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Physics 113
Zettili 3-16

In this problem we are considering a system whose Hamiltonian H and an operator A are given by the matrices

$$H = \epsilon_0 \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 2i \\ 0 & -2i & 0 \end{pmatrix},$$

$$A = a_0 \begin{pmatrix} 0 & -i & 0 \\ i & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

For part (a) we are want to see what values we would obtain if we measured the energy of the system. To do this, we take find the eigenvalues of the matrix H. These are $\lambda_1 = 0, \lambda_2 = \epsilon_0\sqrt{5}$, and $\lambda_3 = -\epsilon_0\sqrt{5}$. The eigenvectors corresponding to these possible energy values are

$$\vec{v_1} = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix},$$

$$\vec{v_2} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ i\sqrt{5} \\ 2 \end{pmatrix}, \text{and}$$

$$\vec{v_3} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ -i\sqrt{5} \\ 2 \end{pmatrix}.$$

Then, in part (b) we measure the energy and get a value of $\epsilon_0\sqrt{5}$. Immediately after this measurement, we measure A. We want to find the values we could obtain for A and the probabilities corresponding to each value. So, we first find the eigenvalues of the matrix A. These are found to be $a_1 = 0, a_2 = 2a_0$, and $a_3 = -a_0$.

Now, if we measure the energy first to be $epsilon_0\sqrt{5}$, then we are left in the state of $\vec{v_2}$ found in part (a). Therefore, the probabilities are calculated in the state of $\vec{v_2}$. Then,

$$\begin{aligned} P(a_1) &= |\langle a_1|v_2\rangle|^2 \\ &= |\frac{1}{\sqrt{10}}(\frac{1}{\sqrt{2}})(1 \;\; 0 \;\; i) \cdot \vec{v_2}|^2 \\ &= \frac{1}{20}|1-2i|^2 \\ &= \frac{1}{4} \end{aligned}$$

$$
\begin{aligned}
P(a_2) &= |\langle a_2|v_2\rangle|^2 \\
&= |\frac{1}{\sqrt{10}}(\frac{1}{\sqrt{6}})(-i \;\; 2 \;\; 1)\cdot \vec{v_2}|^2 \\
&= \frac{1}{60}|i(1+2\sqrt{5})+2|^2 \\
&= \frac{6-\sqrt{5}}{15}
\end{aligned}
$$

$$
\begin{aligned}
P(a_3) &= |\langle a_3|v_2\rangle|^2 \\
&= |\frac{1}{\sqrt{10}}(\frac{1}{\sqrt{3}})(i \;\; 1 \;\; -1)\cdot \vec{v_2}|^2 \\
&= \frac{1}{30}|i(1+\sqrt{5})-2|^2 \\
&= \frac{5}{+}\sqrt{5}15
\end{aligned}
$$

Then, for part (c) we have that $\langle A\rangle = P_1E_1 + P_2E_2 + P_3E_3 = 0 + 2a_0(\frac{6}{15} - \frac{\sqrt{5}}{15}) + (-a_0)(\frac{5}{15} + \frac{\sqrt{5}}{15}) = -\frac{a_0}{15}(3\sqrt{5}+7)$.

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Physics 113
Zettili 3-17

For this problem we are considering a physical system whose Hamiltonian and initial state are given by

$$H = \epsilon_0 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$|\psi_0\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},$$

where ϵ_0 has the dimensions of energy. To get the values that will be obtained when measuring the energy, we find the eigenvalues of the matrix H. These are $\lambda_1 = 0, \lambda_2 = 2\epsilon_0$, and $\lambda_3 = -\epsilon_0$. The eigenvalues give respective eigenvectors of

$$\vec{v_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

$$\vec{v_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \text{and}$$

$$\vec{v_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then, we can write the initial state in terms of these vectors. So,

$$|\psi_0\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},$$

$$= \tfrac{1}{\sqrt{3}}|v_1\rangle + \tfrac{2}{\sqrt{6}}|v_3\rangle.$$

The probabilities for measuring each energy value is then the square of each coefficient in the new representation of the initial state. Therefore, $P(E_1) = \frac{1}{3}, P(E_2) = 0$, and $P(E_3) = \frac{2}{3}$.

For part (b) we are asked to calculate the expectation value of the Hamiltonian, $\langle \hat{H} \rangle$. Since $\langle \psi_0 | \psi_0 \rangle = 1$, the expectation value is just $\langle \hat{H} \rangle = P_1 E_1 + P_2 E_2 + P_3 E_3$. Thus, $\langle \hat{H} \rangle = \frac{1}{3} \cdot 0 + 0 \cdot (2\epsilon_0) + \frac{2}{3} \cdot (-\epsilon_0) = -\frac{2}{3}\epsilon_0$.

