Physics 113 Spring 2010
Quantum Theory Seminar #14

Readings:

Zettili - Chapter 11
Boccio - Chapter 13 (Section 13.1 (ALL))
Chapter 14 (Sections 14.7(ALL), 14.8(ALL))

Presentations:

Section 14.7(ALL) - Schrodinger's Cat (Rachael)
Section 14.9(ALL) - The Quantum Eraser (Sandy)

Zettili Problems:

1. **Z11-06: Born Approximation (EVERYONE)**
   Scattering from exponential potential
2. **Z11-07: Born Approximation (EVERYONE)**
   Scattering from from double delta function potential
3. **Z11-12: Partial Waves (EVERYONE)**
   S-Wave scattering of identical spin 1/2 particles

Boccio Scattering Problem:

1. S-Wave Phase Shift(James and Jen) - We wish to find an approximate expression for the s-wave phase shift, δ_0, for scattering of low energy particles from the potential

$$V(r) = \frac{C}{r^4} \quad , \quad C > 0$$

(a) For low energies, $k \approx 0$, the radial Schrodinger equation for $\ell = 0$ may be approximated by dropping the energy:

$$\left[-\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr} + \frac{2mC}{\hbar^2 r^4} \right] R_{\ell=0}^{inside}(r) = 0$$

By making the transformations

$$R(r) = \frac{1}{\sqrt{r}}\phi(r) \quad , \quad r = \frac{i}{\hbar}\frac{\sqrt{2mC}}{x}$$

show that the radial equation may be solved in terms of Bessel functions. Find an approximate solution, taking into account behavior at r = 0.

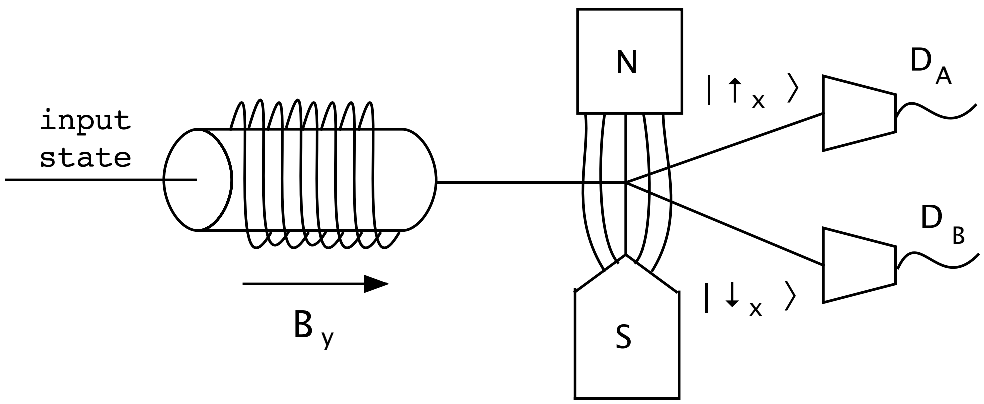
(b) By matching this to $R_{\ell=0}^{outside}(r)$ at r = a, (where a is chosen such that $\hbar a \gg \sqrt{2mC}$ and ka << 1) show that

$$\delta_0 = -k\frac{\sqrt{2mC}}{\hbar}$$

which is independent of a.

Boccio Quantum Measurement Problems:

1. Measurement of a Spin-1/2 Particle(Jen and Ben) - A spin-1/2 electron is sent through a solenoid with a uniform magnetic field in the y direction and then measured with a Stern-Gerlach apparatus with field gradient in the x direction as shown below:



The time spent inside the solenoid is such that $\Omega t = \phi$, where $\Omega = 2\mu_B B/\hbar$ is the Larmor precision frequency.

(a) Suppose the input state is the pure state $|\uparrow_z\rangle$. Show that the probability for detector D_A to fire as a function of ϕ is

$$P_{D_A} = \frac{1}{2}(\cos(\phi/2)+\sin(\phi/2))^2 = \frac{1}{2}(1+\sin\phi)$$

Repeat for the state $|\downarrow_z\rangle$ and show that

$$P_{D_A} = \frac{1}{2}(\cos(\phi/2)-\sin(\phi/2))^2 = \frac{1}{2}(1-\sin\phi)$$

(b) Now suppose the input is a pure coherent superposition of these two states,

$$|\uparrow_x\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle + |\downarrow_z\rangle)$$

