Physics 113 **Spring 2009**

Quantum Theory Seminar #10

Readings:

Zettili - Chapter - 9.1,9.2,9.2.1,9.2.2
Boccio - Chapter - 10.1(all),10.2(all),10.3(all),10.4,10.9

This week we begin studying time-independent perturbation theory; the major approximation method in quantum mechanics.

Presentations:

Non-Degenerate Perturbation Theory Rayleigh-Schrodinger (Chapter 10.1(all)) _Ed________

Degenerate Perturbation Theory Rayleigh-Schrodinger (Chapter 10.2) _Rachael_

Spin-Orbit, Thomas Precession and Relativity Corrections (Chapter 10.3(all)) _Elizabeth_

2nd-order degenerate theory (Chapter 10.9) _James_____

Zettili Problems:

1. **Z9-4 Given a Hamiltonian matrix(EVERYONE)**

2. **Z9-9 A perturbed oscillator energy to 2nd order(EVERYONE)**

Boccio Problems:

1. **Box with a "Sagging Bottom"(Ben)** - Consider a particle in a 1-dimensional box with a "sagging bottom" given by

$$V(x) = \begin{cases} -V_0 \sin(\pi x / L) & \text{for } \; 0 \le x \le L \\ \infty & \text{for } \; x < 0 \;\; and \;\; x > L \end{cases}$$

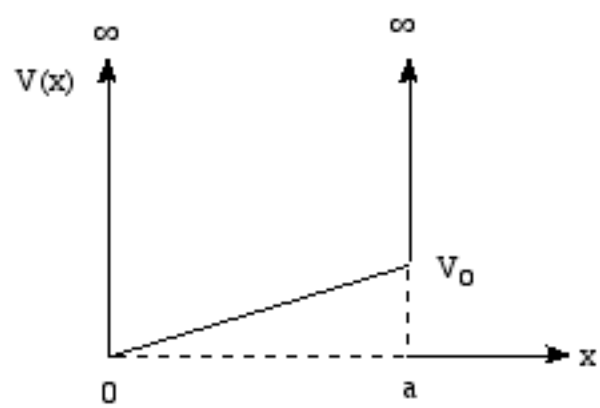
(a) For small V_0 this potential can be considered as a small perturbation of a "box" with a flat bottom, for which we have already solved the Schrodinger equation. What is the perturbation potential?

(b) Calculate the energy shift due to the sagging for the particle in the nth stationary state to first order in the perturbation.

2. Perturbing the Infinite Square Well(Elizabeth) - Calculate the first order energy shift for the first three states of the infinite square well in one dimension due to the perturbation

$$V(x) = V_0 \frac{x}{a}$$

as shown in the figure

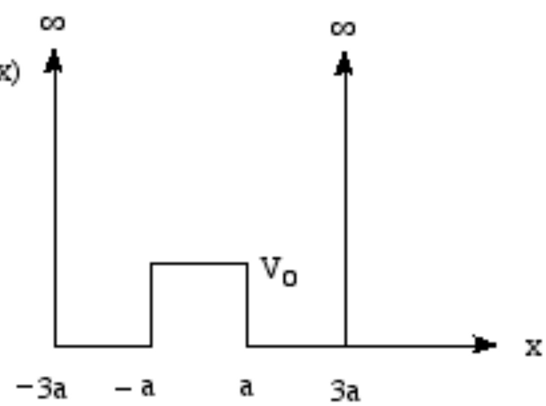


3. Weird Perturbation of an Oscillator(Max) - A particle of mass m moves in one dimension subject to a harmonic oscillator potential $\frac{1}{2} m\omega^2 x^2$. The particle is perturbed by an additional weak anharmonic force described by the potential $\Delta V = \lambda \sin kx$, $\lambda << 1$. Find the corrected ground state.

4. Perturbing the Infinite Square Well Again(Jen) - A particle of mass m moves in a one dimensional potential box

$$V(x)=\begin{cases}\infty & \text{for } |x|>3a\\ 0 & \text{for } a<\text{x}<3\text{a}\\ 0 & \text{for } -3\text{a}<\text{x}<\text{-a}\\ V_0 & \text{for } |x|<a\end{cases}$$

as shown in the figure



Use first order perturbation theory to calculate the new energy of the ground state.

5. Perturbing the 2-dimensional Infinite Square Well(Sandy)
Consider a particle in a 2-dimensional infinite square well given by

$$V(x)=\begin{cases}0 & \text{for } 0\le x\le a\,,0\le y\le a\\ \infty & \text{otherwise}\end{cases}$$

(a) What are the energy eigenvalues and eigenkets for the three lowest levels?

