the state where prob(1) = prob(-1) $= \frac{1}{4}$ and prob(0) $= \frac{1}{2}$. This means that the particle is in the state with the basis of L_z. Then, we know that the generalized state for this particle is $|\psi\rangle = \frac{1}{2}|L_z = 1\rangle + \frac{1}{\sqrt{2}}|L_z = 0\rangle + \frac{1}{2}|L_z = -1\rangle$. We know this because the probabilities listed are simply the squares of the coefficients of the basis states. However, we need to include phase factors for each term, because these phase factors will cancel when L_z is operatingo n the state. So, it is easy to see that the most general, normalized state with this property is $|\psi\rangle = \frac{e^{i\delta_1}}{2}|L_z = 1\rangle + \frac{e^{-i\delta_2}}{\sqrt{2}}|L_z = 0\rangle + \frac{e^{-i\delta_3}}{2}|L_z = -1\rangle$.

Then, we want to determine whether the phase factors are necessary. To see this, let's calculate $P(L_x = 0)$. So, $P(L_x = 0|\psi) = |\langle L_x = 0|\psi\rangle|^2 = |\frac{e^{-i\delta_1} - e^{-i\delta_3}}{2\sqrt{2}}|^2 = \frac{1}{4}(1 - \cos(\delta_1 - \delta_3))$. Therefore, the relative phase differences are not irrelevant. The arbitrary phase factors must be included when the state doesn't commute with $\hat{L}_z$.

Maggie Regan
Physics 113
Boccio 9.7.7.

In this problem the z-component of the spin of an electron is measured and found to be $\frac{\hbar}{2}$.

For part (a), we want to know that if a measurement is then made of the x-component of the spin, what the possible values could be. If one component of the spin is known, then the other components cannot be known. Also, there are only ever two values of spin that are possible. Therefore, we have the possible values of $\frac{\hbar}{2}$ and $\frac{-\hbar}{2}$.

Then, for part (b) we are asked to find the probabilities of finding these various results. So, we have that there are equal possibilities for the two values, and therefore the probabilities are both $\frac{1}{2}$. However, for a more mathematical method, we have that $\langle +x| = \frac{1}{\sqrt{2}}(1\ \ 1)$ and $\langle -x| = \frac{1}{\sqrt{2}}(1\ \ -1)$. Thus, the probability of measuring the up spin of the x-component is $||+z\rangle\langle +x||^2 = \frac{1}{2}||+z\rangle(1\ \ 1)|^2 = \frac{1}{2}$. Then, the probability of measuring the down spin of the x-component is $||+z\rangle\langle -x||^2 = \frac{1}{2}||+z\rangle(1\ \ -1)|^2 = \frac{1}{2}$.

In part (c), the axis defining the measured spin direction an angle θ with respect to the z-axis. We then want to find the probabilities of the various possible results. So, we have that

$$|+\hat{n}\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}$$

and

$$|-\hat{n}\rangle = \begin{pmatrix} \sin(\theta/2) \\ -e^{i\phi}\cos(\theta/2) \end{pmatrix}$$

.

Then, we have $|+z\rangle = \cos(\theta/2)|\hat{n}\rangle + \sin(\theta/2)|\hat{n}\rangle$. Using this, we have that $P(\frac{\hbar}{2}|z) = |\langle +n|z\rangle|^2 = \cos^2(\theta/2)$ and $P(\frac{-\hbar}{2}|z) = |\langle -n|z\rangle|^2 = \sin^2(\theta/2)$.