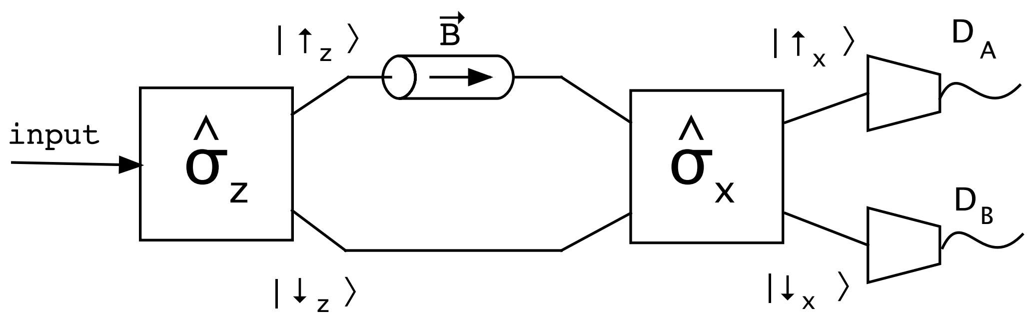
Find and sketch the probability for detector D_A to fire as a function of ϕ.

(c) Now suppose the input state is the completely mixed state

$$\hat{\rho} = \frac{1}{2}\left(\left|\uparrow_z\right\rangle\left\langle\uparrow_z\right| + \left|\downarrow_z\right\rangle\left\langle\downarrow_z\right|\right)$$

Find and sketch the probability for detector D_A to fire as a function of ϕ. Comment.

2. Mixed States vs. Pure States and Interference(Max and Elizabeth) - A "spin-interferometer" is shown below:



Spin-1/2 electrons prepared in a given state (pure or mixed) are separated into two paths by a Stern-Gerlach apparatus(gradient field along z). In one path, the particle passes through a solenoid, with a uniform magnetic field along the x-axis. The two paths are then recombined, sent through another Stern-Gerlach apparatus with field gradient along x, and the particles are counted in detectors in the two emerging ports.

The strength of the magnetic field is chosen so that $\Omega t = \phi$, for some phase ϕ, where $\Omega = 2\mu_B B/\hbar$ is the Larmor frequency and t is the time spent inside the solenoid.

(a) Derive the probability of electrons arriving at detector D_A as a function of ϕ for the following pure state inputs:

$$(i)\ \left|\uparrow_z\right\rangle \ ,\ (ii)\left|\downarrow_z\right\rangle \ ,\ (iii)\ \left|\uparrow_x\right\rangle \ ,\ (iv)\ \left|\downarrow_x\right\rangle$$

Comment on your results.

(b) Remember that for a mixed state we have

$$\hat{\rho} = \sum_i p_i \left|\psi_i\right\rangle\left\langle\psi_i\right|$$

where p_i is the probability of $\left|\psi_i\right\rangle$.

This is a **statistical mixture** of the states $\{\left|\psi_i\right\rangle\}$**, not a coherent superposition** of states. We should think of it "**classically**", i.e., we **have** one of the set $\{|\psi_i\rangle\}$, we just **do not know** which one.

Prove that

$$P_{D_A} = Tr\left[|\uparrow_x\rangle\langle\uparrow_x|\hat{\rho}\right] = \sum_i p_i \left|\langle\uparrow_x|\psi_i\rangle\right|^2$$

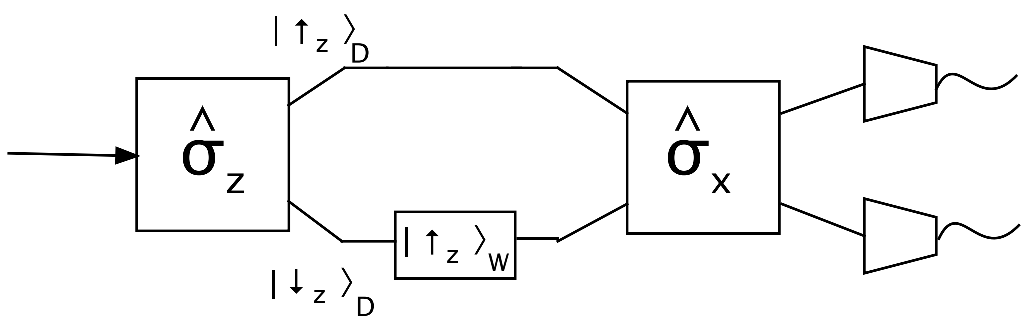
where $\left|\langle\uparrow_x|\psi_i\rangle\right|^2$ = the probability of detector B firing for the given input state (we figured these out in part (a)).

