We are given the following kets, where $|\phi_1\rangle$, $|\phi_2\rangle$, and $|\phi_3\rangle$ are orthonormal.

$$|\psi\rangle = i\,|\phi_1\rangle + 3i\,|\phi_2\rangle - |\phi_3\rangle \tag{1}$$

$$|\chi\rangle = |\phi_1\rangle - i\,|\phi_2\rangle + 5i\,|\phi_3\rangle \tag{2}$$

These can be rewritten in matrix form as column vectors as follows

$$|\psi\rangle = \begin{pmatrix} i \\ 3i \\ -1 \end{pmatrix} \tag{3}$$

$$|\chi\rangle = \begin{pmatrix} 1 \\ -i \\ 5i \end{pmatrix} \tag{4}$$

We find the bras by taking the conjugate transpose of the kets, so

$$\langle\psi| = \begin{pmatrix} -i & -3i & -1 \end{pmatrix} \tag{5}$$

$$\langle\chi| = \begin{pmatrix} 1 & i & -5i \end{pmatrix} \tag{6}$$

# a

We now perform the indicated computations

$$\langle\psi|\,\psi\rangle = \begin{pmatrix} -i & -3i & -1 \end{pmatrix} \begin{pmatrix} i \\ 3i \\ -1 \end{pmatrix} = 1 + 9 + 1 = 11 \tag{7}$$

$$\langle\chi|\,\chi\rangle = \begin{pmatrix} 1 & i & -5i \end{pmatrix} \begin{pmatrix} 1 \\ -i \\ 5i \end{pmatrix} = 1 + 1 + 25 = 27 \tag{8}$$

$$\langle\psi|\,\chi\rangle = \begin{pmatrix} -i & -3i & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -i \\ 5i \end{pmatrix} = -i - 3 - 5i = -3 - 6i \tag{9}$$

$$\langle\chi|\,\psi\rangle = \begin{pmatrix}1 & i & -5i\end{pmatrix}\begin{pmatrix}i\\3i\\-1\end{pmatrix} = i - 3 + 5i = -3 + 6i \tag{10}$$

Now we can use these results to compute

$$\langle\psi+\chi|\,\psi+\chi\rangle = \langle\psi|\,\psi\rangle + \langle\chi|\,\chi\rangle + \langle\psi|\,\chi\rangle + \langle\chi|\,\psi\rangle = 11 + 27 - 3 - 6i - 3 + 6i = 32 \tag{11}$$

We can see that

$$\langle\psi|\,\chi\rangle \neq \langle\chi|\,\psi\rangle \tag{12}$$

and instead

$$\langle\psi|\,\chi\rangle = \langle\chi|\,\psi\rangle * \tag{13}$$

# b

Now we compute

$$|\psi\rangle\,\langle\chi| = \begin{pmatrix}i\\3i\\-1\end{pmatrix}\begin{pmatrix}1 & i & -5i\end{pmatrix} = \begin{pmatrix}i & -1 & 5\\3i & -3 & 15\\-1 & -i & 5i\end{pmatrix} \tag{14}$$

which has a trace of $-3+6i$. Next we compute

$$|\chi\rangle\,\langle\psi| = \begin{pmatrix}1\\-i\\5i\end{pmatrix}\begin{pmatrix}-i & -3i & -1\end{pmatrix} = \begin{pmatrix}-i & -3i & -1\\-1 & -3 & i\\5 & 15 & -5i\end{pmatrix} \tag{15}$$

which has a trace of $-3-6i$, so the traces are conjugates.

## c

We perform additional computations

$$|\psi\rangle^\dagger = \begin{pmatrix} -i & -3i & -1 \end{pmatrix} = \langle\psi| \tag{16}$$

$$|\chi\rangle^\dagger = \begin{pmatrix} 1 & i & -5i \end{pmatrix} = \langle\chi| \tag{17}$$

$$(|\psi\rangle\langle\chi|)^\dagger = \begin{pmatrix} -i & -3i & -1 \\ -1 & -3 & i \\ 5 & 15 & -5i \end{pmatrix} = |\chi\rangle\langle\psi| \tag{18}$$

$$(|\chi\rangle\langle\psi|)^\dagger = \begin{pmatrix} i & -1 & 5 \\ 3i & -3 & 15 \\ -1 & -i & 5i \end{pmatrix} = |\psi\rangle\langle\chi| \tag{19}$$

