Quantum Mechanics Postulates (using the color and hardness world)

There are **five parts**(assumptions) to the **Quantum Mechanics algorithm**.

(A) Physical States

All physical systems are represented by ket vectors normalized to 1.

They are called "**state vectors**" → $|\psi\rangle$

This literally means ALL. An electron is represented by a vector. An atom is represented by a vector. A banana is represented by a vector. A car is represented by a vector. YOU are represented by a vector.

(B) Measurable Properties

Measurable properties(dynamical variables) of physical systems are called "**observables**".

Observables are represented by linear operators.

If the vector associated with a particular physical state $|\psi\rangle$ happens to be an eigenvector, with eigenvalue α, of an operator $\hat{A}$ associated with some particular measurable property of the system, i.e., if

$$\hat{A}|\psi\rangle = \alpha|\psi\rangle$$

then the system in that state **has** the value α of that particular measurable property.

That is, if you were to perform a measurement corresponding to $\hat{A}$ on a system in a state represented by $|\psi\rangle$, then with certainty (probability = 1) you would measure the value α.

Since the eigenvalues of operators representing observables are supposed to be measurable numbers, they must also always be **real**.

This means that we can only choose a certain kind of operator, namely, **HERMITIAN**, which is guaranteed to always have real eigenvalues.

This requirement produces an added bonus for quantum theory.

It turns out that the eigenvectors of any HERMITIAN operator always comprise a **complete, orthonormal set**.

This means that they always comprise a set of **mutually orthonormal vectors** that can be used as a **basis** (same number as the dimension of the space).

It also turns out that **any** HERMITIAN operator will represent a possible observable of the physical system.

Turning this statement around, we can say that the **eigenvectors of any observable can be used as the basis vectors** for our space.

This feature will have an important connection to measurement.

Example: back to color and hardness.......

Since the operators $\hat{H}$ and $\hat{C}$ that represent the observables **hardness** and **color** must be Hermitian, we can use either set of corresponding eigenvectors as a basis set for the quantum theory of color and hardness.

One such basis is then

$$|hard\rangle = |h\rangle \quad , \quad |soft\rangle = |s\rangle$$

where

$$\hat{H}|h\rangle = |h\rangle \to eigenvalue = 1 \quad (by\ definition)$$

$$\hat{H}|s\rangle = -|s\rangle \to eigenvalue = -1 \quad (by\ definition)$$

or, in words, $|h\rangle$ is a state with a measured value of hardness = 1 and $|s\rangle$ is a state with a measured value of hardness = -1.

Since these states form a basis (orthonormal set) they satisfy

$$\langle h|h\rangle = 1 = \langle s|s\rangle$$

$$\langle h|s\rangle = 0 = \langle s|h\rangle$$

The operator $\hat{H}$ represents the entire hardness box.

Since any operator can be written in terms of its eigenvalues and projection operators we have

$$\hat{H} = |h\rangle\langle h| - |s\rangle\langle s|$$

so that

$$\hat{H}|h\rangle = (|h\rangle\langle h| - |s\rangle\langle s|)|h\rangle = |h\rangle\langle h|h\rangle - |s\rangle\langle s|h\rangle = |h\rangle(1) - |s\rangle(0) = |h\rangle$$

$$\hat{H}|s\rangle = (|h\rangle\langle h| - |s\rangle\langle s|)|s\rangle = |h\rangle\langle h|s\rangle - |s\rangle\langle s|s\rangle = |h\rangle(0) - |s\rangle(1) = -|s\rangle$$

The eigenvector/eigenvalue equations above just say that a hardness box (hardness operator acting on a state vector) does not change $|h\rangle$ and $|s\rangle$ states (we will see shortly that an overall minus sign does not change any measurable properties of the $|s\rangle$ state) as required.

We can write the matrices representing these objects in the hardness basis (called a **matrix representation**)

$$|h\rangle = \begin{pmatrix} \langle h|h\rangle \\ \langle s|h\rangle \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad |s\rangle = \begin{pmatrix} \langle h|s\rangle \\ \langle s|s\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$[H] = \begin{pmatrix} \langle h|\hat{H}|h\rangle & \langle h|\hat{H}|s\rangle \\ \langle s|\hat{H}|h\rangle & \langle s|\hat{H}|s\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where we have used $\langle h|\hat{H}|s\rangle = -\langle h|s\rangle = 0$, etc.

Remember that in terms of matrices, Hermitian means the following:

If $[A] = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then $[A^+] = \begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix}$ = Hermitian conjugate = $[A]$

Similarly, another such basis(equivalent) is then

$$|magenta\rangle = |m\rangle \quad , \quad |green\rangle = |g\rangle$$

where

$$\hat{C}|g\rangle = |g\rangle \rightarrow eigenvalue = 1 \quad (by\ definition)$$

$$\hat{C}|m\rangle = -|m\rangle \rightarrow eigenvalue = -1 \quad (by\ definition)$$

or, in words, $|g\rangle$ is a state with a measured value of color = 1 and $|m\rangle$ is a state with a measured value of color = -1. The operator $\hat{C}$ represents the entire color box. Since any operator can be written in terms of its eigenvalues and projection operators we have

$$\hat{C} = |g\rangle\langle g| - |m\rangle\langle m|$$

Since these states form a basis (orthonormal set) they satisfy

$$\langle g|g\rangle = 1 = \langle m|m\rangle$$

$$\langle g|m\rangle = 0 = \langle m|g\rangle$$

The eigenvector/eigenvalue equations above just say that a color box does not change $|g\rangle$ and $|m\rangle$ states (again, we will see shortly that an overall minus sign does not change any measurable properties of the $|m\rangle$ state).