For part (d) we want to find the expectation value of the spin measurement of part (c). We can find this by multiplying the eigenvalues by their respective probabilities. So, $\langle\text{measurement}\rangle = \frac{\hbar}{2}\cos^2(\theta/2) + \left(-\frac{\hbar}{2}\right)\sin^2(\theta/2) = \frac{\hbar}{2}(\cos^2(\theta/2) - \sin^2(\theta/2)) = \frac{\hbar}{2}\cos(\theta)$.

a

We know that the probability of $|0,0\rangle$ is $\frac{3}{4}$ and the probability of $|1,-1\rangle$ is $\frac{1}{4}$. So, we can write this state as

$$|\phi\rangle = \frac{\sqrt{3}}{2}|0,0\rangle + \frac{e^{i\theta}}{2}|1,-1\rangle$$

where θ is an arbitary phase angle.

b

In order to determine the effect of a measurement of $\hat{L}_x$, we must switch to the basis of the eigenvectors of L_x. From problem 9.7.7, we know that in an $L = 1$ state, the matrix L_x is given by

$$L_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

which has eigenvalues

$$\lambda_1 = -1$$
$$\lambda_2 = 1$$
$$\lambda_3 = 0$$

and corresponding eigenvectors

$$\vec{v}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

$$\vec{v}_2 = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$$\vec{v}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

in the L_z basis. Let

$$\vec{w}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

This is the eigenvector associated with the eigenvalue of L_z of -1, which is the state we are in. So, we can write the state $|1, -1\rangle$ in the L_x basis as

$$|1, -1\rangle = |L_z = -1\rangle = \langle \vec{w}_1 | \, \vec{v}_1 \rangle \, \vec{v}_1 + \langle \vec{w}_1 | \, \vec{v}_2 \rangle \, \vec{v}_2 + \langle \vec{w}_1 | \, \vec{v}_3 \rangle \, \vec{v}_3 = \frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2 + \frac{1}{\sqrt{2}}\vec{v}_3$$

So,

$$prob(-1) = prob(\lambda_1) = prob(\vec{v}_1) = \frac{1}{4}$$
$$prob(1) = prob(\lambda_2) = prob(\vec{v}_2) = \frac{1}{4}$$
$$prob(0) = prob(\lambda_3) = prob(\vec{v}_3) = \frac{1}{2}$$

C

We can write $|1, -1\rangle$ in the L_x basis as spatial wavefunctions by simply converting to the spherical harmonics. So,

$$\langle \theta, \phi | \, (\frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2 + \frac{1}{\sqrt{2}}\vec{v}_3) = \frac{1}{2}Y_{1,-1} + \frac{1}{2}Y_{1,1} + \frac{1}{\sqrt{2}}Y_{1,0}$$

We are given

$$\Psi(\theta,\phi) = \sqrt{\tfrac{3}{8}}Y_{11}(\theta,\phi) + \sqrt{\tfrac{1}{8}}Y_{10}(\theta,\phi) + AY_{1-1}(\theta,\phi)$$

We can quickly see that $Y_{11}(\theta,\phi)$ is the Hilbert basis of $\Psi(\theta,\phi)$ therefore we can rewrite Ψ in terms of its ket vectors.

$$\Psi(\theta,\phi) = \sqrt{\tfrac{3}{8}}\,|1,1\rangle + \sqrt{\tfrac{1}{8}}\,|1,0\rangle + A\,|1,-1\rangle$$

(a) To normalize Ψ the square of the coefficients infront of the kets must sum to one.

$$\tfrac{3}{8} + \tfrac{1}{8} + A^2 = 1$$

$$A = \sqrt{\tfrac{1}{2}}$$

(b) Next, we want to find $\hat{L}_+\Psi(\theta,\psi)$. First, lets define the operater $\hat{L}_+$.

$$\hat{L}_+Y_{l,m}(\theta,\phi) = \hbar\sqrt{l(l+1) - m(m+1)}Y_{l,m+1}(\theta,\phi)$$

We can now apply this operator to $\Psi(\theta,\phi)$ defined above.

