Physics 113 Spring 2010

Quantum Theory Seminar #2

This week we continue to learn the **mathematical language of quantum mechanics. Learning the language of quantum mechanics BEFORE studying quantum mechanics is important.** The only way to do this is **lots of problems** so we continue in that mode.

Readings:

Zettili - Chapter(s) - 2
Sections 2.4.3, 2.4.6-2.4.9, 2.5, 2.6
Boccio - Chapter(s) - 4(pages 38-81),5(pages 12-26)

Topics:

Mathematics of Quantum Mechanics
Probability

Presentations: **Discuss content of the selected readings emphasizing whatever you think are the important concepts.**

Infinite Dimensions(Ch 4 pages 46-50)	**__Orion______**
Functions of Operators(Ch 4 pages 63-69)	**__Karen______**
Operators with a Continuous Spectrum (Ch 4 pages 72-76)	**__Dan________**
Bayesian ideas(Ch 5 pages 12-26)	**__Andrew Z___**

These presentations should, at least, cover the basic concepts of the readings(also see if you can find new material from any other sources). Include as examples where appropriate.

Zettili Exercises:

1. **Z2-23 Matrices and Projection Operators(Jean)**
2. **Z2-28 Operator Identity(Jono)**
3. **Z2-45 Delta Function(Ari)**
4. **Z2-49 An Operator and Its Properties(Andrew K)**
5. **Z2-54 A Pair of Hermitian Operators(Sarah)**
6. **Z2-55 The Permutation Operator(Jonah)**

Boccio Problems:

1. Spectral Decomposition(EVERYONE)- Find the eigenvalues and eigenvectors of the matrix

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Construct the corresponding projection operators, and verify that the matrix can be written in terms of its eigenvalues and eigenvectors. This is the **spectral decomposition** for this matrix.

2. Measurement Results(Kevin) - Given particles in state

$$|\alpha\rangle = \frac{1}{\sqrt{83}}\left(-3|1\rangle + 5|2\rangle + 7|3\rangle\right)$$

where $\{|1\rangle, |2\rangle, |3\rangle\}$ form an orthonormal basis, what are the possible experimental results for a measurement of

$$\hat{Y} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

(written in this basis) and with what probabilities do they occur?

3. Expectation Values(Ari) - Let

$$R = \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}$$

represent an observable, and

$$|\Psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$$

be an arbitrary state vector(with $|a|^2 + |b|^2 = 1$). Calculate $\langle R^2 \rangle$ in two ways:

(a) Evaluate $\langle R^2 \rangle = \langle \Psi | R^2 | \Psi \rangle$ directly

(b) Find the eigenvalues and eigenvectors of. Expand the state vector as a linear combination of the eigenvectors and evaluate $\langle R^2 \rangle = r_1^2 |c_1|^2 + r_2^2 |c_2|^2$

4. Eigenket Properties(Robert) - Consider a 3-dimensional ket space. If a certain set of orthonormal kets, say $|1\rangle$, $|2\rangle$ and $|3\rangle$ are used as the basis kets, the operators $\hat{A}$ and $\hat{B}$ are represented by

$$\hat{A} \to \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} \quad , \quad \hat{B} \to \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$$

where a and b are both real numbers.

(a) Obviously $\hat{A}$ exhibits a degenerate spectrum. Does $\hat{B}$ also exhibit a degenerate spectrum?

(b) Show that $\hat{A}$ and $\hat{B}$ commute.

(c) Find a new set of orthonormal kets which are simultaneous eigenkets of both $\hat{A}$ and $\hat{B}$. Specify the eigenkets of $\hat{A}$ and $\hat{B}$. Does your specification of eigenvalues completely characterize each eigenket?

5. Hardness World(Orion)) - Let us define a state using the hardness basis $(|h\rangle, |s\rangle)$, where $\hat{O}_{HARDNESS}|h\rangle = |h\rangle$, $\hat{O}_{HARDNESS}|s\rangle = -|s\rangle$ and the "hardness" operator is represented by (in this basis)

$$\hat{O}_{HARDNESS} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We have

$$|A\rangle = \cos\theta|h\rangle + e^{i\phi}\sin\theta|s\rangle$$

where θ and ϕ are real constants.

