

Quantum Theory Seminar #8

Readings:

- Zettili - Chapter - 7.1-7.3  
 Boccio - Chapter - 9.4-9.5  
 Website - Hoyer Mathematica Notebook  
 on Addition of Angular Momentum

Presentations:

- Magnetic Resonance \_Ari\_  
 (Sections 9.4,9.4.1)
- Addition of Angular Momentum \_Dan\_  
 (Sections 9.5,9.5.1,9.5.2)

Zettili Problems:

1. Z7-7(Jean) Matrix elements
2. Z7-15(Karen) Operator algebra
3. Z7-9 (EVERYONE) Eigenvalues and Eigenvectors  
 (do exactly and take limits)
4. Z7-30(Andrew K) Energy levels: 2 spin 1/2 particles
5. Z7-32(Jonah) More Energy levels: 3 spin 1/2 particles

Boccio Problems:

1. Addition of Angular Momentum(Jono) - Two atoms with  $J_1=1$  and  $J_2=2$  are coupled, with an energy described by  $\hat{H} = \epsilon \hat{J}_1 \cdot \hat{J}_2$ ,  $\epsilon > 0$ . Determine all of the energies and degeneracies for the coupled system
2. Spin = 1 systems(EVERYONE)

We now consider a spin = 1 system.

- (a) Use the spin = 1 states  $|1,1\rangle$ ,  $|1,0\rangle$  and  $|1,-1\rangle$  (eigenstates of  $\hat{S}_z$ ) as a basis to form the matrix representation (3x3) of the angular momentum operators

$$\hat{S}_x, \hat{S}_y, \hat{S}_z, \hat{S}^2, \hat{S}_+, \hat{S}_-$$

(b) Determine the eigenstates of  $\hat{S}_x$  in terms of the eigenstates  $|1,1\rangle$ ,  $|1,0\rangle$  and  $|1,-1\rangle$  of  $\hat{S}_z$ .

(c) A spin = 1 particle is in the state

$$|\psi\rangle = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3i \end{pmatrix} \text{ in the } \hat{S}_z \text{ basis}$$

(1) What are the probabilities that a measurement of  $\hat{S}_z$  will yield the values  $\hbar$ ,  $0$ , or  $-\hbar$  for this state? What is  $\langle \hat{S}_z \rangle$ ?

(2) What is  $\langle \hat{S}_x \rangle$  in this state?

(3) What is the probability that a measurement of  $\hat{S}_x$  will yield the value  $\hbar$  for this state?

(d) A particle with spin = 1 has the Hamiltonian

$$\hat{H} = A\hat{S}_z + \frac{B}{\hbar}\hat{S}_x^2$$

(1) Calculate the energy levels of this system.

(2) If, at  $t = 0$ , the system is in an eigenstate of  $\hat{S}_x$  with eigenvalue  $\hbar$ , calculate the expectation value of the spin  $\langle \hat{S}_z \rangle$  at time  $t$ .

**3. Deuterium Atom (Sarah)** - Consider a deuterium atom (composed of a nucleus of spin 1 and a electron). The electronic angular momentum is  $\hat{J} = \hat{L} + \hat{S}$ , where  $\hat{L}$  is the orbital angular momentum of the electron and  $\hat{S}$  is its spin. The total angular momentum of the atom is  $\hat{F} = \hat{J} + \hat{I}$  where  $\hat{I}$  is the nuclear spin. The eigenvalues of  $\hat{J}^2$  and  $\hat{F}^2$  are  $J(J+1)\hbar^2$  and  $F(F+1)\hbar^2$  respectively.

(a) What are the possible values of the quantum numbers  $J$  and  $F$  for the deuterium atom in the  $1s(L=0)$  ground state?

(b) What are the possible values of the quantum numbers  $J$  and  $F$  for a deuterium atom in the  $2p(L=1)$  excited state?

**4. Spherical Harmonics (Robert)** - Consider a particle in a state described by

$$\psi = N(x + y + 2z)e^{-\alpha r}$$

where N is a normalization factor.