$$\hat{L}_+\Psi(\theta,\phi) = \sqrt{\tfrac{1}{8}}\hbar\sqrt{2}\,|1,1\rangle + \sqrt{\tfrac{1}{2}}\hbar\sqrt{2}\,|1,0\rangle$$

$$= \tfrac{1}{2}\hbar\,|1,1\rangle + \hbar\,|1,0\rangle$$

(c) Now, we want to solve for $\hat{L}_x\Psi(\theta,\phi)$. The operator $\hat{L}_x$ is defined as $\hat{L}_x = \frac{1}{2}(\hat{L}_+ + \hat{L}_-$. The operator $\hat{L}_-$ is,

$$\hat{L}_-Y_{l,m}(\theta,\phi) = \hbar\sqrt{l(l+1) - m(m-1)}Y_{l,m-1}(\theta,\phi)$$

Therefore,

$$\hat{L}_-\Psi(\theta,\phi) = \sqrt{\tfrac{3}{8}}\hbar\sqrt{2}\,|1,0\rangle + \sqrt{\tfrac{1}{8}}\hbar\sqrt{2}\,|1,-1\rangle$$

$$= \tfrac{\sqrt{3}}{2}\hbar\,|1,0\rangle + \tfrac{1}{2}\hbar\,|1,-1\rangle$$

So,

$$\hat{L}_x = \tfrac{1}{2}(\hat{L}_+ + \hat{L}_- = \tfrac{1}{2}(\tfrac{1}{2}\hbar\,|1,1\rangle + \hbar\,|1,0\rangle + \tfrac{\sqrt{3}}{2}\hbar\,|1,0\rangle + \tfrac{1}{2}\hbar\,|1,-1\rangle)$$

$$= \tfrac{1}{4}\hbar\,|1,1\rangle + \tfrac{\sqrt{3}+2}{4}\hbar\,|1,0\rangle + \tfrac{1}{4}\hbar\,|1,-1\rangle$$

The probability of finding the angular momentum in the x direction while in state Ψ is $\langle\Psi|\,\hat{L}_x\,|\Psi\rangle$.

$$\langle\Psi|\,\hat{L}_x\,|\Psi\rangle = \tfrac{\sqrt{3}+2}{4}\sqrt{\tfrac{1}{8}}\hbar + \tfrac{1}{4}\sqrt{\tfrac{3}{8}}\hbar + \tfrac{1}{4}\sqrt{\tfrac{1}{2}}\hbar$$

$$= \tfrac{\sqrt{3}+2}{4\sqrt{2}}\hbar$$

We can also solve for $\langle\Psi|\,\hat{L}^2\,|\Psi\rangle$.

$$\hat{L}^2 Y_{l,m} = \hbar^2 l(l+1) Y_{l,m}(\theta,\phi)$$

$$\hat{L}^2\Psi(\theta,\phi) = 2\hbar^2\sqrt{\tfrac{3}{8}}Y_{11} + 2\hbar^2\sqrt{\tfrac{1}{8}}Y_{10} + 2\hbar^2\sqrt{\tfrac{1}{2}}Y_{1-1}$$

$$\langle\Psi|\,\hat{L}^2\,|\Psi\rangle = 2\hbar^2\tfrac{3}{8} + 2\hbar^2\tfrac{1}{8} + 2\hbar^2\tfrac{1}{2}$$

(d) We can solve for the probability of having angular momentum in the z direction. Angular momentum in the z direction, L_z cooresponds to Y_{10}. Therefore, $|\,\langle Y10|\,\Psi(\theta,\phi)\rangle\,|^2 = \frac{1}{8}$.

(e) Now, if we are given $\Phi(\theta,\phi) = \sqrt{\frac{8}{15}}Y_{11} + \sqrt{\frac{4}{15}}Y_{10} + \sqrt{\frac{3}{15}}Y_{1-1}$. We also know that the operator $\hat{L}_z Y_{l,m} = m\hbar Y_{l,m}$.

So,

$$\langle\Phi|\,\hat{L}_z\,|\Psi\rangle = \hbar\sqrt{\tfrac{1}{5}}$$

We can also solve,

$$\langle\Phi|\,\hat{L}_-\,|\Psi\rangle = \hbar\sqrt{\tfrac{1}{5}}$$

Maggie Regan
Physics 113
Zettili 5.31.

