Physics 113 Spring 2010

Quantum Theory Seminar #8

Readings:

Zettili - Chapter - 7.1-7.3
Boccio - Chapter - 9.4-9.5
Website - Hoyer Mathematica Notebook on Addition of Angular Momentum

Presentations:

Magnetic Resonance (Sections 9.4,9.4.1) _Max_

Addition of Angular Momentum (Sections 9.5,9.5.1,9.5.2) _Jen_

Zettili Problems:

1. **Z7-7(Ben) Matrix elements**
2. **Z7-15(Ed) Operator algebra**
3. **Z7-9 (EVERYONE) Eigenvalues and Eigenvectors (do exactly and take limits)**
4. **Z7-30(Elizabeth) Energy levels: 2 spin 1/2 particles**
5. **Z7-32(Max) More Energy levels: 3 spin 1/2 particles**

Boccio Problems:

1. Addition of Angular Momentum(Jen) - Two atoms with $J_1 = 1$ and $J_2 = 2$ are coupled, with an energy described by $\hat{H} = \varepsilon \hat{\vec{J}}_1 \cdot \hat{\vec{J}}_2$, $\varepsilon > 0$. Determine all of the energies and degeneracies for the coupled system

2. Spin = 1 systems(EVERYONE)

We now consider a **spin = 1** system.

(a) Use the spin = 1 states $|1,1\rangle$, $|1,0\rangle$ and $|1,-1\rangle$ (eigenstates of $\hat{S}_z$) as a basis to form the matrix representation (3x3) of the angular momentum operators

$$\hat{S}_x\ ,\hat{S}_y\ ,\hat{S}_z\ ,\hat{S}^2\ ,\hat{S}_+\ ,\hat{S}_-$$

(b) Determine the eigenstates of $\hat{S}_x$ in terms of the eigenstates $|1,1\rangle$, $|1,0\rangle$ and $|1,-1\rangle$ of $\hat{S}_z$.

(c) A spin = 1 particle is in the state

$$|\psi\rangle = \frac{1}{\sqrt{14}}\begin{pmatrix} 1 \\ 2 \\ 3i \end{pmatrix} \text{ in the } \hat{S}_z \text{ basis}$$

(1) What are the probabilities that a measurement of $\hat{S}_z$ will yield the values $\hbar$, 0, or $-\hbar$ for this state? What is $\langle \hat{S}_z \rangle$?

(2) What is $\langle \hat{S}_x \rangle$ in this state?

(3) What is the probability that a measurement of $\hat{S}_x$ will yield the value $\hbar$ for this state?

(d) A particle with spin = 1 has the Hamiltonian

$$\hat{H} = A\hat{S}_z + \frac{B}{\hbar}\hat{S}_x^2$$

(1) Calculate the energy levels of this system.

(2) If, at t = 0, the system is in an eigenstate of $\hat{S}_x$ with eigenvalue $\hbar$, calculate the expectation value of the spin $\langle \vec{S}_Z \rangle$ at time t.

3. Deuterium Atom(Ben) - Consider a deuterium atom (composed of a nucleus of spin 1 and a electron). The electronic angular momentum is $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$, where $\hat{\vec{L}}$ is the orbital angular momentum of the electron and $\hat{\vec{S}}$ is its spin. The total angular momentum of the atom is $\hat{\vec{F}} = \hat{\vec{J}} + \hat{\vec{I}}$ where $\hat{\vec{I}}$ is the nuclear spin. The eigenvalues of $\hat{J}^2$ and $\hat{F}^2$ are $J(J+1)\hbar^2$ and $F(F+1)\hbar^2$ respectively.

(a) What are the possible values of the quantum numbers J and F for the deuterium atom in the 1s(L=0) ground state?

(b) What are the possible values of the quantum numbers J and F for a deuterium atom in the 2p(L=1) excited state?

4. Spherical Harmonics(Rachael) - Consider a particle in a state described by

$$\psi = N(x + y + 2z)e^{-\alpha r}$$

where N is a normalization factor.