We are asked to repeat 3-17-b using the density operator method. We want to find the expectation value $\langle \hat{H} \rangle$ using the trace of the density matrix times the operator $\hat{H}$. From before, we know that we are in the state

$$|\phi_o\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \tag{1}$$

and the operator matrix

$$H = \epsilon_o \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{2}$$

We begin by computing the density operator from the state $|\phi_o\rangle$

$$W = |\phi_o\rangle \langle\phi_o| = \frac{1}{6} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix} \tag{3}$$

Now we multiply the matrices

$$WH = \frac{\epsilon_o}{6} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \frac{\epsilon_o}{6} \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & -4 \end{pmatrix} \tag{4}$$

Therefore the trace is

$$Tr(WH) = \frac{-2}{3} \epsilon_o \tag{5}$$

This means that

$$\langle \hat{H} \rangle = Tr(WH) = \frac{-2}{3} \epsilon_o \tag{6}$$

which agrees with the previous attempt.

Given two observables,

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(a) To find the probability of obtaining the $A = \sqrt{2}$ and $B = -1$ we first find the eigenvalues and eigenvectors cooresponding to $\sqrt{2}$ and -1 for A and B respectively.

Eigenvalues of A: $\sqrt{2}, \sqrt{2}, 0$

$$\left| A = \sqrt{2} \right\rangle = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \left| A = \sqrt{2} \right\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

Eigenvalues of B: $-1, 1, 1$

$$|B = -1\rangle = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \tag{2}$$

(b) Next, we can measure the probability of B then A.

$$|\langle B = -1| \left| A = \sqrt{2} \right\rangle|^2 = 0 \tag{3}$$

And we can measure the probability of A then B.

$$|\left\langle A = \sqrt{2} \right| |B = -1\rangle|^2 = 0 \tag{4}$$

(c) The probability of A then B and B then A are both zero. Therefore, A and B commute.

Measuring Energies: Measuring Two Observables

Consider a system whose state $|\psi(t)\rangle$ and two observables A and B are given by

$$|\psi(t)\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix}-i\\2\\0\end{pmatrix}, \qquad A = \begin{pmatrix}1 & i & 1\\-i & 0 & 0\\1 & 0 & 0\end{pmatrix}, \qquad B = \begin{pmatrix}3 & 0 & 0\\0 & 1 & i\\0 & -1 & 0\end{pmatrix}.$$

(a) Are A and B compatible? Which among the sets of operators $\{\hat{A}\}$, $\{\hat{B}\}$, and $\{\hat{A}, \hat{B}\}$ form a complete set of commuting operators?

(b) Measuring A first and then B immediately afterwards, find the probability of obtaining a value of -1 for A and a value of 3 for B.

(c) Now, measuring B and then A immediately afterwards, find the probability of obtaining 3 for B and -1 for A. Compare this result with the probability obtained in (b).

Solution

A and B are compatible if their matrices commute. Since

$$\begin{aligned} AB - BA &= \begin{pmatrix}1 & i & 1\\-i & 0 & 0\\1 & 0 & 0\end{pmatrix}\begin{pmatrix}3 & 0 & 0\\0 & a & i\\0 & -1 & 0\end{pmatrix} - \begin{pmatrix}3 & 0 & 0\\0 & a & i\\0 & -1 & 0\end{pmatrix}\begin{pmatrix}1 & i & 1\\-i & 0 & 0\\1 & 0 & 0\end{pmatrix} \qquad (1)\\ &= \begin{pmatrix}0 & -3i & -4\\-3i & 0 & 0\\4 & 0 & 0\end{pmatrix}\\ &\neq 0, \end{aligned}$$

and so A and B are not compatible. The eigenvalues of A are

$$\lambda_a = -1, 0, 1 \qquad (2)$$

and so its eigenvectors form a nondegenerate basis. Similaraly, the eigenvalues for B are

$$\lambda_b = 3, \frac{1}{2}(1 \pm \sqrt{5}) \qquad (3)$$

and so it's eigenvectors are nondegenerate as well. Thus $\{A\}$ and $\{B\}$ form complete sets of commutating operators. However, $\{A, B\}$ does not form a CSCO since A and B do not commute.