In this problem there is a spin $\frac{3}{2}$ particle. We are given the Hamiltonian of $\hat{H} = \frac{\epsilon_0}{\hbar^2}(\hat{S}_x^2 - \hat{S}_y^2 - \hat{S}_z^2)$. From the text, we have that

$$\hat{S}_z = \frac{\hbar}{2}\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$\hat{S}_- = \hbar\begin{bmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

$$\hat{S}_+ = \hbar\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then, we can calculate $\hat{S}_x$, $\hat{S}_y$, $\hat{S}_x^2, \hat{S}_y^2$, and $\hat{S}_z^2$.

$$\hat{S}_x = \frac{\hbar}{2}\begin{bmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

$$\hat{S}_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{bmatrix}$$

$$\hat{S}_x^2 = \frac{\hbar^2}{4}\begin{bmatrix} 3 & 0 & 2\sqrt{3} & 0 \\ 0 & 7 & 0 & 2\sqrt{3} \\ 2\sqrt{3} & 0 & 7 & 0 \\ 0 & 2\sqrt{3} & 0 & 3 \end{bmatrix}$$

$$\hat{S}_y^2 = \frac{\hbar^2}{4}\begin{bmatrix} 3 & 0 & -2\sqrt{3} & 0 \\ 0 & 7 & 0 & -2\sqrt{3} \\ -2\sqrt{3} & 0 & 7 & 0 \\ 0 & -2\sqrt{3} & 0 & 3 \end{bmatrix}$$

$$\hat{S}_z^2 = \frac{\hbar}{2}\begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

Thus,

$$\hat{H} = \frac{\epsilon_0}{4}\begin{bmatrix} -9 & 0 & 4\sqrt{3} & 0 \\ 0 & 1 & 0 & 4\sqrt{3} \\ 4\sqrt{3} & 0 & 1 & 0 \\ 0 & 4\sqrt{3} & 0 & -9 \end{bmatrix}$$

Using mathematica, we are then able to find the eigenvalues, which correspond to the possible energy values of this Hamiltonian. These are $\frac{-13}{4}\epsilon_0$, $\frac{-13}{4}\epsilon_0$, $\frac{3}{4}\epsilon_0$, and $\frac{3}{4}\epsilon_0$. Therefore, the eigenvectors are

$$\vec{v_1} = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

,

$$\vec{v_2} = \frac{1}{2}\begin{pmatrix} 0 \\ -1 \\ 0 \\ \sqrt{3} \end{pmatrix}$$

,

$$\vec{v_3} = \frac{1}{2}\begin{pmatrix} -1 \\ 0 \\ -\sqrt{3} \\ 0 \end{pmatrix}$$

, and

$$\vec{v_4} = \frac{1}{2}\begin{pmatrix} 0 \\ -\sqrt{3} \\ 0 \\ -1 \end{pmatrix}$$

.

Given the hamiltonian

$$\hat{H} = \frac{\epsilon_0}{\hbar^2}(\hat{S}_x^2 + \hat{S}_y^2) + \frac{\epsilon}{\hbar}\hat{S}_z$$

A spin $\frac{5}{2}$ particle has possible magnetic numbers of $m = [-\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}]$. If we use the identity that $\hat{S}_x^2 + \hat{S}_y^2 = \hat{S}^2 - \hat{S}_z^2$, we can write

$$E_m = \left\langle \frac{5}{2}, m \right| \hat{h} \left| \frac{5}{2}, m \right\rangle = \frac{\epsilon_0}{\hbar^2}(\hbar^2(\frac{5}{2}(\frac{5}{2} + 1) - \hbar^2 m^2) + \frac{\epsilon}{\hbar}\hbar m$$
$$= \epsilon_0(\frac{35}{4} - m^2 + m)$$

The possible energy eigenvaules can be solve for by plugging in for the various m values.

m	$-\frac{5}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
E	0	$5\epsilon_0$	$8\epsilon_0$	$9\epsilon_0$	$8\epsilon_0$	$5\epsilon_0$