(a) Show, by rewriting the  $Y_1^{\pm 1,0}$  functions in terms of  $x, y, z$  and  $r$  that

$$Y_1^{\pm 1} = \mp \left( \frac{3}{4\pi} \right)^{1/2} \frac{x \pm iy}{\sqrt{2}r}$$

$$Y_1^0 = \left( \frac{3}{4\pi} \right)^{1/2} \frac{z}{r}$$

(b) Using this result, show that for a particle described by  $\psi$  above

$$P(L_z = 0) = 2/3, P(L_z = \hbar) = 1/6 \text{ and } P(L_z = -\hbar) = 1/6$$

**5. Spin in Magnetic Field (Kevin/Robert/Orion)** - Suppose that we have a spin-1/2 particle interacting with a magnetic field via the Hamiltonian

$$\hat{H} = \begin{cases} -\vec{\mu} \cdot \vec{B} & , \vec{B} = B\hat{e}_z & 0 \leq t < T \\ -\vec{\mu} \cdot \vec{B} & , \vec{B} = B\hat{e}_y & T \leq t < 2T \end{cases}$$

where  $\vec{\mu} = \mu_B \vec{\sigma}$  and the system is in the initial ( $t = 0$ ) state

$$|\psi(0)\rangle = |x+\rangle = \frac{1}{\sqrt{2}}(|z+\rangle + |z-\rangle)$$

Determine the probability that the state of the system at  $t = 2T$  is

$$|\psi(2T)\rangle = |x+\rangle$$

in three ways:

- (1) Using the Schrodinger equation (solving differential equations)
- (2) Using the time development operator (using operator algebra)
- (3) Using the density operator formalism

**6. What happens in the Stern-Gerlach box? (Orion)** - An atom with spin = 1/2 passes through a Stern-Gerlach apparatus adjusted so as to transmit atoms that have their spins in the +z direction. The atom spends time T in a magnetic field B in the x-direction.

(a) At the end of this time what is the probability that the

atom would pass through a Stern-Gerlach selector for spins in the  $-z$  direction?

(b) Can this probability be made equal to one, if so, how?

**7. Spin = 1 particle in a magnetic field [use Boccio #2] (Dan) -**

A particle with intrinsic spin = 1 is placed in a uniform magnetic field  $\vec{B} = B_0 \hat{e}_x$ . The initial spin state is

$$|\psi(0)\rangle = |1,1\rangle$$

Take the spin Hamiltonian to be

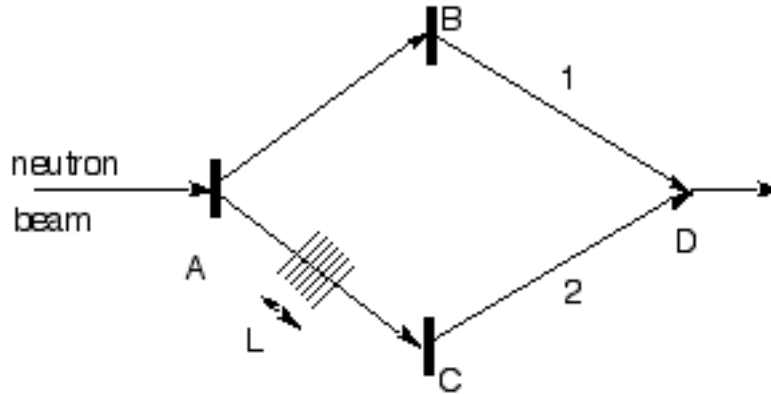
$$\hat{H} = \omega_0 \hat{S}_x$$

and determine the probability that the particle is in the state  $|\psi(t)\rangle = |1,-1\rangle$  at time  $t$ .

**8. Multiple magnetic fields (EVERYONE) -** A spin-1/2 system with magnetic moment  $\vec{\mu} = \mu_0 \vec{\sigma}$  is located in a uniform time-independent magnetic field  $B_0$  in the positive  $z$ -direction. For the time interval  $0 < t < T$  an additional uniform time-independent field  $B_1$  is applied in the positive  $x$ -direction. During this interval, the system is again in a uniform constant magnetic field, but of different magnitude and direction  $z'$  from the initial one. At and before  $t = 0$ , the system is in the  $m = 1/2$  state with respect to the  $z$ -axis.

