Physics 14 Quantum Physics Assignment #7 Spring 2005

Readings for week #3 of Quantum Mechanics

Pages 57-84 in Boccio Wave Mechanics Notes
Chapter 2 in French Textbook

Exercises #5, #6, #8, and #10 at end of Lecture Notes

[5] For $f(x) = \begin{cases} 0 & \text{for} \quad x < 0 \text{ and } x > b \\ 1/\sqrt{b} & \text{for} \quad 0 < x < b \end{cases}$

(a) Find the Fourier series of $f(x)$ in the interval $[-a/2, a/2]$ where $a/2 > b$
(b) Find the Fourier transform of $f(x)$
(c) Explicitly relate the Fourier transform of $f(x)$ to the Fourier coefficients c_n when $a \to \infty$

[6] Let $f(\vec{x}) = \begin{cases} 0 & \text{for} \quad r > a \\ C & \text{for} \quad r \le a \end{cases}$ Find the Fourier transform $F(\vec{k})$

[8] Find $\lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\varepsilon^2 + x^2} = \begin{cases} 0 & \text{for } x \ne 0 \\ \infty & \text{for } x = 0 \end{cases}$ in terms of $\delta(x)$

[10] Using the uncertainty principle, estimate how long a time a pencil can be balanced on its point.

Textbook Problems:

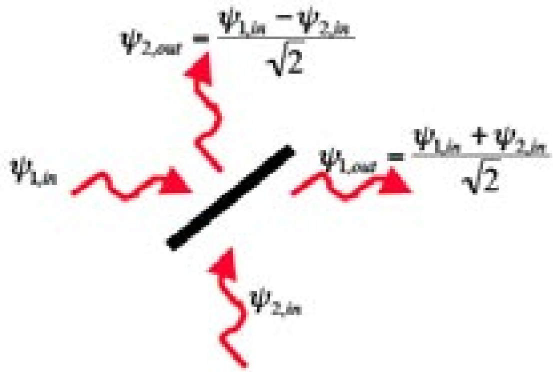
2-02 Group velocity of localized waves
2-15 Bohr atom derived from de Broglie's relation
2-16 Hydrogen: a structure of minimum energy

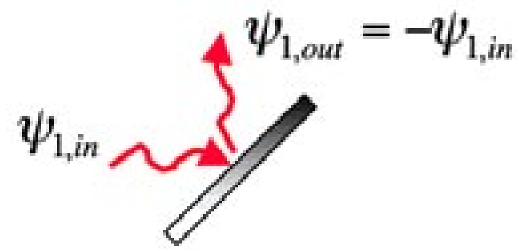
Challenging Problems:

1. The Mach-Zender Interferometer and Quantum Interference

Background information: Consider a single photon incident on a 50-50 beam splitter (that is, a partially transmitting, partially reflecting mirror, with equal coefficients). Whereas classical electromagnetic energy divides equally, the photon is indivisible. That is, if a photon-counting detector is placed at each of the output ports (see figure below), only **one** of them clicks. Which one clicks is completely random (that is, we have no better guess for one over the other).

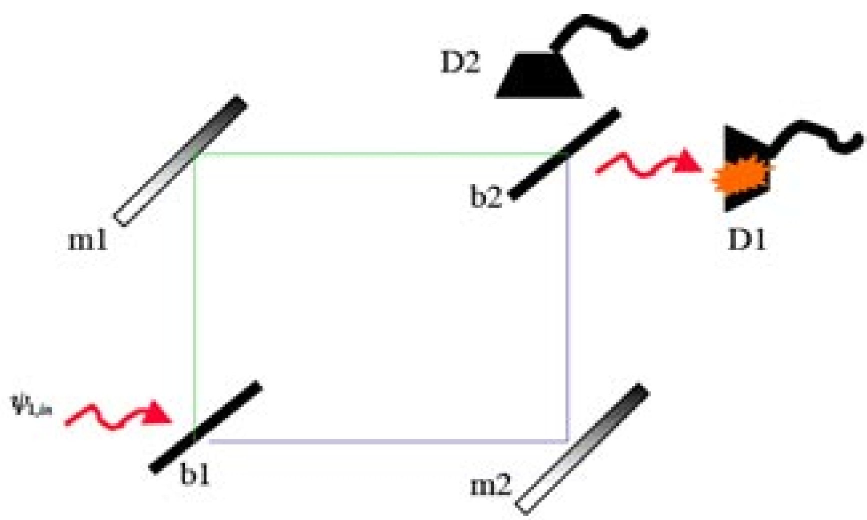
The input-output transformation of the waves incident on 50-50 beam splitters and perfectly reflecting mirrors are shown in the figure below.





(a) Show that with these rules, there is a 50-50 chance of either of the detectors shown in the first figure above to click.

(b) Now we set up a Mach-Zender interferometer(shown below):



The wave is split at beam-splitter b1, where it travels either path b1-m1-b2(call it the green path) or the path b1-m2-b2 (call it the blue path). Mirrors are then used to recombine the beams on a second beam splitter, b2. Detectors D1 and D2 are placed at the two output ports of b2.

Assuming the paths are perfectly balanced (that is equal length), show that the probability for detector D1 to click is 100% - **no randomness!**

(c) Classical logical reasoning would predict a probability for D1 to click given by

$$P_{D1} = P(transmission\ at\ b1 \mid green\ path)P(green\ path) + P(reflection\ at\ b2 \mid blue\ path)P(blue\ path)$$

Calculate this and compare to the quantum result. **Explain.**

(d) **Extra Credit:** How would you set up the interferometer so that detector D2 clicked with 100% probability? How about making them click at random? Leave the **basic geometry the same**, that is, do not change the direction of the beam splitters or the direction of the incident light.

(2) **The Poisson Probability Distribution**

The arrival time of rain drops on the roof or photons from a laser beam on a detector are completely random, with no correlation from count to count. If we count for a certain time interval we won't always get the same number - it will fluctuate from shot-to-shot. This kind of noise is sometimes known as "shot noise" or counting statistics.

Suppose the particles arrive at an average rate R. In a small time interval $\Delta t << 1/R$ no more than one particle can arrive. We seek the probability for n particles to arrive after a time t, $P(n,t)$.

(a) Show that the probability to detect zero photons exponentially decays, that is, $P(0,t) = e^{-Rt}$.

(b) Obtain the differential equation as a recursion relation

$$\frac{d}{dt}P(n,t) + RP(n,t) = RP(n-1,t)$$

(c) Solve this to find the Poisson distribution, $P(n,t) = \frac{(Rt)^n}{n!}e^{-Rt}$.

Plot a histogram for $Rt = 0.1\,, 1.0\ and\ 10.0$/

(d) Show that the mean and standard deviation in the number of counts are:

$$\langle n \rangle = Rt \quad , \quad \sigma_n = \sqrt{Rt} = \sqrt{\langle n \rangle}$$

HINT: to find the variance, consider $\langle n(n-1) \rangle$.

Fluctuations of this type, going as the square root of the mean are characteristic of counting statistics.