We want to prove that the $det(exp(A)) = exp(Tr(A))$. Matrix $A$ is diagonalizable, therefore, $A = P\lambda P^{-1}$. In that case,

$$det(exp(A)) = det(Pexp(\lambda)P^{-1}) \tag{1}$$

The determinate of products equals the product of the determinates. The determinate of $P$ and $P^{-1}$ is one so,

$$det(exp(A)) = det(exp(\lambda)) \tag{2}$$

We can also show that the determinate of an exponent of a diagonal matrix is equal to the exponent of the trace of the same matrix.

$$det(exp(\lambda)) = exp(Tr(\lambda)) \tag{3}$$

The trace of the eigenvalue matrix is equal to the trace of the original matrix. So we see that

$$det(exp(A)) = exp(Tr(A)) \tag{4}$$

Determine the eigenvalues and eigenstates of the following matrix

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

Using Gram-Schmidt, construct an orthonormal basis set from the eigenvectors of this operator. To determine the eigenvalues I first solve for $\lambda$ in $det(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0}$.

$$det \begin{pmatrix} 2-\lambda & 2 & 0 \\ 1 & 2-\lambda & 1 \\ 1 & 2 & 1-\lambda \end{pmatrix} = (2-\lambda)((2-\lambda)(1-\lambda)-2) - 2((1-\lambda)-1) = 0 \tag{1}$$

$$det(A) = \lambda((2-\lambda)(\lambda-3)+2) = 0 \tag{2}$$

$$\lambda = 0 \ and \ \lambda^2 - 5\lambda + 4 = 0 \tag{3}$$

$$\lambda = 4 \ and \ 1 \tag{4}$$

There are three eigenvalues, $\lambda_1 = 0, \lambda_2 = 4, and \lambda_3 = 1$. We can find the associated eigenstates.

$$\mathbf{A} - \lambda_\mathbf{1}\mathbf{I} = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \tag{5}$$

This matrix row reduces to,

$$\mathbf{A} - \lambda_\mathbf{1}\mathbf{I} = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{6}$$

So we obtain the equations $2x + 2y = 0$ and $x + 2y + z = 0$, therefore, we have an eigenstate

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \tag{7}$$

$$\mathbf{A} - \lambda_2 \mathbf{I} = \begin{pmatrix} -2 & 2 & 0 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{pmatrix} \tag{8}$$

This matrix row reduces to

$$\mathbf{A} - \lambda_2 \mathbf{I} = \begin{pmatrix} -2 & 2 & 0 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{9}$$

So we obtain the equations $-2x + 2y = 0$ and $x - 2y + z = 0$, therefore, we have an eigenstate

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{10}$$

$$\mathbf{A} - \lambda_3 \mathbf{I} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \tag{11}$$

This matrix row reduces to

$$\mathbf{A} - \lambda_3 \mathbf{I} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{12}$$

So we obtain the equations $x + 2y = 0$ and $x + y + z = 0$, therefore, we have an eigenstate

$$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \tag{13}$$

The normalized eigenstates are

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \tag{14}$$

To find the construct an orthonormal basis set I first want to generate an orthonormal set by projecting the nonorthoganal eigenstates onto each other and substracting the similar components. The second and third eigenstates are already orthogonal, so I only need to change with the first eigenstate.

$$\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix} = \frac{1}{3} \tag{15}$$

The overlapping amount, $\frac{1}{3}$ by the direction vector of the second eigenstate.

$$\frac{1}{3}\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} \tag{16}$$

I use the same procedure to find the component of the first eigenstate on the third eigenstate.

$$\frac{1}{\sqrt{6}}\begin{bmatrix}-2\\1\\1\end{bmatrix} - \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix} = \frac{-2}{3\sqrt{3}} \tag{17}$$

The overlapping amount, $\frac{-2}{3\sqrt{3}}$ by the direction vector of the third eigenstate.

$$\frac{-2}{3\sqrt{3}}\frac{1}{\sqrt{6}}\begin{bmatrix}-2\\1\\1\end{bmatrix} \tag{18}$$

I then subtract the two overlapping components from the first eigenstate.

$$\frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix} - \frac{1}{3}\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} + \frac{1}{3}\frac{1}{\sqrt{3}}\begin{bmatrix}-2\\1\\1\end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix}0\\-1\\1\end{bmatrix} \tag{19}$$

The normalized vector is

$$\frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix} \tag{20}$$

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**Boccio 4.22.7**

An operator is given to us as $\hat{H} = a(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|)$, for a two state system, where $a$ is a number. We want to find the eigenvalues and the corresponding eigenvectors.

First, we represent the operator in matrix form. So,

$$\hat{H} = \begin{pmatrix} a & a \\ a & -a \end{pmatrix}$$

.