(a) Show, by rewriting the $Y_1^{\pm 1,0}$ functions in terms of x,y,z and r that

$$Y_1^{\pm 1} = \mp\left(\frac{3}{4\pi}\right)^{1/2}\frac{x \pm iy}{\sqrt{2}r}$$

$$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\frac{z}{r}$$

(b) Using this result, show that for a particle described by ψ above

$$P(L_z = 0) = 2/3\,, P(L_z = \hbar) = 1/6 \text{ and } P(L_z = -\hbar) = 1/6$$

5. Spin in Magnetic Field(Sandy/Max/Rachael) - Suppose that we have a spin-1/2 particle interacting with a magnetic field via the Hamiltonian

$$\hat{H} = \begin{cases} -\vec{\mu}\cdot\vec{B} & , \quad \vec{B} = B\hat{e}_z \qquad 0 \le t < T \\ -\vec{\mu}\cdot\vec{B} & , \quad \vec{B} = B\hat{e}_y \qquad T \le t < 2T \end{cases}$$

where $\vec{\mu} = \mu_B\vec{\sigma}$ and the system is in the initial(t = 0) state

$$|\psi(0)\rangle = |x+\rangle = \frac{1}{\sqrt{2}}\left(|z+\rangle + |z-\rangle\right)$$

Determine the probability that the state of the system at t = 2T is

$$|\psi(2T)\rangle = |x+\rangle$$

in three ways:

(1) Using the Schrodinger equation (solving differential equations)
(2) Using the time development operator (using operator algebra)
(3) Using the density operator formalism

6. What happens in the Stern-Gerlach box?(Sandy) - An atom with spin = 1/2 passes through a Stern-Gerlach apparatus adjusted so as to transmit atoms that have their spins in the +z direction. The atom spends time T in a magnetic field B in the x-direction.

(a) At the end of this time what is the probability that the atom would pass through a Stern-Gerlach selector for spins in the -z direction?
(b) Can this probability be made equal to one, if so, how?

7. Spin = 1 particle in a magnetic field [use Boccio #2](James) - A particle with intrinsic spin = 1 is placed in a uniform magnetic field $\vec{B} = B_0 \hat{e}_x$. The initial spin state is

$$|\psi(0)\rangle = |1,1\rangle$$

Take the spin Hamiltonian to be

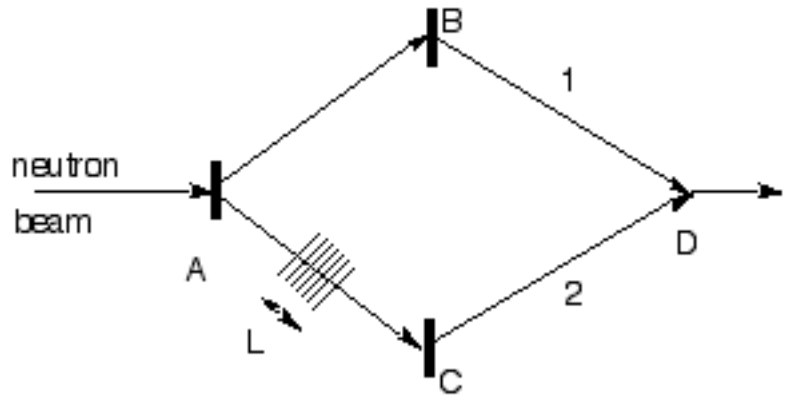
$$\hat{H} = \omega_0 \hat{S}_x$$

and determine the probability that the particle is in the state $|\psi(t)\rangle = |1,-1\rangle$ at time t.

8. Multiple magnetic fields(EVERYONE) - A spin-1/2 system with magnetic moment $\vec{\mu} = \mu_0 \vec{\sigma}$ is located in a uniform time-independent magnetic field B_0 in the positive z-direction. For the time interval 0 < t < T an additional uniform time-independent field B_1 is applied in the positive x-direction. During this interval, the system is again in a uniform constant magnetic field, but of different magnitude and direction z' from the initial one. At and before t = 0, the system is in the m = 1/2 state with respect to the z-axis.