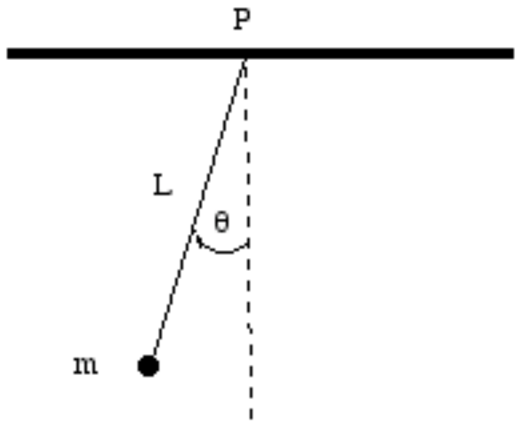
(b) We now add a perturbation given by

$$V_1(x)=\begin{cases}\lambda\text{xy} & \text{for } 0\le x\le a\,,0\le y\le a\\ 0 & \text{otherwise}\end{cases}$$

Determine the first order energy shifts for the three levels.

(c) Draw an energy diagram with and without the perturbation for the three energy states, Make sure to specify which unperturbed state is connected to which perturbed state.

6. **Not So Simple Pendulum(Ed) -** A mass m is attached by a massless rod of length L to a pivot P and swings in a vertical plane under the influence of gravity as shown in the figure below



(a) In the small angle approximation find the quantum mechanical energy levels of the system.
(b) Find the lowest order correction to the ground state energy resulting from the inaccuracy of the small angle approximation.

7. **1-Dimensional Anharmonic Oscillator(Rachael) -** Consider a particle of mass m in a 1-dimensional anharmonic oscillator potential with potential energy

$$V(x) = \frac{1}{2}m\omega^2 x^2 + \alpha x^3 + \beta x^4$$

(a) Calculate the 1st-order correction to the energy of the n^{th} perturbed state. Write down the energy correct to 1^{st} order.
(b) Evaluate all the required matrix elements of x^3 and x^4 to determined the perturbed energy levels and the the wave function of the n^{th} state perturbed to 1^{st} order.

8. Relativistic Correction for Harmonic Oscillator(Max) - A particle of mass m moves in a 1-dimensional oscillator potential

$$V(x) = \frac{1}{2} m\omega^2 x^2$$

In the nonrelativistic limit, where the kinetic energy and the momentum are related by

$$T = \frac{p^2}{2m}$$

the ground state energy is well known to be $E_0 = \frac{1}{2}\hbar\omega$.

Relativistically, the kinetic energy and the momentum are related by

$$T = E - mc^2 = \sqrt{m^2c^4 + p^2c^2} - mc^2$$

(a) Determine the lowest order correction to the kinetic energy (a p^4 term).

(b) Consider the correction to the kinetic energy as a perturbation and compute the relativistic correction to the ground state energy.

9. Degenerate perturbation theory on a spin = 1 system(Sandy)
Consider the spin Hamiltonian for a system of spin = 1

$$\hat{H} = A\hat{S}_z^2 + B(\hat{S}_x^2 - \hat{S}_y^2) \quad , \quad B << A$$

This corresponds to a spin=1 ion located in a crystal with rhombic symmetry.

(a) Solve the unperturbed problem for $\hat{H}_0 = A\hat{S}_z^2$.

(b) Find the perturbed energy levels to first order.

(c) Solve the problem exactly by diagonalizing the Hamiltonian matrix in some basis. Compare to perturbation results.

10. **Perturbation Theory in Two-Dimensional Hilbert Space(James)**
Consider a spin-1/2 particle in the presence of a static magnetic field along the z and x directions,

$$\vec{B} = B_z \hat{e}_z + B_x \hat{e}_x$$

(a) Show that the Hamiltonian is

$$\hat{H} = \hbar\omega_0 \hat{\sigma}_z + \frac{\hbar\Omega}{2}\hat{\sigma}_x$$

where $\hbar\omega_0 = \mu_B B_z$ and $\hbar\Omega = 2\mu_B B_x$.

(b) If $B_x = 0$, the eigenvectors are $\left|\uparrow_z\right\rangle$ and $\left|\downarrow_z\right\rangle$ with eigenvalues $\pm\hbar\omega_0$ respectively. Now turn on a weak x field with $B_x \ll B_z$. Use perturbation theory to find the new eigenvectors and eigenvalues to lowest order in B_x / B_z.

(c) Suppose now $B_z = 0$. What are the eigenvectors and eigenvalues in terms of $\left|\uparrow_z\right\rangle$ and $\left|\downarrow_z\right\rangle$. Relate this to degenerate perturbation theory.

(d) This problem can actually be solved exactly. Find the eigenvectors and eigenvalues for all $\vec{B}$. Show that these agree with your results in parts (b) and (c) by taking appropriate limits.

(e) Plot the energy eigenvalues as a function of B_z for fixed B_x. Label the eigenvectors on the curves when $B_z = 0$ and $B_z \to \pm\infty$.

11. Finite Spatial Extent of the Nucleus(EVERYONE) - In most discussions the nucleus is treated as a positively charged point particle. In fact, the nucleus does possess a finite size with a radius given approximately by the empirical formula

$$R \approx r_0 A^{1/3}$$

where $r_0 = 1.2\times 10^{-13} cm$ (i.e., 1.2 fermi) and A is the atomic weight (essentially the number of protons and neutrons in the nucleus). A reasonable assumption is to take the total nuclear charge +Ze as being uniformly distributed over the entire nuclear volume (assumed to be a sphere).