Solution.b

The eigenvector of A corresponding to the eigenvalue -1

$$|a_{-1}\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ -i \\ 1 \end{pmatrix} \tag{4}$$

And the eigenvector of B corresponding to the eigenvalue 3 is

$$|b_3\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{5}$$

The probability of first measuring -1 for A is

$$P(A = -1|\psi) = |\langle a_{-1}|\psi\rangle|^2 = \left|\frac{1}{\sqrt{5}}\frac{1}{\sqrt{3}}(-i - 2i)\right|^2 = \frac{3}{5}. \tag{6}$$

The probability of then measuring 3 for B is.

$$P(B = 3|A = -1) = |\langle b_3|a_{-1}\rangle|^2 = \frac{1}{3} \tag{7}$$

Thus the probability of both events occurring in sequence is

$$P(A = -1 \text{ then } B = 3) = P(A = -1)P(B = 3|A = -1) = \frac{1}{5} \tag{8}$$

Solution.c

Now suppose we measure B first. The The probability of first measuring 3 for B is

$$P(B = 3|\psi) = |\langle b_3|\psi\rangle|^2 = \left|\frac{1}{\sqrt{5}}(-i)\right|^2 = \frac{1}{5}. \tag{9}$$

The probability of then measuring -1 for A is.

$$P(A=-1|B=3) = |\langle a_{-1}|b_3\rangle|^2 = \frac{1}{3} \tag{10}$$

And finally, the probability of both events occurring in sequence is

$$P(B=3 \text{ then } A=-1) = P(B=3)P(B=3|A=-1) = \frac{1}{15} \tag{11}$$

As expected, these probabilities are not equal since A and B don't commute.

We are given that the fixed mean is \$2.50 and that the price of tofu is \$8.00. Let q_i be the price of item i. Then we can define the following functions

$$f(p_i) = \sum_{i=1}^{4} q_i p_i - 2.5 = 0 \tag{1}$$

$$g(p_i) = \sum_{i=1}^{4} p_i - 1 = 0 \tag{2}$$

The first constraint equation f gives that the fixed mean is \$2.50 and the second constraint equation g gives that the total probability is 1. Now we can write

$$\frac{\partial S}{\partial p_i} + \lambda_f \frac{\partial f}{\partial p_i} + \lambda_g \frac{\partial g}{\partial p_i} = 0 \tag{3}$$

Taking the derivatives, like in the previous problem, gives

$$-(log(p_i) + 1) + \lambda_f q_i + \lambda_g = 0 = -(log(p_{i+1}) + 1) + \lambda_f q_{i+1} + \lambda_g \tag{4}$$

So we can cancel and rearrange to get

$$log(\frac{p_{i+1}}{p_i}) = \lambda_f (q_{i+1} - q_i) \tag{5}$$

so

$$\frac{log(\frac{p_{i+1}}{p_i})}{q_{i+1} - q_i} = \lambda_f \tag{6}$$

where λ_f is constant. Now we let

$$\alpha = \frac{p_2}{p_1} = \frac{p_3}{p_2} \tag{7}$$

$$\beta = \frac{p_4}{p_3} \tag{8}$$

This implies that

$$log(\alpha) = \frac{log(\beta)}{5} \tag{9}$$

so

$$\beta = \alpha^5 \tag{10}$$

Now we can compute

$$\frac{\sum_{i=1}^{4} q_i p_i}{\sum_{i=1}^{4} p_i} = 2.5 = \frac{p_1}{p_1} \frac{1 + 2\alpha + 3\alpha^2 + 8\alpha^7}{1 + \alpha + \alpha^2 + \alpha^7} \tag{11}$$

so

$$5.5\alpha^7 + 0.5\alpha^2 - 0.5\alpha - 1.5 = 0 \tag{12}$$

Solving numerically gives

$$\alpha = 0.836 \tag{13}$$

So

$$p_1 = \frac{1}{1 + \alpha + \alpha^2 + \alpha^7} = 0.355 = P(B) \tag{14}$$

and

$$p_2 = \alpha p_1 = 0.296 = P(C) \tag{15}$$

$$p_3 = \alpha p_2 = 0.248 = P(F) \tag{16}$$

$$p_4 = \alpha^5 p_3 = 0.101 = P(T) \tag{17}$$

(a) The expectation value is equal to the price of the food times the probability of the food being ordered, $1.75 = P(B) + 2P(C) + 3P(F)$.

