Physics 131 **Spring 2009**

Particle Physics Seminar #13

Textbook: Griffiths - Introduction to Elementary Particles
Website: (all notes referred to below are on web site)

http://chaos.swarthmore.edu/courses/Phys131_2007/index.html

Topic(s):

(1) Quantum Electrodynamics

Problems:

EP-15 Noether's theorem (translation)
EP-16 Symmetries and Conservation Laws
EP-17 Maxwell's equations
EP-18 The Proca equation
EP-19 Compton effect - spinless particles
EP-20 Scalar electrodynamics
EP-21 Proton decay

EP-15 - Noether's theorem (for translations)

(a) Assume a space-time translation: $x^\mu \to x'^\mu = x^\mu + a^\mu$, with infinitesimal displacement a_μ . By equating $\delta L(\phi, \partial_\mu \phi) = a^\mu \partial_\mu L$ to the differential development of δL in ϕ and $\partial_\mu \phi$, show that that there exists a conserved tensor

$$T^{\mu\nu} = \frac{\partial L}{\partial(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} L$$

It is called the stress-energy-momentum tensor.

HINT: Use the Euler-Lagrange equations to simplify δL and remember that L here is the Lagrangian density.

(b) Show that T^{00} is conserved (assume that $T^{0i}, i = 1,2,3$ vanish at ∞). What conserved quantity is represented by T^{00}?

This problem is a proof of **Noether's theorem** for translation. The general statement is:

To every continuous symmetry(invariance) of a

field theory Lagrangian, there corresponds a conserved quantity

EP-16 Symmetries and Conservation Laws

Here we look again at EP-14 (Scalar Electrodynamics) where we studied some of the properties of the dynamics of a charged (complex) scalar field $\phi(x)$ coupled to the electromagnetic field $A_\mu(x)$. Recall that the Lagrangian density for this system is

$$L = \left(D_\mu \phi(x)\right)^* \left(D_\mu \phi(x)\right) - m_0^2 \left|\phi(x)\right|^2 - \frac{\lambda}{4!}\left(\left|\phi(x)\right|^2\right)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

where D_μ is the covariant derivative

$$D_\mu = \partial_\mu + ieA_\mu$$

e is the electric charge and * denotes complex conjugation.

(a) Derive an expression for the locally conserved current $j_\mu(x)$, associated with the global symmetry

$$\phi(x) \to \phi'(x) = e^{i\theta}\phi(x)$$
$$\phi^*(x) \to \phi^*{}'(x) = e^{-i\theta}\phi^*(x)$$
$$A_\mu(x) \to A'_\mu(x) = A_\mu(x)$$

in terms of the fields of the theory.

(b) Show that the conservation of j_μ implies the existence of a constant of motion. Find an explicit form for this constant of motion.

(c) Consider now the case of the local (or gauge) transformation

$$\phi(x) \to \phi'(x) = e^{i\theta(x)}\phi(x)$$
$$\phi^*(x) \to \phi^*{}'(x) = e^{-i\theta(x)}\phi^*(x)$$
$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x)$$

where $\theta(x)$ and $\Lambda(x)$ are two functions. What should be the relation between $\theta(x)$ and $\Lambda(x)$ for this transformation to be a symmetry of the Lagrangian of the system?

(d) Show that, if the system has the local symmetry of part (c), there is a locally conserved gauge current $J_\mu(x)$. Find an explicit expression for J_μ and discuss in which way it is different from the current j_μ of part(a). Find an explicit expression for the associated constant of motion and discuss its physical meaning.

(e) Find the energy-momentum tensor $T^{\mu\nu}$ for this system Show that it can be written as the sum of two terms

$$T^{\mu\nu} = T^{\mu\nu}(A) + T^{\mu\nu}(\phi, A)$$

where $T^{\mu\nu}(A)$ is the energy-momentum tensor for the free electromagnetic field and $T^{\mu\nu}(\phi,A)$ is the tensor which results by modifying the energy-momentum tensor for the decoupled complex scalar field ϕ by the **minimal coupling** procedure.

(f) Find explicit expression for the Hamiltonian $\tilde{H}(x)$ and the linear momentum $\tilde{P}(x)$ densities for this system. Give a physical interpretation for all of the terms that you found for each quantity.

(g) Consider now the case of an infinitesimal Lorentz transformation

$$x_\mu \to x'_\mu = x_\mu + \omega_{\mu\nu} x^\nu$$

where $\omega_{\mu\nu}$ is infinitesimal and antisymmetric. Show that the invariance of the Lagrangian of this system under these Lorentz transformations leads to the existence of a conserved tensor $M_{\mu\nu\lambda}$. Find an explicit form for this tensor. Give an interpretation for its spatial components. Does the conservation of this tensor impose any restriction on the properties of the energy-momentum tensor $T^{\mu\nu}$? Explain.
WARNING: Be very careful in how you treat the fields. Not all the fields are scalars!

EP-17 - Maxwell's Equations (see my notes)

Consider the Lagrangian

$$L_1 = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \qquad \text{i.e.,} \qquad S[A] = \int d^4x\, L_1$$

where

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

(a) Treating A_μ as the field, find the equations of motion.

(b) Show that

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

This is called a "Bianchi identity".

(c) Define

$$F_{0i} \equiv E_i \quad , \quad F_{ij} \equiv \varepsilon_{ijk} B^k$$

Obtain the standard Maxwell equations.

