Ampere's Law Workshop

Ampere's Law

$$\oint \vec{B}(\vec{r}) \bullet d\vec{s} = \frac{4\pi}{c} \sum_i I_i = \frac{4\pi}{c} \iint \vec{J}(\vec{r}) \bullet d\vec{a} \quad \text{(cgs units)}$$

$$\oint \vec{B}(\vec{r}) \bullet d\vec{s} = \mu_0 \sum_i I_i = \mu_0 \iint \vec{J}(\vec{r}) \bullet d\vec{a} \quad \text{(mks units)}$$

Stokes' Theorem

$$\oint_C \vec{F}(\vec{r}) \bullet d\vec{s} = \iint_S \left(\nabla \times \vec{F}(\vec{r})\right) \bullet d\vec{a}$$

$\vec{F}(\vec{r})$ is a general vector function and C is the curve that bounds the surface S.

Differential Form of Ampere's Law

Let $\vec{F}(\vec{r})$ in Stokes theorem be a magnetic field $\vec{B}(\vec{r})$. Since the surface S is completely arbitrary and can be as small as one likes,

$$\nabla \times \vec{B}(\vec{r}) = \frac{4\pi}{c} \vec{J}(\vec{r}) \quad \text{(cgs units)}$$

$$\nabla \times \vec{B}(\vec{r}) = \mu_0 \vec{J}(\vec{r}) \quad \text{(mks units)}.$$

Problems

1. Consider the vector field $\vec{F} = \hat{j} F_0 e^{-y^2/\lambda^2}$.

   (a) Sketch a set of vectors in the x-y plane that show the magnitude and direction of this field everywhere.

   (b) Do you expect $\nabla \times \vec{F}$ to be zero or nonzero? (Could this field be an electrostatic field?)

   (c) Calculate $\nabla \times \vec{F}$. Is the direction of $\nabla \times \vec{F}$ consistent with the direction of the circulation of the field around a closed path? Does the magnitude make sense physically?

2. Consider the vector field $\vec{F} = \hat{j} F_0 e^{-x^2/\lambda^2}$.

   (a) Sketch a set of vectors in the x-y plane that show the magnitude and direction of this field everywhere.

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(c) Calculate $\nabla \times \vec{F}$. Is the direction of $\nabla \times \vec{F}$ consistent with the direction of the circulation of the field around a closed path? Does the magnitude make sense physically?

3. (a) A solid metal cylinder of radius R carries a current I. Find an expression for the magnetic field everywhere (inside as well as outside the cylinder) assuming that the current density is constant throughout the cylinder. Then take the curl of this magnetic field to make sure that you recover the correct current density everywhere in space.
   (b) A vector potential $\vec{A}(\vec{r})$ can be defined, the curl of which is the magnetic field. This is analogous to the electric potential, the negative gradient of which is the electric field. The vector potential for this solid metal cylinder can be written as follows:
$$\vec{A}(x,y,z) = -\frac{I}{cR^2}\left(x^2 + y^2\right)\hat{k} \qquad r < R$$
$$\vec{A}(x,y,z) = -\frac{I}{c}\ln\left(x^2 + y^2\right)\hat{k} \qquad r > R$$
   Show that the $\nabla \times \vec{A} = \vec{B}$ everywhere.
   (c) A useful relationship for the curl of a curl of a vector is
$$\nabla \times \left(\nabla \times \vec{A}\right) = \nabla\left(\nabla \bullet \vec{A}\right) - \nabla^2\vec{A},$$
   where the meaning of the last term is $\nabla^2\vec{A} = \left(\nabla^2 A_X\right)\hat{i} + \left(\nabla^2 A_Y\right)\hat{j} + \left(\nabla^2 A_Z\right)\hat{k}$. When this relationship is combined with the differential form of Ampere's law, one obtains
$$\nabla^2\vec{A}(\vec{r}) = -\frac{4\pi}{c}\vec{J}(\vec{r}).$$
   Verify that this relationship is true everywhere.

4. For the vector field $\vec{F}(\vec{r}) = \left(2xz + 3y^2\right)\hat{j} + 4yz^2\hat{k}$, compute
$$\oint_C \vec{F}(\vec{r}) \bullet d\vec{s} = \iint_S \left(\nabla \times \vec{F}(\vec{r})\right) \bullet d\vec{a} \ ,$$
where the closed path C is the path bounding the unit square in the y-z plane with one corner at the origin.