(b) The sum of the probabilities of ordering the food will sum to one, $P(B)+P(C)+P(F) = 1$. To find the ranges of each of the probabilities I first set $P(F) = 0$. In this case, $P(C)$ will be at a maximum and $P(B)$ will be at its minimum value.

$$1.75 = P(B) + 2P(C) \tag{1}$$

$$1 = P(B) + P(C) \tag{2}$$

$$1.75 = 1 - P(C) + 2P(C) \tag{3}$$

$$P(C) = 0.75 \tag{4}$$

$$P(B) = 0.25 \tag{5}$$

To find the range of the probability of $P(F)$, I set $P(C) = 0$. In this case, both $P(B)$ and $P(F)$ will be at a maximum value.

$$1.75 = P(B) + 3P(F) \tag{6}$$

$$1 = P(B) + P(F) \tag{7}$$

$$1.75 = 1 - P(F) + 3P(F) \tag{8}$$

$$P(C) = 0.375 \tag{9}$$

$$P(B) = 0.625 \tag{10}$$

Therefore, we have

$$0.25 \leq P(B) \leq 0.625$$
$$0 \leq P(C) \leq 0.75$$
$$0 \leq P(F) \leq 0.375$$

(c) We can now solve for $P(C)$ and $P(B)$ in terms of $P(F)$. We get...

$$P(C) = 0.75 - 2P(F) \tag{11}$$

$$P(B) = 0.25 + P(F) \tag{12}$$

To solve for the value $P(B)$, $P(C)$, and $P(F)$ which maximize the entropy we use the equation,

$$S = \sum_i P(A_i) log_2 \frac{1}{P(A_i)} \tag{13}$$

$$S = (0.25+P(F))log_2(\frac{1}{0.25+P(F)})+(0.75+2P(F))log_2(\frac{1}{0.75-2P(F)})+P(F)log_2(P(F)) \tag{14}$$

To solve for $P(F)$ which maximizes the entropy, we take the derivative of S and then set it equal to zero. The result is $P(F) = 0.216$, $P(B) = 0.466$, and $P(C) = 0.318$.

(d) To solve for the expectation value of the calories and cold food, we just multiply the number of calories or the probability of the food being cold by the probability of the food being ordered.

$$\langle calories \rangle = (0.466)(1000) + (0.318)(600) + (0.216)(400) = 743 \tag{15}$$

$$\langle cold \rangle = (0.466)(0.5) + (0.318)(0.2) + (0.216)(0.1) = 0.318 \tag{16}$$

Pure and Nonpure States

Consider an observable σ that can only take on two values $+1$ and -1. The eigenvectors of the corresponding operator are denoted by $|+\rangle$ and $|-\rangle$. Now consider the following states.

(a) The one-parameter family of pure states that are represented by the vectors

$$|\theta\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{e^{i\theta}}{\sqrt{2}}|-\rangle$$

(b) The nonpure state

$$\rho = \frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-|$$

Show that $\langle\sigma\rangle = 0$ for both of these states. What, if any, are the the physical differences between these various states, and how could they be measured?

Solution

The expectation value for $|\theta\rangle$ is

$$\begin{aligned}
\langle\theta|\hat{\sigma}|\theta\rangle &= \left(\frac{1}{\sqrt{2}}\langle+| + \frac{e^{i\theta}}{\sqrt{2}}\langle-|\right)\hat{\sigma}\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{e^{i\theta}}{\sqrt{2}}|-\rangle\right) \\
&= \frac{1}{2}\langle+|+\rangle + \frac{1}{2}\langle-|-\rangle \\
&= 0,
\end{aligned} \tag{1}$$

where I have used the fact that $|+\rangle$ and $|-\rangle$ are orthonormal. Meanwhile, for ρ we have

$$\begin{aligned}
\langle\hat{\sigma}\rangle &= \mathrm{Tr}(\hat{\rho}\hat{\sigma}) \\
&= \mathrm{Tr}\left(\frac{1}{2}|+\rangle\langle+| + \frac{1}{2}|-\rangle\langle-|\right)\left(|+\rangle\langle+| - |-\rangle\langle-|\right) \\
&= \mathrm{Tr}\left(\frac{1}{2}|+\rangle\langle+| - \frac{1}{2}|-\rangle\langle-|\right) \\
&= \frac{1}{2}\langle+|+\rangle\langle+|+\rangle - \frac{1}{2}\langle-|-\rangle\langle-|-\rangle \\
&= 0
\end{aligned} \tag{2}$$

Although both expectation values of σ are zero, these states are physically different. We can measure their physical differences by measuring another observable in these states.

Given the operator

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

I can calculate the probability, $Prob(M = 0|\rho)$ for the state operators,

$$\rho = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \quad \rho = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \quad \rho = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

First, we begin by finding the eigenvector which cooresponds to the eigenvalue where $\lambda = 0$.

$$\mathbf{v} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \tag{1}$$

Therefore, the $P(M = 0)$ is

$$P(M = 0) = \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \tag{2}$$

To find the probability of $M = 0$ given ρ is $Tr(P(M = 0)\rho)$.