(a) At t = 0+, what are the amplitudes for finding the system with spin projections m'= ±1/ 2 with respect to the z'-axis?
(b) What is the time development of the energy eigenstates with respect to the z' direction, during the time interval 0 < t < T?
(c) What is the probability at t = T of observing the system in the spin state m = -1/2 along the original z-axis? [Express answers in terms of the angle θ between the z and z' axes and the frequency $\omega_0 = \mu_0 B_0 / \hbar$]

9. Neutron interferometer(James) - In a classic table-top experiment (neutron interferometer), a monochromatic neutron beam ($\lambda = 1.445\ \mathring{A}$) is split by Bragg reflection at point A of an interferometer into two beams which are then recombined (after another reflection) at point D as in the figure below:



One beam passes through a region of transverse magnetic field of strength B (direction shown by lines)for a distance L. Assume that the two paths from A to D are identical except for the region of magnetic field.

(a) Find the explicit expression for the dependence of the intensity at point D on B, L and the neutron wavelength, with the neutron polarized parallel or anti-parallel to the magnetic field.
(b) Show that the change in the magnetic field that produces two successive maxima in the counting rates is given by

$$\Delta B = \frac{8\pi^2 \hbar c}{|e| g_n \lambda L}$$

where $g_n\,(=-1.91)$ is the neutron magnetic moment in units of $-e\hbar/2m_n c$. This calculation was a PRL publication in 1967.

10. Magnetic Resonance(Jen) - A particle of spin 1/2 and magnetic moment μ is placed in a magnetic field $\vec{B} = B_0\hat{z} + B_1\hat{x}\cos\omega t - B_1\hat{y}\sin\omega t$, which is often employed in magnetic resonance experiments. Assume that the particle has spin up along the +z-axis at t = 0 $(m_z = +1/2)$. Derive the probability to find the particle with spin down $(m_z = -1/2)$ at time t > 0.

11. Addition of angular momentum(EVERYONE) - Consider a system of two particles with $j_1 = 2$ and $j_2 = 1$. Determine the $|j,m,j_1,j_2\rangle$ states listed below in the $|j_1,m_1,j_2,m_2\rangle$ basis.

$$|3,3,j_1,j_2\rangle, |3,2,j_1,j_2\rangle, |3,1,j_1,j_2\rangle, |2,2,j_1,j_2\rangle, |2,1,j_1,j_2\rangle, |1,1,j_1,j_2\rangle$$

12. Clebsch-Gordan Coefficients(Elizabeth) - Work out the Clebsch-Gordan coefficients for the combination

$$\frac{3}{2}\otimes\frac{1}{2}$$

13. Spin-1/2 and Density Matrices(Ed) - Let us consider the application of the density matrix formalism to the problem of a spin-1/2 particle in a static external magnetic field. In general, a particle with spin may carry a magnetic moment, oriented along the spin direction (by symmetry). For spin-1/2, we have that the magnetic moment (operator) is thus of the form:

$$\hat{\mu}_i = \frac{1}{2}\gamma\hat{\sigma}_i$$

where the $\hat{\sigma}_i$ are the Pauli matrices and γ is a constant giving the strength of the moment, called the gyromagnetic ratio. The term in the Hamiltonian for such a magnetic moment in an external magnetic field, $\vec{B}$ is just:

$$H = -\vec{\mu}\cdot\vec{B}$$

The spin-1/2 particle has a spin orientation or "polarization" given by

$$\vec{P} = \langle\vec{\sigma}\rangle$$

Let us investigate the motion of the polarization vector in the external field. Recall that the expectation value of an operator may be computed from the density matrix according to

$$\langle A\rangle = Tr(\rho A)$$

In addition the time evolution of the density matrix is given by

$$i\frac{\partial\hat{\rho}}{\partial t} = \left[\hat{H}(t),\hat{\rho}(t)\right]$$

Determine the time evolution $d\vec{P}/dt$ of the polarization vector. Do not make any assumption concerning the purity of the state.