- (a) At  $t = 0+$ , what are the amplitudes for finding the system with spin projections  $m' = \pm 1/2$  with respect to the  $z'$ -axis?
- (b) What is the time development of the energy eigenstates with respect to the  $z'$  direction, during the time interval  $0 < t < T$ ?
- (c) What is the probability at  $t = T$  of observing the system in the spin state  $m = -1/2$  along the original  $z$ -axis?  
[Express answers in terms of the angle  $\theta$  between the  $z$  and  $z'$  axes and the frequency  $\omega_0 = \mu_0 B_0 / \hbar$ ]

**9. Neutron interferometer (Ari) -** In a classic table-top experiment (neutron interferometer), a monochromatic neutron beam ( $\lambda = 1.445 \text{ \AA}$ ) is split by Bragg reflection at point A of an interferometer into two beams which are then recombined (after another reflection) at point D as in the figure below:



One beam passes through a region of transverse magnetic field of strength  $B$  (direction shown by lines) for a distance  $L$ . Assume that the two paths from  $A$  to  $D$  are identical except for the region of magnetic field.

(a) Find the explicit expression for the dependence of the intensity at point  $D$  on  $B$ ,  $L$  and the neutron wavelength, with the neutron polarized parallel or anti-parallel to the magnetic field.

(b) Show that the change in the magnetic field that produces two successive maxima in the counting rates is given by

$$\Delta B = \frac{8\pi^2 \hbar c}{|e| g_n \lambda L}$$

where  $g_n (= -1.91)$  is the neutron magnetic moment in units of  $-\hbar/2m_n c$ . This calculation was a PRL publication in 1967.

**10. Magnetic Resonance (Andrew Z)** - A particle of spin  $1/2$  and magnetic moment  $\mu$  is placed in a magnetic field

$\vec{B} = B_0 \hat{z} + B_1 \hat{x} \cos \omega t - B_1 \hat{y} \sin \omega t$ , which is often employed in magnetic resonance experiments. Assume that the particle has spin up along the  $+z$ -axis at  $t = 0$  ( $m_z = +1/2$ ). Derive the probability to find the particle with spin down ( $m_z = -1/2$ ) at time  $t > 0$ .

**11. Addition of angular momentum (EVERYONE)** - Consider a system of two particles with  $j_1 = 2$  and  $j_2 = 1$ . Determine the  $|j, m, j_1, j_2\rangle$  states listed below in the  $|j_1, m_1, j_2, m_2\rangle$  basis.

$$|3, 3, j_1, j_2\rangle, |3, 2, j_1, j_2\rangle, |3, 1, j_1, j_2\rangle, |2, 2, j_1, j_2\rangle, |2, 1, j_1, j_2\rangle, |1, 1, j_1, j_2\rangle$$

**12. Clebsch-Gordan Coefficients(Karen)** - Work out the Clebsch-Gordan coefficients for the combination

$$\frac{3}{2} \otimes \frac{1}{2}$$

**13. Spin-1/2 and Density Matrices(Jonah)** - Let us consider the application of the density matrix formalism to the problem of a spin-1/2 particle in a static external magnetic field. In general, a particle with spin may carry a magnetic moment, oriented along the spin direction (by symmetry). For spin-1/2, we have that the magnetic moment (operator) is thus of the form:

$$\hat{\mu}_i = \frac{1}{2} \gamma \hat{\sigma}_i$$

where the  $\hat{\sigma}_i$  are the Pauli matrices and  $\gamma$  is a constant giving the strength of the moment, called the gyromagnetic ratio. The term in the Hamiltonian for such a magnetic moment in an external magnetic field,  $\vec{B}$  is just:

$$H = -\vec{\mu} \cdot \vec{B}$$

The spin-1/2 particle has a spin orientation or "polarization" given by

$$\vec{P} = \langle \vec{\sigma} \rangle$$

Let us investigate the motion of the polarization vector in the external field. Recall that the expectation value of an operator may be computed from the density matrix according to

$$\langle A \rangle = \text{Tr}(\rho A)$$

In addition the time evolution of the density matrix is given by

$$i \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}(t), \hat{\rho}(t)]$$

Determine the time evolution  $d\vec{P}/dt$  of the polarization vector. Do not make any assumption concerning the purity of the state.