Physics 113 Spring 2009

Quantum Theory Seminar #6

Readings:

- Zettili - Chapter - 4
- Boccio - Chapter - 8.2,8.3,8.4,8.5(pp 27-31, 46-48), 8.6, 8.7, 8.8, 8.9, 8.10, 8.11, 8.12, 8.13

Topics:

- 1-Dimensional Systems

Presentations: (5-10 minutes MAX)

Wave Packets and Stationary Phase (Chapter 8.2,8.5(pp 27-31, 46-48))	_Ben_______
Meaning of Indeterminacy Relations Time-Energy Uncertainty Meaning of Wave Function (Chapter 8.3, 8.4)	_Elizabeth_
Harmonic Oscillator-Algebraic Method (Chapter 8.6.2 pp 61-64)	_Sandy_____
Coherent States (Chapter 86.2 pp 64-70)	_Ed________
Green Functions (Chapter 8.7)	_Rachael___
E&M and Aharonov-Bohm Effect (Chapter 8.8)	_Elizabeth_
Translation Invariant Potential (Chapter 8.11)	_Dougal____
Quantum Interference with Slits (Chapter 8.13)	_Jen_______

Zettili Problems:

1. **(Ben)**
   (a) Z4-31 A wall suddenly moves....
   (b) Z4-32 A spring constant suddenly changes.....

Boccio Problems:

1. Coherent States(MIDTERM #3) - Coherent states are defined to be eigenstates of the annihilation or lowering operator in the harmonic oscillator potential. Each coherent state has a complex label z and is given by $|z\rangle = e^{z\hat{a}^+}|0\rangle$

(a) Show that $\hat{a}|z\rangle = z|z\rangle$

(b) Show that $\langle z_1|z_2\rangle = e^{z_1^* z_2}$

(c) Show that the completeness relation takes the form

$$\hat{I} = \int \frac{dxdy}{\pi} |z\rangle\langle z| e^{-z^* z}$$

where $\hat{I}$ is the identity operator, z = x + iy, and the integration is taken over the whole x-y plane, for example in polar coordinates.

2. Correlation function(Jen) - Consider a function, known as the **correlation function**, defined by

$$C(t) = \langle \hat{x}(t)\hat{x}(0)\rangle$$

where $\hat{x}(t)$ is the position operator in the Heisenberg picture.

Evaluate the correlation function explicitly for the ground-state of the one dimensional simple harmonic oscillator.

3. A matrix element(Dougal) - Show for the one dimensional simple harmonic oscillator

$$\langle 0|e^{ik\hat{x}}|0\rangle = \exp\left[-k^2 \langle 0|\hat{x}^2|0\rangle / 2\right]$$

where $\hat{x}$ is the position operator.

4. A confined particle(Jen) - A particle of mass m is confined to a space $0 < x < a$ in one dimension by infinitely high walls at $x = 0$ and $x = a$. At t = 0 the particle is initially in the left half of the well with a wave function given by

$$\psi(x,0) = \begin{cases} \sqrt{2/a} & 0 < x < a/2 \\ 0 & a/2 < x < a \end{cases}$$

(a) Find the time-dependent wave function $\psi(x,t)$.

(b) What is the probability that the particle is in the n^{th} eigenstate at time t ?

(c) Derive an expression for average value of particle energy.

5. K-Meson oscillations(Sandy) - An additional effect to worry about when thinking about the time development of K-meson states is that the $|K_L\rangle$ and $|K_S\rangle$ states decay with time. Thus, we expect that these states should have the time dependence

$$|K_L(t)\rangle = e^{-i\omega_L t - t/2\tau_L}|K_L\rangle \quad , \quad |K_S(t)\rangle = e^{-i\omega_S t - t/2\tau_S}|K_S\rangle$$

where

$$\omega_L = E_L/\hbar \quad , \quad E_L = \left(p^2c^2 + m_L^2c^4\right)^{1/2}$$

$$\omega_S = E_S/\hbar \quad , \quad E_S = \left(p^2c^2 + m_S^2c^4\right)^{1/2}$$

and

$$\tau_S \approx 0.9\times 10^{-10}\,\text{sec} \quad , \quad \tau_L \approx 560\times 10^{-10}\,\text{sec}$$

Suppose that a pure K_L beam is sent through a thin absorber whose only effect is to change the relative phase of the K_0 and $\bar{K}_0$ amplitudes by 10°. Calculate the number of K_S decays, relative to the incident number of particles, that will be observed in the first 5 cm after the absorber. Assume the particles have momentum = mc.