Then, to find the eigenvalues we have:

$$\det(\hat{H}) = \begin{vmatrix} a-\lambda & a \\ a & -a-\lambda \end{vmatrix} = 0.$$

Completing the algebra we have:

$$(a-\lambda)(-a-\lambda) - a^2 = 0$$
$$\lambda^2 - 2a^2 = 0$$
$$\lambda_1 = a\sqrt{2}, \lambda_2 = -a\sqrt{2}.$$

These are our eigenvectors. Plugging these back into the matrix, we want to find the vectors for which $\hat{H} - \lambda\hat{I} = 0$, where $\hat{I}$ is the identity matrix and there are two lambda's as stated above.

From this, we are given vectors $\vec{v_1} = \binom{1}{-1+\sqrt{2}}$ and $\vec{v_2} = \binom{1}{-1-\sqrt{2}}$.

We are given the operator $\hat{Q}$ such that

$$\hat{Q}\,|q\rangle = q\,|q\rangle \tag{1}$$

Now we compute the eigenvalues for several related functions.

$$\hat{Q}^2\,|q\rangle = \hat{Q}(\hat{Q}\,|q\rangle) = \hat{Q}q\,|q\rangle = q(\hat{Q}\,|q\rangle) = q^2\,|q\rangle \tag{2}$$

Using this result and induction it is clear that

$$\hat{Q}^n\,|q\rangle = q^n\,|q\rangle \tag{3}$$

Next we compute

$$e^{\hat{Q}}\,|q\rangle = \sum_{n=1}^{\infty} \frac{\hat{Q}^n}{n!}\,|q\rangle = \sum_{n=1}^{\infty} \frac{q^n}{n!}\,|q\rangle = e^q\,|q\rangle \tag{4}$$

## B.4.22.5

If the states $\{|1\rangle\ |2\rangle, |3\rangle\}$ form an orthonormal basis and if the operator $\hat{K}$ has the properties

$$\hat{K}|1\rangle = 2|1\rangle$$
$$\hat{K}|2\rangle = 3|2\rangle$$
$$\hat{K}|3\rangle = -6|3\rangle$$

(a) Write an expression for $\hat{K}$ in terms of its eigenvalues and eigenvectors (projection operators). Use this expression to derive the matrix representing $\hat{K}$ in the $|1\rangle, |2\rangle, |3\rangle$ basis.

(b) What is the *expectation* or *average value* of $\hat{K}$, defined as $\langle\alpha|\hat{K}|\alpha\rangle$, in the state

$$\alpha = \tfrac{1}{\sqrt{83}}\big(-3|1\rangle + 5|2\rangle + 7|3\rangle\big)$$

## Solution.a

It is striaght forward to see that

$$\hat{K} = 2|1\rangle\langle 1| + 3|2\rangle\langle 2| - 6|3\rangle\langle 3| \tag{1}$$

satisfies the conditions stated above. In the $|1\rangle, |2\rangle, |3\rangle$ basis, the coefficients attached to the projection operators are the diagonal terms in the $\hat{K}$ matrix.

$$\hat{K} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \tag{2}$$

## Solution.b

Let's have $\hat{K}$ operating first on the ket $|\alpha\rangle$. We can write both $\hat{K}$ and $|\alpha\rangle$ in terms of the basis vectors.

$$\langle\alpha|\hat{K}|\alpha\rangle = \langle\alpha|\frac{1}{\sqrt{83}}\Big(2|1\rangle\langle 1| + 3|2\rangle\langle 2| - 6|3\rangle\langle 3|\Big)\Big(-3|1\rangle + 5|2\rangle + 7|3\rangle\Big) \tag{3}$$

Because the basis vectors are orthogonormal, we have the condition

$$\langle i|j\rangle = \delta_{ij}. \tag{4}$$

We can now easily multiply the two terms enclosed on the right side of equation 3 using the distributive property. Writing $\langle\alpha|$ in terms of the basis vectors, equation 3 becomes

$$\frac{1}{83}\Big(-3\langle 1| + 5\langle 2| + 7\langle 3|\Big)\Big(-6|1\rangle + 15|2\rangle - 42|3\rangle\Big) = \frac{-201}{83} \tag{5}$$

## B.4.22.11

Let

$$A = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

Calculate $\exp(\alpha A)$, $\alpha$ real.

## Solution

We can find the eigenvalues of $A$ by solving its characteristic polynomial

$$(-1-\lambda)(-1-\lambda) - 4 = (\lambda - 1)(\lambda + 3) = 0, \tag{6}$$

which gives the solutions

$$\lambda = 1, -3. \tag{7}$$

We can then find the corresponding eigenvectors

$$|1\rangle = \ker(A - \lambda_1 I) = \ker\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{8}$$

$$|-3\rangle = \ker(A - \lambda_{-3} I) = \ker\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{9}$$

Let D be a diagonal matrix with eigenvalues 1 and -3 along the diagonal. And let U be an orthogonal matrix whose columns are $|1\rangle$ and $|-3\rangle$ multiplied by a normalization constant.