Repeat part (a) for the following mixed state inputs:

$$(i)\ \hat{\rho} = \frac{1}{2}|\uparrow_z\rangle\langle\uparrow_z| + \frac{1}{2}|\downarrow_z\rangle\langle\downarrow_z| \quad , \quad (ii)\ \hat{\rho} = \frac{1}{2}|\uparrow_x\rangle\langle\uparrow_x| + \frac{1}{2}|\downarrow_x\rangle\langle\downarrow_x| \quad , \quad (iii)\ \hat{\rho} = \frac{1}{3}|\uparrow_z\rangle\langle\uparrow_z| + \frac{2}{3}|\downarrow_z\rangle\langle\downarrow_z|$$

3. Which-path information,Entanglement, and Decoherence(James and Max) - If we can determine which path a particle takes in an interferometer, then we do not observe quantum interference fringes. But how does this arise?

Consider the interferometer shown below:



Into one arm of the interferometer we place a "which-way" detector in the form of another spin-1/2 particle prepared in the state $|\uparrow_z\rangle_W$. If the electron which travels through the interferometer, and is ultimately detected (denoted by subscript D), interacts with the "which-way" detector, the which-way electron flips the spin $|\uparrow_z\rangle_W \Rightarrow |\uparrow_z\rangle_W$, i.e, the "which-way" detector works such that

If $|\psi\rangle_D = |\uparrow_z\rangle_D$ nothing happens to $|\uparrow_z\rangle_W$

If $|\psi\rangle_D = |\downarrow_z\rangle_D$ then $|\uparrow_z\rangle_W \rightarrow |\downarrow_z\rangle_W$ (a spin flip)

Thus, as a composite system

$$|\uparrow_z\rangle_D |\uparrow_z\rangle_W \rightarrow |\uparrow_z\rangle_D |\uparrow_z\rangle_W$$
$$|\downarrow_z\rangle_D |\uparrow_z\rangle_W \rightarrow |\downarrow_z\rangle_D |\downarrow_z\rangle_W$$

(a) The electron D is initially prepared in the state $|\uparrow_x\rangle_D = \left(|\uparrow_z\rangle_D + |\downarrow_z\rangle_D\right)/\sqrt{2}$. Show that before detection, the two electrons D and W are in an entangled state

$$|\Psi_{DW}\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle_D |\uparrow_z\rangle_W + |\downarrow_z\rangle_D |\downarrow_z\rangle_W\right)$$

(b) Only the electron D is detected. Show that its "marginal state", ignoring the electron W, is the completely mixed state,

$$\hat{\rho}_D = \frac{1}{2}|\uparrow_z\rangle_{D\,D}\langle\uparrow_z| + \frac{1}{2}|\downarrow_z\rangle_{D\,D}\langle\downarrow_z|$$

This can be done by calculating

$$\text{Prob}(m_D) = \sum_{m_W} \left|\langle m_D, m_W | \Psi_{DW}\rangle\right|^2$$

for some observable.

As we showed in 12(b), this state shows no interference between $|\uparrow_z\rangle_D$ and $|\downarrow_z\rangle_D$. Thus, the "which-way" detector removes the coherence between states that existed in the input.

(c) Suppose now the which-way detector does not function perfectly and does not completely flip the spin, but rotates it by an angle θ about x so that

$$|\uparrow_\theta\rangle_W = \cos(\theta/2)|\uparrow_z\rangle_W + \sin(\theta/2)|\downarrow_z\rangle_W$$

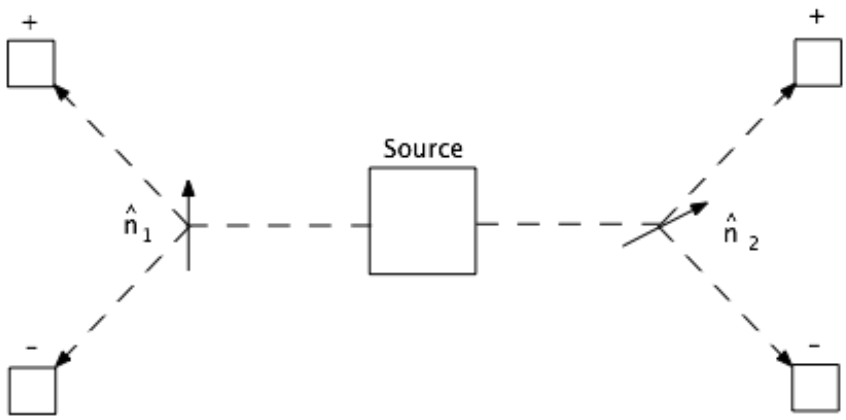
Show that in this case the marginal state is

$$\hat{\rho}_D = \frac{1}{2}|\uparrow_z\rangle_{D\,D}\langle\uparrow_z| + \frac{1}{2}|\downarrow_z\rangle_{D\,D}\langle\downarrow_z| + \cos(\theta/2)|\uparrow_z\rangle_{D\,D}\langle\downarrow_z| + \sin(\theta/2)|\downarrow_z\rangle_{D\,D}\langle\uparrow_z|$$

Comment on the limits $\theta \rightarrow 0$ and $\theta \rightarrow \pi$.