6. A Josephson Junction(Max)

A Josephson junction is formed when two superconducting wires are separated by an insulating gap of capacitance C. The quantum states $\psi_i, i=1,2$ of the two wires can be characterized by the numbers n_i of Cooper pairs (charge $-2e$) and phases θ_i, such that $\psi_i = \sqrt{n_i}e^{i\theta_i}$ (Ginzburg-Landau approximation). The (small) amplitude that a pair tunnel across a narrow insulating barrier is $-E_J/n_0$ where $n_0 = n_1 + n_2$ and E_J is the the so-called Josephson energy. The interesting physics is expressed in terms of the differences

$$n = n_2 - n_1 \quad , \quad \phi = \theta_2 - \theta_1$$

We consider a junction where $n_1 \approx n_2 \approx n_0/2$.

When there exists a nonzero difference n between the numbers of pairs of charge $-2e$. where $e>0$, on the two sides of the junction, there is net charge $-ne$ on side 2 and net charge ne on side 1. Hence a voltage difference ne/C arises, where the voltage on side 1 is higher than that on side 2 if $n = n_2 - n_1 > 0$. Taking the zero of the voltage to be at the center of the junction, the electrostatic energy of the Cooper pair of charge $-2e$ on side 2 is ne^2/C, and that of a pair on side 1 is $-ne^2/C$. The total electrostatic energy is $C(\Delta V)^2/2 = Q^2/2C = (ne)^2/2C$.

The equations of motion for a pair in the two-state system {1,2} are

$$i\hbar\frac{d\psi_1}{dt} = U_1\psi_1 - \frac{E_J}{n_0}\psi_2 = -\frac{ne^2}{C}\psi_1 - \frac{E_J}{n_0}\psi_2$$

$$i\hbar\frac{d\psi_2}{dt} = U_2\psi_2 - \frac{E_J}{n_0}\psi_1 = \frac{ne^2}{C}\psi_2 - \frac{E_J}{n_0}\psi_1$$

(a) Discuss the physics of the terms in these equations.

(b) Using $\psi_i = \sqrt{n_i}e^{i\theta_i}$, show that the equations of motion for n and ϕ are given by

$$\dot{\phi} = \dot{\theta}_2 - \dot{\theta}_1 \approx -\frac{2ne^2}{\hbar C}$$

$$\dot{n} = \dot{n}_2 - \dot{n}_1 \approx \frac{E_J}{\hbar}\sin\phi$$

(c) Show that the pair(electric current) from side 1 to side 2 is given by

$$J_S = J_0\sin\phi \quad , \quad J_0 = \frac{\pi E_J}{\varphi_0}$$

(d) Show that

$$\ddot{\phi} \approx -\frac{2e^2 E_J}{\hbar^2 C}\sin\phi$$

For E_J positive, show that this implies there are oscillations about $\phi = 0$ whose angular frequency (called the Josephson plasma frequency)is given by

$$\omega_J = \sqrt{\frac{2e^2 E_J}{\hbar^2 C}}$$

for small amplitudes.

If E_J is negative, then there are oscillations about $\phi = \pi$.

(e) If a voltage $V = V_1 - V_2$ is applied across the junction(by a battery), a charge $Q_1 = VC = (-2e)(-n/2) = en$ is held on side 1, and the negative of this on side 2. Show that we then have

$$\dot{\phi} \approx -\frac{2eV}{\hbar} \equiv -\omega$$

which gives $\phi = \omega t$.

The battery holds the charge difference across the junction fixed at $VC = en$, but can be a source or sink of charge such that a current can flow in the circuit. Show that in this case, the current is given by

$$J_S = -J_0 \sin \omega t$$

i.e., the DC voltage of the battery generates an AC pair current in circuit of frequency

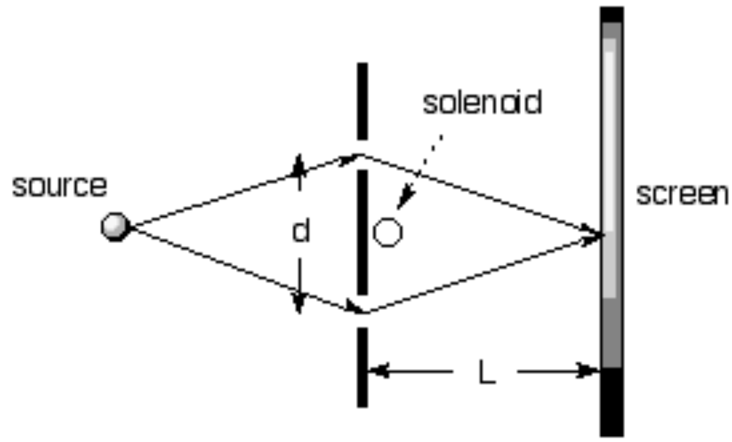
$$\omega = \frac{2eV}{\hbar}$$

7. Matrix Elements for Harmonic Oscillator(Sandy) - Compute the following matrix elements

$$\langle m | \hat{x}^3 | n \rangle \quad , \quad \langle m | \hat{x}\hat{p} | n \rangle$$

8. Aharonov-Bohm experiment(Rachael) - Consider an infinitely long solenoid which carries a current I so that there is a constant magnetic field inside the solenoid. Suppose that in the region outside the solenoid the motion of a particle with charge e and mass m is described by the Schrodinger equation. Assume that for I = 0 , the solution of the equation is given by

$$\psi_0(\vec{r},t) = e^{iE_0 t/\hbar} \psi_0(\vec{r})$$



(a) Write down and solve the Schrodinger equation in the region outside the solenoid in the case $I \neq 0$.