(a) Is this state normalized? Show your work.

(b) Find the state $|B\rangle$ that is orthogonal to $|A\rangle$. Make sure $|B\rangle$ is normalized.

(c) Express $|h\rangle$ and $|s\rangle$ in the $(|A\rangle, |B\rangle)$ basis

(d) What are the possible outcomes of a hardness measurement on state $|A\rangle$ and with what probability will each occur?

(e) Express the hardness operator in the $(|A\rangle, |B\rangle)$ basis

6. Things in Hilbert Space (EVERYONE) - For all parts of this problem, let H be a Hilbert space spanned by the basis kets $\{|0\rangle,|1\rangle,|2\rangle,|3\rangle\}$, and let a and b be arbitrary complex constants.

(a) Which of the following are Hermitian operators on H?

1. $|0\rangle\langle 1| + i|1\rangle\langle 0|$
2. $|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 3| + |3\rangle\langle 2|$
3. $(a|0\rangle + |1\rangle)^{+}(a|0\rangle + |1\rangle)$
4. $\left((a|0\rangle + b^{*}|1\rangle)^{+}(b|0\rangle - a^{*}|1\rangle)\right)|2\rangle\langle 1| + |3\rangle\langle 3|$
5. $|0\rangle\langle 0| + i|1\rangle\langle 0| - i|0\rangle\langle 1| + |1\rangle\langle 1|$

(b) Find the spectral decomposition of the following operator on H:

$$\hat{K} = |0\rangle\langle 0| + 2|1\rangle\langle 2| + 2|2\rangle\langle 1| - |3\rangle\langle 3|$$

(c) Let $|\Psi\rangle$ be a normalized ket in H, and let $\hat{I}$ denote the identity operator on H. Is the operator

$$\hat{B} = \frac{1}{\sqrt{2}}(\hat{I} + |\Psi\rangle\langle\Psi|)$$

a projection operator?

(d) Find the spectral decomposition of the operator $\hat{B}$ from part (c).

7. Playing Cards(Karen) - Two cards are drawn at random from a shuffled deck and laid aside without being examined. Then a third card is drawn. Show that the probability that the third card is a spade is 1/4 just as it was for the first card. HINT: Consider all the (mutually exclusive) possibilities (two discarded cards spades, third card spade or not spade, etc).

8. Birthdays(Dan) - What is the probability that you and a friend have different birthdays? (for simplicity let a year have 365 days). What is the probability that three people have different birthdays? Show that the probability that n people have different birthdays is

$$p = \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right)\left(1 - \frac{3}{365}\right).....\left(1 - \frac{n-1}{365}\right)$$

Estimate this for n << 365 by calculating $\ell n(p)$ (use the fact that $\ell n(1+x) \approx x$ for x << 1). Find the smallest integer n for which $p < 1/2$. Hence show that for a group of 23 people or more, the probability is greater than 1/2 that two of them have the same birthday.

9. Is there life?(Andrew Z) - The number of stars in our galaxy is about $N = 10^{11}$. Assume that the probability that a star has planets is $p = 10^{-2}$, the probability that the conditions on the planet are suitable for life is $q = 10^{-2}$, and the probability of life evolving, given suitable conditions, is $r = 10^{-2}$. These numbers are rather arbitrary.

(a) What is the probability of life existing in an arbitrary solar system (a star and planets, if any)?
(b) What is the probability that life exists in at least one solar system?

10. Law of large Numbers(Robert) - This problem illustrates the law of large numbers.

(a) Assuming the probability of obtaining "heads" in a coin toss is 0.5, compare the probability of obtaining "heads" in 5 out of 10 tosses with the probability of obtaining "heads" in 50 out of 100 tosses.
(b) For a set of 10 tosses and for a set of 100 tosses, calculate the probability that the fraction of "heads" will be between 0.445 and 0.555.

11. Bayes (EVERYONE) - Suppose that you have 3 nickels and 4 dimes in your right pocket and 2 nickels and a quarter in your left pocket. You pick a pocket at random and from it select a coin at random. If it is a nickel, what is the probability that it came from your right pocket? Use Baye's formula.