(a) Derive the following expression for the electrostatic potential energy of an electron in the field of the "finite" nucleus:

$$V(r) = \begin{cases} -\dfrac{Ze^2}{r} & r > R \\ \dfrac{Ze^2}{R}\left(\dfrac{r^2}{2R^2} - \dfrac{3}{2}\right) & r < R \end{cases}$$

Draw a graph comparing this potential energy and the point nucleus potential energy.

(b) Since you know the solution of the point nucleus problem, choose this as the unperturbed Hamiltonian $\hat{H}_0$ and construct a perturbation Hamiltonian $\hat{H}_1$ such that the total Hamiltonian contains the $V(r)$ derived above. Write an expression for $\hat{H}_1$.

(c) Calculate the 1st-order perturbed energy for the 1s (nlm)=(100) state obtaining an expression in terms of Z and fundamental constants. How big is this result compared to the ground state energy of hydrogen?

Remember $R \ll a_0 = Bohr\ radius$

12. Spin-Oscillator Coupling(James) - Consider a Hamiltonian describing a spin-1/2 particle in a harmonic well as given below:

$$\hat{H}_0 = \frac{\hbar\omega}{2}\hat{\sigma}_z + \hbar\omega\left(\hat{a}^+\hat{a} + 1/2)\right)$$

(a) Show that $\left\{|n\rangle \otimes |\downarrow\rangle = |n,\downarrow\rangle, |n\rangle \otimes |\uparrow\rangle = |n,\uparrow\rangle\right\}$ are energy eigenstates with eigenvalues $E_{n,\downarrow} = n\hbar\omega$ and $E_{n,\uparrow} = (n+1)\hbar\omega$, respectively.

(b) The states associated with the ground-state energy and the first excited energy level are $\left\{|0,\downarrow\rangle, |1,\downarrow\rangle, |0,\uparrow\rangle\right\}$. What is(are) the ground state(s)? What is(are) the first excited state(s)? Note: two states are degenerate.

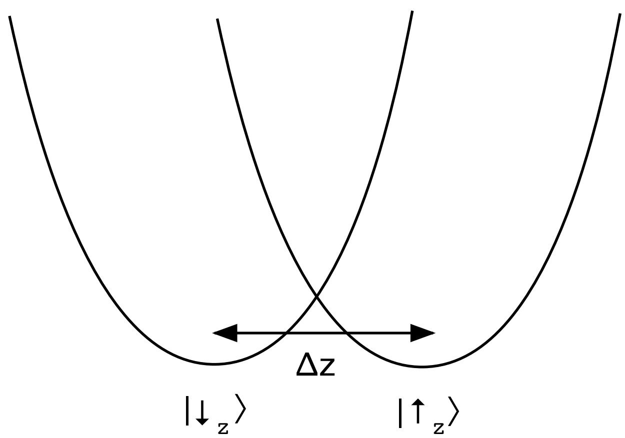
(c) Now consider adding an interaction between the motion and the spin, described by the Hamiltonian

$$\hat{H}_1 = \frac{\hbar\Omega}{2}\left(\hat{a}\hat{\sigma}_+ + \hat{a}^+\hat{\sigma}_-\right)$$

so that the total Hamiltonian is now $\hat{H} = \hat{H}_0 + \hat{H}_1$. Write a matrix representation of $\hat{H}$ in the subspace of the ground and first excited states in the ordered basis given in part (b).

(d) Find the first order correction to the ground state and excited state energy eigenvalues for the subspace above.

13. Motion in spin-dependent traps(Jen) - Consider an electron moving in one dimension, in a spin-dependent trap as shown below:



If the electron is in a spin-up state (with respect to the z-axis), it is trapped in the right harmonic oscillator well and if it is in a spin-down state (with respect to the z-axis), it is trapped in the left harmonic oscillator well. The Hamiltonian that governs its dynamics can be written as:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega_{osc}^2(\hat{z} - \Delta z/2)^2 \otimes \left|\uparrow_z\right\rangle\left\langle\uparrow_z\right| + \frac{1}{2}m\omega_{osc}^2(\hat{z} + \Delta z/2)^2 \otimes \left|\downarrow_z\right\rangle\left\langle\downarrow_z\right|$$

(a) What are the energy levels and stationary states of the system? What are the degeneracies of these states? Sketch an energy level diagram for the first three levels and label the degeneracies.

A small, constant "transverse field" B_x is now added with

$$\left|\mu_B B_x\right| << \hbar\omega_{osc}$$

(b) Qualitatively sketch how the energy plot in part (a) is modified.

(c) Now calculate the perturbed energy levels for this system

(d) What are the new eigenstates in the ground-state doublet? For Δz macroscopic, these are sometimes called Schrodinger cat states. Explain why.