EP-18 - The Proca equation - The Proca equation for a **massive** photon is

$$\left(\partial^\nu\partial_\nu + m^2\right)A^\mu = 4\pi J^\mu$$

where $A^\mu = (A^0, \vec{A})$ is the 4-potential and $J^\mu = (\rho, \vec{J})$ is the 4-current. Suppose that the 4-current consists of a single electron at rest at the origin, with $\rho = \delta(\vec{r})$ and $\vec{J} = 0$. Show that, in units $\hbar = c = 1$, the time-independent solution to this equation is

$$A^0 = \frac{C}{r}e^{-mr}$$

where C is a normalization constant that you do not need to worry about.

Suppose that $m = 10^{-19}eV$. Insert appropriate factors of $\hbar$ and c into the exponential, then calculate the distance at which A^0 differs from a Coulomb potential by a factor 1/e.

EP-19 Compton effect - spinless particles

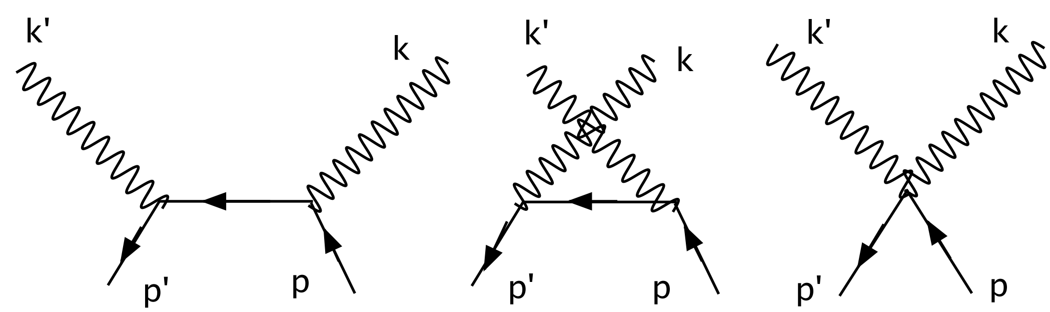
Consider a free scalar field, with Lagrangian

$$L_0 = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi$$

Its interaction with the electromagnetic field is given by minimal coupling, i.e., by the prescription

$$\partial_\mu\phi \to \left(\partial_\mu - ieA_\mu\right)\phi$$

(a) Derive the amplitude for the process $\gamma + \phi \to \gamma + \phi$, to order e^2 in perturbation theory using the three diagrams below.



(b) Calculate the energy distribution of ϕ particles in the laboratory frame, after they have undergone Compton scattering. Show that this distribution differs from that of spin-1/2.

(c) Compare the angular distributions of the processes

$$e^+ + e^- \to \mu^+ + \mu^- \quad , \quad e^+ + e^- \to \phi + \phi^*$$

Extra Problem 20 - Scalar Electrodynamics

The dynamics of a **charged** (complex) scalar field $\phi(x)$ coupled to the electromagnetic field $A_\mu(x)$ is governed by the Lagrangian density $\mathscr{L}$.

$$\mathscr{L} = \frac{1}{2}\left(D_\mu \phi(x)\right)^* \left(D_\mu \phi(x)\right) - \frac{m_0^2}{2}|\phi(x)|^2 - \frac{\lambda}{4!}\left(|\phi(x)|^2\right)^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

where D_μ is the **covariant derivative**

$$D_\mu = \partial_\mu + ieA_\mu$$

e is the electric charge and * denotes complex conjugation.

(1) Show that this Lagrangian density is invariant under the local gauge transformations

$$\phi'(x) = \phi(x)e^{-ie\Lambda(x)}$$

$$\phi'^*(x) = \phi^*(x)e^{+ie\Lambda(x)}$$

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x)$$

(2) Derive the classical equations of motion in a manifestly relativistically covariant form.

(3) Find the Hamiltonian density for this system.

(4) Write the complex field $\phi(x)$ in its polar components

$$\phi(x) = \rho(x)e^{i\theta(x)}$$

and find the equations of motion obeyed by the **real** fields $\rho(x)$ and $\theta(x)$. The gauge fixing condition (London gauge: $\partial_\mu A'^\mu = 0$) is:

$$\partial_\mu A^\mu(x) + \frac{1}{e}\partial_\mu \partial^\mu \theta(x) = 0$$

Write these equations of motion in the gauge $\theta = 0$, known as the London or Unitary gauge. Find the Lagrangian for the field $\rho(x)$.

(5) Determine the equations of motion for $\rho(x)$ and $A^\mu(x)$ (derived from the effective Lagrangian; note that "gauge fixing" allows you to absorb phase factors into the potential).

Use the equation of motion for $A^\mu(x)$ to show that the coefficient of this quadratic term can be interpreted as a **photon mass**.

Extra Problem 21 - This problem asks you to do an "order of magnitude" calculation of proton decay. You do not need to do detailed calculations, but should be able to estimate things using the techniques of Chapter 6.

(A) Draw the Feynman diagram for the weak decay $\pi^+ \to \mu^+ + \nu_\mu$ (see section 2.4 if you need help).

(B) In the Grand Unified Theory, there exists a supermassive gauge boson called X. X is hypothesized to have charge -4/3 and can decay as $X \to d + e^-$ or as $X \to \bar{u} + \bar{u}$. By drawing the appropriate Feynman diagram, show that the existence of the X particle permits the proton decay mode $p \to \pi^0 + e^+$.

(C) The lifetime of the π^+ is $2 \times 10^{-8}\,\mathrm{sec}$. Assume that the X particle has a mass of $10^{14} GeV$ and that its coupling constants to quarks and leptons are of the same magnitude as the couplings of the weak force carriers to quarks and leptons. Do an order of magnitude calculation of the lifetime of the proton by scaling the pion's lifetime appropriately using the amplitudes of the matrix elements for the two decays. (HINT: consider the form of the propagator for a massive force carrier, as in Chapter 6; ignore phase space factors and the spins of the particles, and differences in the internal structure of a pion and a proton).