Uncertainty Analysis

Uncertainty analysis is a sometimes tedious but important part of modern science. We divide our discussion into three sections: first, we address methods for determining uncertainties; second, we talk about how these uncertainties propagate into our calculated results; and third, we discuss issues of presentation.

Uncertainty Estimates

Individual Measurements: For stable measurements, use one-half of the smallest measurement division available, or, if the precision of a particular device is explicitly given, use that value instead. When measurements fluctuate while they are being made, use your best estimate of the range of the instability, or, if possible, repeat the measurement several times as described immediately below.

Repeated Measurements: when practical it is best to repeat unstable measurements several times. When this is done, the *mean* and *standard deviation of the mean* should be used to estimate the true value and its uncertainty. For *N* measurements (x_1, x_2, x_3, … , x_n), the mean <x> is

$$\langle x \rangle = \frac{1}{n}\sum_{i=1}^{n} x_i .$$

The mean is our best estimate of the (unknown) true value of x. The *deviation of the sample* is

$$\sigma_{n-1} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \langle x \rangle\right)^2} ,$$

which is the spread of *individual* measurements about <x>. The *standard deviation of the mean* is

$$\sigma_m = \frac{\sigma_{n-1}}{\sqrt{n}} .$$

This estimates the spread of similar *average* measurements about the true value of x. We use it for the uncertainty in <x>. Compare to the “normal” *standard deviation*

$$\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(x_i - x_0\right)^2} ,$$

defined for the Gaussian probability distribution centered on x_0,

$$P(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-x_0)^2/2\sigma^2}.$$

Here the true mean x_0 is known *a priori,* but in an experiment, this is not the case. We have to measure the mean *and* the standard deviation about it. Since the spread about a *known* mean x_0 is probably smaller than the spread about a *measured* mean <x>, which only approximates the true real value, we use n-1 instead of n in the denominator, giving a slightly larger deviation. In statistical language, we say that "one degree of freedom is subsumed into the mean." In fact definitions with both n and n-1 are common, but we will keep to the more conservative form of σ_{n-1}.

Propagation of Uncertainty

A number of different rules are typically quoted for the propagation of uncertainty, but there is one fundamental relation from which one can derive the rest. Unfortunately this depends upon multivariable analysis, but we'll illustrate it with just one example. For $y = f(x)$ and for small uncertainties, the uncertain in y (Δy) is related to the uncertainty in x (Δx) as follows:

$$\Delta y = \left|\frac{dy}{dx}\right| \Delta x.$$

Multivariable quantities are easily treated if we assume that the individual uncertainties are independent, small, and follow a Gaussian distribution, in which case they add in *quadrature.* For $Q = f(x,y,z\ldots)$, we get, the uncertainty in Q (ΔQ) is

$$\Delta Q = \sqrt{\left|\frac{\partial Q}{\partial x}\right|^2 (\Delta x)^2 + \left|\frac{\partial Q}{\partial y}\right|^2 (\Delta y)^2 + \left|\frac{\partial Q}{\partial z}\right|^2 (\Delta z)^2 + \cdots}.$$

This generates some more familiar specialized rules. For instance when $Q = \pm ax \pm by \pm cz\ldots$

$$\Delta Q = \sqrt{a^2(\Delta x)^2 + b^2(\Delta y)^2 + c^2(\Delta z)^2 + \cdots}.$$

When $Q = a\, x^l\, y^m\, z^n\ldots$

$$\frac{\Delta Q}{Q} = \sqrt{l^2\left(\frac{\Delta x}{x}\right)^2 + m^2\left(\frac{\Delta y}{y}\right)^2 + n^2\left(\frac{\Delta z}{z}\right)^2 + \cdots}.$$

Note that this relationship uses the *relative* error, $\Delta Q/Q$, so the constant a drops out. The relative error is dimensionless and often expressed as a percent. The relationship before this last one, on the other hand, gives the *absolute* error, with the same dimension as Q. The relative error can be converted to absolute error (or vice-versa) by multiplying (dividing) by Q. These last two relationships can be used for most simple cases that come up in experiments; for more complicated problems, however, we have to go back to the first relationship of this section.

In addition to the standard analytical approach above, graphical and computer methods of uncertainty estimation are also used. Usually this means functional fits, which provide parameters and their associated uncertainties from a set of data points. The technique is obviously a bit more involved than the simple propagation rules we derive above, but the basic principles are the same. The details are in any case hidden in software, and we will simply apply the results. Lastly, it is often possible to identify one or two primary contributions to the error in the final result and ignore all other contributions. This can save a great deal of calculation. The uncertainty in physical constants like π, for instance, is much less than the uncertainty in typical experimental measurements, and often one or two uncertainties dominate the uncertainties in the experimental measurements. We focus on just these whenever possible, but we always justify this choice with a short explanation.

Presentation of Uncertainties

Uncertainties are typically quoted to one or two places and the result given to the same absolute precision. There are many equivalent formats, one of which is as follows:

$$L = (12.4 \pm 0.2)\ \text{m}$$

We report the *error* (the difference between experiment and theory) in units of the total absolute uncertainty, which we usually call *sigma* (σ), whether it is σ_{n-1}, σ, or just an estimated quantity. Suppose, for example, that the speed of light is measured to be $c = (3.22 \pm 0.20) \times 10^8$ m/s. The accepted value is 3.00×10^8 m/s. The error is 0.22×10^8 m/s, 1.1 times the uncertainty, or 1.1 σ. In a Gaussian distribution, about 68% of the data fall within one standard deviation of the mean, so this is a perfectly fine result; we expect only about 2/3 of experiments to do better. If the error were 3σ or larger, however, we might be more concerned.