(b) Consider the two-slit diffraction experiment for the particles described above shown in the figure below. Assume that the distance d between the two slits is large compared to the diameter of the solenoid.

Compute the shift ΔS of the diffraction pattern on the screen due to the presence of the solenoid with $I \neq 0$. Assume that $L >> \Delta S$.

9. Oscillator with Delta Function(Ed) - Consider a harmonic oscillator potential with an extra delta function term at the origin, that is,

$$V(x) = \frac{1}{2}m\omega^2 x^2 + \frac{\hbar^2 g}{2m}\delta(x)$$

(a) Using the parity invariance of the Hamiltonian, show that the energy eigenfunctions are even and odd functions and that the simple harmonic oscillator odd-parity energy eigenstates are still eigenstates of the system Hamiltonian, with the same eigenvalues.

(b) Expand the even-parity eigenstates of the new system in terms of the even-parity harmonic oscillator eigenfunctions and determine the expansion coefficients.

(c) Show that the energy eigenvalues that correspond to even eigenstates are solutions of the equation

$$\frac{2}{g} = -\sqrt{\frac{\hbar}{m\pi\omega}}\sum_{k=0}^{\infty}\frac{(2k)!}{2^{2k}(k!)^2}\left(2k+\frac{1}{2}-\frac{E}{\hbar\omega}\right)^{-1}$$

You might need the fact that

$$\psi_{2k}(0) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{\sqrt{(2k)!}}{2^k k!}$$

(d) Consider the following cases:

(1) g > 0 , E > 0

(2) g < 0 , E > 0

(3) g < 0 , E < 0

Show the first and second cases correspond to an infinite number of energy eigenvalues.

Where are they relative to the original energy eigenvalues of the harmonic oscillator?

Show that in the third case, that of an attractive delta function core, there exists a single eigenvalue corresponding to the ground state of the system provided that the coupling is such that

$$\left[\frac{\Gamma(3/4)}{\Gamma(1/4)}\right]^2 < \frac{g^2\hbar}{16m\omega} < 1$$

You might need the series summation:

$$\sum_{k=0}^{\infty}\frac{(2k)!}{4^k(k!)^2}\frac{1}{2k+1-x} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(1/2-x/2)}{\Gamma(1-x/2)}$$

You will need to look up other properties of the gamma function to solve this problem.

10. Instantaneous Force(Ben) - Consider a simple harmonic oscillator in its ground state.

An instantaneous force imparts momentum p_0 to the system such that the new state vector is given by

$$|\psi\rangle = e^{-ip_0x/\hbar}|0\rangle$$

where $|0\rangle$ is the ground-state of the original oscillator.

What is the probability that the system will stay in its ground state?

11. Using the commutator(Elizabeth) - Using the coordinate-momentum commutation relation prove that

$$\sum_n (E_n - E_0)\left|\langle E_n | \hat{x} | E_0 \rangle\right|^2 = \text{constant}$$

where E_n is the energy corresponding to the eigenstate $|E_n\rangle$.

Determine the value of the constant. Assume the Hamiltonian has the general form

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$$

12. Measurement on a Particle in a Box(James) - Consider a particle in a box of width a, prepared in the ground state.

(a) What are then possible values one can measure for:
(1) energy, (2) position, (3) momentum

(b) What are the probabilities for the possible outcomes you found in part (a)?

(c) At some (call it t = 0) we perform a measurement of position. However, our detector has only finite resolution. We find that the particle is in the middle of the box (call it the origin) with an uncertainty $\Delta x = a/2$, that is, we know the position is, for sure, in the range $-a/4 < x < a/4$, but we are completely uncertain where it is within this range. What is the (normalized) post-measurement state?

(d) Immediately after the position measurement what are the possible values for
(1) energy, (2) position, (3) momentum
and with what probabilities?

(e) At a later time, what are the possible values for
(1) energy, (2) position, (3) momentum
and with what probabilities? Comment.

13. 1/x potential(MIDTERM #4) - An electron moves in one dimension and is confined to the right half-space (x > 0) where it has potential energy

$$V(x) = -\frac{e^2}{4x}$$

where e is the charge on an electron.

(a) What is the solution of the Schrodinger equation at large x ?

(b) What is the boundary condition at x = 0 ?

(c) Use the results of (a) and (b) to guess the ground state solution of the equation. Remember the ground state wave function has no zeros except at the boundaries.

(d) Find the ground state energy.

(e) Find the expectation value $\langle \hat{x} \rangle$ in the ground state.