4. Bell's Inequality (Ed and Rachael)

A pair of spin-1/2 particles is produced by a source. The spin state of each particle can be measured using a Stern-Gerlach apparatus (see diagram below).



Einstein-Podolsky-Rosen setup for two spin-1/2 particles emitted by a source. The Stern-Gerlach apparatuses are represented by arrows showing their field directions. The small squares show the observed positions of spin-up and spin-down particles.

(a) Let $\hat{n}_1$ and $\hat{n}_2$ be the field directions of the Stern-Gerlach magnets. Consider the commuting observables

$$\sigma^{(1)} = \frac{2}{\hbar}\hat{n}_1 \cdot \vec{S}_1 \quad , \quad \sigma^{(2)} = \frac{2}{\hbar}\hat{n}_2 \cdot \vec{S}_2$$

corresponding to the spin component of each particle along the direction of the Stern-Gerlach apparatus associated with it. What are the possible values resulting from measurement of these observables and what are the corresponding eigenstates?

(b) Consider the observable

$$\sigma^{(12)} = \sigma^{(1)} \otimes \sigma^{(2)}$$

and write down its eigenvectors and eigenvalues. Assume that the pair of particles is produced in the singlet state

$$|0,0\rangle = \frac{1}{\sqrt{2}}\left(|S_z+\rangle^{(1)}|S_z-\rangle^{(2)} - |S_z-\rangle^{(1)}|S_z+\rangle^{(2)}\right)$$

What is the expectation value of ?

(c) Make the assumption that it is meaningful value to the spin of a particle even when it is not being measured. Assume also that the only possible results of the measurement of a spin component are $\pm\hbar/2$. Then show that the probability of finding the spins pointing in two given directions will be proportional to the overlap of the hemispheres that these two directions define. Quantify this criterion and calculate the expectation value of $\sigma^{(12)}$.

(d) Assume the spin variables depend on a **hidden variable** λ. The expectation value of the spin observable $\sigma^{(12)}$ is determined in terms of the normalized distribution function $f(\lambda)$:

$$\left\langle \sigma^{(12)} \right\rangle = \frac{4}{\hbar^2} \int d\lambda f(\lambda) S_z^{(1)}(\lambda) S_\phi^{(2)}(\lambda)$$

Prove **Bell's inequality**

$$\left| \left\langle \sigma^{(12)}(\phi) \right\rangle \right| - \left| \left\langle \sigma^{(12)}(\phi') \right\rangle \right| \le 1 + \left| \left\langle \sigma^{(12)}(\phi - \phi') \right\rangle \right|$$

(e) Consider Bell's inequality for $\phi' = 2\phi$ and show that it is not true when applied in the context of quantum mechanics.

5. Livio and Oivil (Sandy)- Two scientists (they happen to be twins, named "Oivil" and "Livio", but never mind .....) decide to do the following experiment. They set up a light source, which emits two photons at a time, back-to-back in the laboratory frame. The ensemble is given by

$$\rho = \frac{1}{2}\left(|LL\rangle\langle LL| + |RR\rangle\langle RR| \right)$$

where "L" refers to left-handed polarization and "R" refers to right-handed polarization. Thus, $|LR\rangle$ would refer to the state in which photon number 1 (defined as the photon which is aimed at Oivil, say) is left-handed and photon number 2 (the photon aimed at scientist Livio) is right-handed.

These scientists(one of whom has a diabolical bent) decide to play a game with Nature: Oivil (of course) stays in the lab, while Livio treks to a point a light-year away. The light source is turned on and emits two photons, one directed toward each scientist. Oivil soon measures the polarization of his photon; it is left-handed. He quickly makes a note that his sister is going to see a left-handed photon, about a year from that time.

The year has passed and finally Livio sees her photon, and measures its polarization. She sends a message back to her brother Oivil, who learns in yet another year what he know all along; Livio's photon was left-handed.

Oivil then has a sneaky idea. He secretly changes the apparatus, without telling his forlorn sister. Now the ensemble is

$$\rho = \frac{1}{2}\left(|LL\rangle + |RR\rangle \right)\left(\langle LL| + \langle RR| \right)$$

He causes another pair of photons to be emitted with this new apparatus and repeats the experiment. The result is identical to the first experiment.

(a) Was Oivil lucky, or will he get the right answer every time, for each apparatus? Demonstrate the answer explicitly using the density matrix formalism.

(b) What is the probability that Livio will observe a left-handed photon, or a right-handed photon, for each apparatus? Is there a problem with causality here? How can Oivil know what Livio is going to see, long before she sees it? Discuss what is happening here. Feel free to modify the experiment to illustrate any points you wish to make.