12. A 2-Dimensional Hilbert Space(Jonah) - Consider a 2-dimensional Hilbert space spanned by an orthonormal basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. Let us define the operators

$$\hat{S}_x = \frac{\hbar}{2}\left(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\right) \quad , \quad \hat{S}_y = \frac{\hbar}{2i}\left(|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|\right) \quad , \quad \hat{S}_z = \frac{\hbar}{2}\left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right)$$

(a) Show that each of these operators is Hermitian.
(b) Find the matrix representations of these operators in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$.
(c) Show that $\left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z$, and cyclic permutations. Do this two ways: Using the Dirac notation definitions above and the

matrix representations found in (b). Given these commutators, how do you interpret these operators.

Now let $|\pm\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle \pm |\downarrow\rangle\right)$

(d) Show that these vectors form a new orthonormal basis.

(e) Find the matrix representations of these operators in the basis $\{|+\rangle, |-\rangle\}$.

(f) The matrices found in (b) and (e) are related through a **similarity transformation** by a unitary matrix, U,

$$\hat{S}_x^{(\uparrow\downarrow)} = U^\dagger \hat{S}_x^{(\pm)} U \quad , \quad \hat{S}_y^{(\uparrow\downarrow)} = U^\dagger \hat{S}_y^{(\pm)} U \quad , \quad \hat{S}_z^{(\uparrow\downarrow)} = U^\dagger \hat{S}_z^{(\pm)} U$$

where the superscript denotes the basis in which the operator is represented. Find U and show that it is unitary.

Now let $\hat{S}_\pm = \frac{1}{2}\left(\hat{S}_x \pm i\hat{S}_y\right)$

(g) Express $\hat{S}_\pm$ as outer products in the basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ and show that $\hat{S}_+^\dagger = \hat{S}_-$.

(h) Show that

$$\hat{S}_+|\downarrow\rangle = |\uparrow\rangle \, , \hat{S}_-|\uparrow\rangle = |\downarrow\rangle \, , \hat{S}_-|\downarrow\rangle = 0 \, , \hat{S}_+|\uparrow\rangle = 0$$

and find

$$\langle\uparrow|\hat{S}_+ \, , \langle\downarrow|\hat{S}_+ \, , \langle\uparrow|\hat{S}_- \, , \langle\downarrow|\hat{S}_-$$

13. Psychological Tests(Sarah) - Two psychologists reported on tests in which subjects were given the **prior information:**

I = "In a certain city, 85% of the taxicabs are blue and 15% are green"

and the **data:**

D = "A witness to a crash who is 80% reliable (i.e., who in the lighting conditions prevailing can distinguish between green and blue 80% of the time) reports that the taxicab involved in the crash was green"

The subjects were then asked to judge the probability that the taxicab was actually blue. What is the correct answer?

14. Bayes Rules and Gaussians(Kevin) - Let us consider a classical problem(no quantum uncertainty). Suppose we are trying to measure the position of a particle and we assign a prior probability function,

$$p(x) = \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-(x-x_0)^2/2\sigma_0^2}$$

Our measuring device is not perfect. Due to noise it can only measure with a resolution Δ, i.e., when I measure the position, I must put error bars on this. Thus, if my detector registers the position as y, I assign liklihood that the position was x to a Gaussian,

$$p(y\,|\,x) = \frac{1}{\sqrt{2\pi\Delta^2}} e^{-(y-x)^2/2\Delta^2}$$

Use Bayes theorem to show that, given the new data, I must now update my probability assignment of the position to a new Gaussian,

$$p(x\,|\,y) = \frac{1}{\sqrt{2\pi\sigma'^2}} e^{-(x-x')^2/2\sigma'^2}$$

where

$$x' = x_0 + K_1(y - x_0) \ , \ \sigma'^2 = K_2\sigma_0^2 \ , \ K_1 = \frac{\sigma_0^2}{\sigma_0^2 + \Delta^2} \ , \ K_2 = \frac{\Delta^2}{\sigma_0^2 + \Delta^2}$$

Comment on the behavior as the measurement resolution improves.