(a)

$$\frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{4} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{4} \end{bmatrix} \tag{3}$$

The trace of this matrix is $\frac{3}{8}$.

(b)

$$\frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4}$$

The trace of this matrix is 0.

(c)

$$\frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \tag{5}$$

The trace of this matrix is $\frac{1}{2}$.

To determine whether the given matrices are valid density operators, we check if the trace equals 1 and the determinant is greater than or equal to zero. Then, to see if it represents a pure state we look to see if the square trace is equal to one as well.

$Tr(p_1) = 1$ but $det(p_1) = \frac{-3}{2}$ so p_1 is not a density operator.

$Tr(p_2) = 1$, $det(p_2) = 0$, and $Tr(p_2^2) = 1$ so p_2 is a pure state density operator. By trial and error it is easy to find that

$$p_2 = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix} \frac{1}{5}\begin{pmatrix} 3 & 4 \end{pmatrix} \tag{1}$$

$Tr(p_3) = 1$ but $det(p_3) = -2$ so p_3 is not a density operator.

$Tr(p_4) = 1$, $det(p_4) = \frac{1}{4}$, and $Tr(p_4^2) = \frac{7}{16}$ so p_4 is a density operator and a mixed state.

$Tr(p_5) = 1$, $det(p_5) = 0$, and $Tr(p_5^2) = 1$ so p_5 is a pure state density operator. From trial and error we can find that

$$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{2} \end{pmatrix} \tag{2}$$

More Density Matrices

Suppose we have a system with total angular momentum 1. Pick a basis corresponding to the three eigenvectors of the z-component of the angular momentum, J_z, with eigenvalues $+1, 0, -1$, respectively. We are given an ensemble of such systems described by the density matrix

$$\rho = \frac{1}{4}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(a) Is ρ a permissible density matrix? Give your reasoning. For the remainder of this problem, assume that it is permissible. Does it describe a pure or mixed state? Give your reasoning.

(b) Given the ensemble described by ρ, what is the average value of J_z?

(c) What is the spread (standard deviation) in the measured values of J_z?

Solution

(a) ρ is a permissible density matrix since its trace is 1, and its eigenvalues, 0, 1/4, and 3/4, fit the requirements $\sum \lambda_k = 1$ and $0 \geq \lambda \geq 1$. A pure state has eigenvalues only equal to 0 or 1, so ρ is a mixed state.

(b) Consider the basis vectors $|+\rangle$, $|0\rangle$ and $|-\rangle$ such that

$$\hat{J}_z|+\rangle = |+\rangle \qquad \hat{J}_z|0\rangle = 0 \qquad \hat{J}_z|-\rangle = -|-\rangle. \tag{1}$$

We can write the matrix form of $\hat{J}_z$ in this basis

$$\hat{J}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2}$$

The expectation value of $\hat{J}_z$ is then

$$\mathrm{Tr}(\hat{\rho}\hat{J}_z) = \frac{1}{4}\mathrm{Tr}\begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \frac{1}{4} \tag{3}$$

(c) The expectation value of J_z^2 is

$$\mathrm{Tr}(\hat{\rho}\hat{J}_z^2) = \frac{1}{4}\mathrm{Tr}\begin{pmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \frac{3}{4} \tag{4}$$

The standard deviation in the measurement of J_z is therefore

$$\delta = \langle J_z^2 \rangle + \langle J_z \rangle^2 = \frac{\sqrt{11}}{16}. \tag{5}$$

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For part (a) of this problem we have that $\hat{H}$ is a Hermitian operator. We want to show that $U = e^{iH}$ is a unitary operator.

So, if $U = e^{iH}$, then $U^t = e^{-iH^t} = e^{-iH}$. Thus, $U^tU = e^{-iH} \cdot U^{iH} = 1$. Therefore, U is a unitary operator.

Then, in part (b) we want to show that $det(U) = e^{itr(H)}$. So, first we choose a basis where H is diagonal. So,

$$H = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \lambda_{n-1} & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix},$$

. Therefore,

$$U = \begin{bmatrix} e^{i\lambda_1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & e^{i\lambda_{n-1}} & 0 \\ 0 & \cdots & 0 & e^{i\lambda_n} \end{bmatrix},$$

.

This means that $det(U) = e^{i\lambda_1}e^{i\lambda_2}\cdots e^{i\lambda_n} = e^{i(\lambda_1+\cdots+\lambda_n)} = e^{itr(H)}$.