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}, \qquad U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{10}$$

We now have

$$A = UDU^{-1} \tag{11}$$

Raising $A$ to the power of $n$ is now easy to calculate since

$$A^n = (UDU^{-1})^n = UDU^{-1}UDU^{-1}UDU^{-1}.... \tag{12}$$

$$= UD(U^{-1}U)D(U^{-1}U)D(U^{-1}U).... \tag{13}$$

$$= UD^nU^{-1}. \tag{14}$$

And so

$$\exp(\alpha A) = \sum_{n=1}^{\infty} \frac{(\alpha A)^n}{n!} = \sum_{n=1}^{\infty} \frac{(\alpha UDU^{-1})^n}{n!} \tag{15}$$

$$= U \left( \sum_{n=1}^{\infty} \frac{(\alpha D)^n}{n!} \right) U^{-1} = U \exp(\alpha D) U^{-1}. \tag{16}$$

Because $D$ is diagonal, we see that

$$\exp(\alpha D) = \begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-3\alpha} \end{pmatrix}. \tag{17}$$

We can now use straight forward matrix multiplication.

$$\exp \alpha A = U \exp(\alpha D) U^{-1} = \frac{1}{2} \begin{pmatrix} e^{\alpha} + e^{-3\alpha} & e^{\alpha} - e^{-3\alpha} \\ e^{\alpha} - e^{-3\alpha} & e^{\alpha} + e^{-3\alpha} \end{pmatrix} \tag{18}$$

## B.5.6.1.c

Which of these two events is more likely?

1. four rolls of a die yield at least one six

2. twenty-four rolls of two dice yield at least one double six

## Solution

Let $A$ be the event that a six is rolled and $R^n$ be the event that the dice is rolled $n$ times. The probability of not rolling a six in one roll is

$$P(\sim A|R^1) = 1 - P(A|R^1) = \frac{5}{6}. \tag{19}$$

The probability of not rolling a six in four rolls is

$$P(\sim A|R^4) = P(\sim A|R^1)^4 \approx 0.482. \tag{20}$$

Therefore, the probability of rolling a six in four rolls is

$$P(A|R^4) = 1 - P(\sim A|R^4) \approx 0.518. \tag{21}$$

## B.5.6.1.h

A box contains a double-headed coin, a double-tailed coin and a conventional coin. A coin is picked at random and flipped. It shows a head. What is the probability that it is the double-headed coin?

## Solution

Let $H$ be the event that the coin shows a head and $D$ be the event that the coin is double heads. From Bayes theorem, we have

$$P(D|H) = P(H|D)\frac{P(D)}{P(H)} = (1)\left(\frac{1/3}{1/2}\right) = \frac{2}{3}. \tag{22}$$

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**Physics 113**
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In this problem there are two dice. One in your left hand and one in your right hand. We are first asked to find the probability that the right-hand die shows a larger number than the left-hand die after they are both rolled. To do this, we want to add all of the probabilities that given a certain number for the right-hand die the left-hand die is larger multiplied by the probability of getting that the right-hand die to roll that number. This last probability is always the same. There is always a $\frac{1}{6}$ probability that a number will show up when the die is rolled. Then, the probability of the left-hand die showing a higher number depends on what number is rolled for the right-hand die.

If a 1 is rolled, then there is a $\frac{5}{6}$ probability that the left-hand die shows a higher number, because there are five options (out of six) for which the left-hand die can be greater than 1, all with an equal probability of being rolled ($\frac{1}{6}$). Therefore, if a 2 is rolled by the right-hand die then there is a $\frac{4}{6}$ probability that the left-hand die will show a larger number.

Continuing in this way we have the summation:

$$\begin{aligned} P &= \frac{1}{6}\left(\frac{5}{6}+\frac{4}{6}+\frac{3}{6}+\frac{2}{6}+\frac{1}{6}+0\right) \\ &= \frac{15}{36} \\ &= \frac{5}{12}. \end{aligned}$$

Thus, the probability that the left-hand die will show a number larger than that rolled by the right-hand die is $\frac{5}{12}$.

The next part of the question asks what the probability of the right-hand die showing a larger number is if you have already rolled a 5 with the left-hand die. This is simply found by realizing that the only way the right-hand die can show a larger number is if it shows a 6. Then, the probability of rolling a 6 with the right-hand die is as found before, $\frac{1}{6}$. Therefore, the answer to this part of the question is a probability of $\frac{1}{6}$.