Physics 14 Quantum Physics Assignment #12 Spring 2005

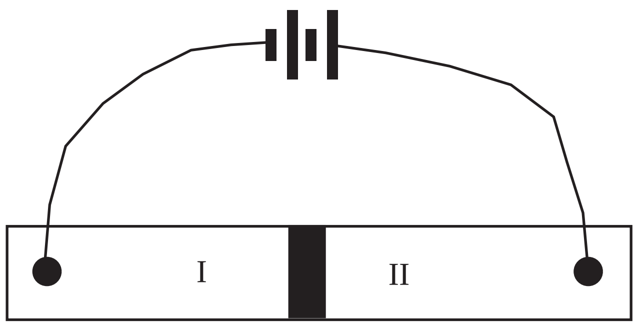
Readings for week #8 of Quantum Mechanics

Chapter 6 - Boccio Textbook
Chapters 6,7 in French

Exercises #38 and #40

[38] Josephson junctions

Consider superconducting metals I and II separated by a very thin insulating layer, such that the electron wave functions can overlap between the metals (this is a Josephson junction). A battery V is connected across the junction to ensure an average charge neutrality (see figure) .



This situation can be described by means of the coupled Schrodinger equations:

$$i\hbar\frac{\partial\psi_1}{\partial t} = U_1\psi_1 + K\psi_2 + K\frac{\psi_1\psi_2^*}{\psi_1^*}$$

$$i\hbar\frac{\partial\psi_2}{\partial t} = U_2\psi_2 + K\psi_1 + K\frac{\psi_2\psi_1^*}{\psi_2^*}$$

where ψ_1 and ψ_2 are the probability amplitudes for an electron in regions I and II respectively, U_1 and U_2 are the electric potential energies in regions I and II respectively, K is the coupling constant due to the thin insulating layer (related to wavefunction overlap), and

$$K\frac{\psi_1\psi_2^*}{\psi_1^*} \quad \text{and} \quad K\frac{\psi_2\psi_1^*}{\psi_2^*}$$

describe the battery as a source of electrons.

(a) Show that $\rho_1 = |\psi_1|^2$ and $\rho_2 = |\psi_2|^2$ are constant in time.
(b) Assuming that $\rho_1 = \rho_2 = \rho_0$ (same metals) and expressing the probability amplitudes in the form

$$\psi_1 = \sqrt{\rho_0}e^{i\theta_1} \quad , \quad \psi_2 = \sqrt{\rho_0}e^{i\theta_2}$$

find the differential equations for θ_1 and θ_2.
(c) Show that the battery current

$$I = \frac{K}{i\hbar}\left(\psi_1\psi_2^* - \psi_1^*\psi_2\right)$$

oscillates, and find the frequency of these oscillations.

[40] Neutrino Oscillations

It is generally recognized that there are at least three different kinds of neutrinos. They can be distinguished by the reactions in which the neutrinos are created or absorbed. Let us call these three types of neutrino ν_e, ν_μ and ν_τ. It has been speculated that each of these neutrinos has a small but finite rest mass, possibly different for each type. Let us suppose, for this exam question, that there is a small perturbing interaction between these neutrino types, in the absence of which all three types of neutrinos have the same nonzero rest mass M_0 . The Hamiltonian of the system can be written as

$$\hat{H} = \hat{H}_0 + \hat{H}_1$$

where

$$\hat{H}_0 = \begin{pmatrix} M_0 & 0 & 0 \\ 0 & M_0 & 0 \\ 0 & 0 & M_0 \end{pmatrix} \to \text{no interactions present}$$

and

$$\hat{H}_1 = \begin{pmatrix} 0 & \hbar\omega_1 & \hbar\omega_1 \\ \hbar\omega_1 & 0 & \hbar\omega_1 \\ \hbar\omega_1 & \hbar\omega_1 & 0 \end{pmatrix} \to \text{effect of interactions}$$

where we have used the basis

$$|\nu_e\rangle = |1\rangle \quad , \quad |\nu_\mu\rangle = |2\rangle \quad , \quad |\nu_\tau\rangle = |3\rangle$$

(a) First assume that $\omega_1 = 0$, i.e., no interactions. What is the time development operator? Discuss what happens if the neutrino initially was in the state

$$|\psi(0)\rangle = |\nu_e\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ or } |\psi(0)\rangle = |\nu_\mu\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ or } |\psi(0)\rangle = |\nu_\tau\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

What is happening physically in this case ?

(b) Now assume that $\omega_1 \neq 0$, i.e., interactions are present. Also assume that at $t = 0$ the neutrino is in the state

$$|\psi(0)\rangle = |\nu_e\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

What is the probability as a function of time, that the neutrino will be in each of the other two states ?

(c) An experiment to detect the "neutrino oscillations" is being performed. The flight path of the neutrinos is 2000 meters. Their energy is 100 GeV. The sensitivity of the experiment is such that the presence of 1% of neutrinos different from those present at the start of the flight can be measured with confidence. Let $M_0 = 20\,eV$. What is the smallest value of $\hbar\omega_1$ that can be detected ? How does this depend on M_0? Don't ignore special

Challenging Problems

3. Double Delta Function Potential Well

Consider a potential corresponding to two delta function wells

$$V(x) = -V_0\left[\delta(x+\frac{\ell}{2}) + \delta(x-\frac{\ell}{2})\right]$$

Because the potential is invariant under a parity reflection, the energy eigenfunctions are also eigenfunctions of parity. Thus, we can look for even/odd parity solutions of the Schrodinger equation.

(a) For even parity show that the form of the solution is

$$\psi_{even}(x) = \begin{cases} Be^{+\kappa x} & x < -\ell/2 \\ A(e^{+\kappa x} + e^{-\kappa x}) & -\ell/2 < x < +\ell/2 \\ Be^{-\kappa x} & x > +\ell/2 \end{cases}$$

$$\kappa = \sqrt{\frac{2mE_b}{\hbar^2}} \quad , E_b = -E = \text{binding energy}$$

(b) Use the appropriate boundary conditions to show that we get a transcendental equation for the energy of the form

$$\text{Even parity: } \frac{\kappa}{\kappa_0} = 1 + e^{-\kappa\ell} \quad , \quad \kappa_0 = \frac{mV_0}{\hbar^2}$$

(c) For odd parity show that the form of the solution is

$$\psi_{even}(x) = \begin{cases} -Be^{+\kappa x} & x < -\ell/2 \\ A(e^{+\kappa x} - e^{-\kappa x}) & -\ell/2 < x < +\ell/2 \\ Be^{-\kappa x} & x > +\ell/2 \end{cases}$$

$$\kappa = \sqrt{\frac{2mE_b}{\hbar^2}} \quad , E_b = -E = \text{binding energy}$$

(d) Use the appropriate boundary conditions to show that we get a transcendental equation for the energy of the form

$$\text{Odd parity: } \frac{\kappa}{\kappa_0} = 1 - e^{-\kappa\ell} \quad , \quad \kappa_0 = \frac{mV_0}{\hbar^2}$$

(e) As $\ell \to 0$, the solution goes to the expected case of a single delta function potential. Explain the limit $\ell \to \infty$.

(f) Devise a graphical method which represents the numerical solution to these equations by plotting both the left and right hand sides as a function of κ/κ_0.

(g) Describe the solutions that result using the graphical method. Show that there is only one solution if $\kappa_0\ell < 1$. Is it even parity or odd parity? Explain.