We can write the matrices representing these objects in the color basis (called a representation)

$$|g\rangle = \begin{pmatrix} \langle g|g\rangle \\ \langle g|m\rangle \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad |m\rangle = \begin{pmatrix} \langle m|g\rangle \\ \langle m|m\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$[C] = \begin{pmatrix} \langle g|\hat{C}|g\rangle & \langle g|\hat{C}|m\rangle \\ \langle m|\hat{C}|g\rangle & \langle m|\hat{C}|m\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where we have used $\langle h|\hat{H}|s\rangle = -\langle h|s\rangle = 0$, etc.

Let us now write $|g\rangle$ and $|m\rangle$ in terms of $|h\rangle$ and $|s\rangle$. This is always possible since $|h\rangle$, $|s\rangle$ is a basis and $|g\rangle$, $|m\rangle$ are just some other vectors in the space.

This is just the principle of SUPERPOSITION, that we mentioned earlier, in action. We write

$$|g\rangle = a|h\rangle + b|s\rangle$$
$$|m\rangle = c|h\rangle + d|s\rangle$$

Normalization: the states must be normalized to 1 (we assume a,b,c,d are real numbers) so that

$$\langle g|g\rangle = 1 = \left(a\langle h| + b\langle s|\right)\left(a|h\rangle + b|s\rangle\right) = a^2 + b^2$$
$$\langle m|m\rangle = 1 = \left(c\langle h| + d\langle s|\right)\left(c|h\rangle + d|s\rangle\right) = c^2 + d^2$$

Orthogonality: the states must be orthogonal (eigenvectors of a Hermitian operator) so that

$$\langle g|m\rangle = 0 = \left(a\langle h| + b\langle s|\right)\left(c|h\rangle + d|s\rangle\right) = ac + bd$$

One possible solution to these equations (which as we shall see represents the color states) is

$$a = b = c = -d = \frac{1}{\sqrt{2}}$$

so that we obtain

$$|g\rangle = \frac{1}{\sqrt{2}}|h\rangle + \frac{1}{\sqrt{2}}|s\rangle \quad , \quad |m\rangle = \frac{1}{\sqrt{2}}|h\rangle - \frac{1}{\sqrt{2}}|s\rangle$$

Similarly, we can express $|h\rangle$ and $|s\rangle$ in terms of $|g\rangle$ and $|m\rangle$ as a basis. We obtain

$$|h\rangle = \frac{1}{\sqrt{2}}|g\rangle + \frac{1}{\sqrt{2}}|m\rangle \quad , \quad |s\rangle = \frac{1}{\sqrt{2}}|g\rangle - \frac{1}{\sqrt{2}}|m\rangle$$

So, **sums and differences of vectors represent superpositions of physical states.**

> **States of definite color are superpositions of different hardness states.**
>
> **States of definite hardness are superpositions of different color states.**

The hardness and color operators are incompatible observables in the sense that states of definite hardness (eigenvectors of the hardness operator) apparently have no definite color value (since they are not eigenvectors of the color operator) and vice versa. This means that color and hardness are incompatible and that their representative operators do not commute. We can see that by determining the matrix for $\hat{H}$ in the color basis and then computing the commutator. We have

$$[H] = \begin{pmatrix} \langle g|\hat{H}|g\rangle & \langle g|\hat{H}|m\rangle \\ \langle m|\hat{H}|g\rangle & \langle m|\hat{H}|m\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where we have used

$$\langle g|\hat{H}|m\rangle = \left(\frac{1}{\sqrt{2}}\langle h| + \frac{1}{\sqrt{2}}\langle s|\right)\hat{H}\left(\frac{1}{\sqrt{2}}|h\rangle - \frac{1}{\sqrt{2}}|s\rangle\right) = \frac{1}{2}\left(\langle h|\hat{H}|h\rangle - \langle h|\hat{H}|s\rangle + \langle s|\hat{H}|h\rangle - \langle s|\hat{H}|s\rangle\right)$$

$$= \frac{1}{2}\left(\langle h|h\rangle + \langle h|s\rangle + \langle s|h\rangle + \langle s|s\rangle\right) = \frac{1}{2}(1+0+0+0) = 1$$

We then have

$$\left[\hat{C},\hat{H}\right] = \hat{C}\hat{H} - \hat{H}\hat{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \neq 0$$

So it looks like it will be possible to work out the descriptions of color and hardness and of all the relations between them within the framework of a these 2-dimensional vector space ideas.

The final part of this postulate is

> **if the system is in any arbitrary state $|\psi\rangle$ and we measure the observable $\hat{B}$ (where $|\psi\rangle$ is not an eigenstate of $\hat{B}$), then the only possible results are the eigenvalues of $\hat{B}$, that is, the set $\{b_k\}$.**

(C) Dynamics

There is a dynamics of state vectors.

There are "**deterministic**" laws(you know the kind of laws where if you know everything at the start, then you can predict everything that will happen in the future exactly) about how a state vector of any given system changes with time.

Since every state vector representing a real physical system must have length = 1, the changes of the state vectors dictated by the dynamical laws (called the **Schrodinger equation or time-development equation**) are exclusively changes of direction (and never of length).

We define the "**time evolution or development**" operator that governs how a state vector changes in time by the relationship

$$|A,t+\Delta t\rangle = \hat{T}|A,t\rangle = \|U(t,\Delta t)\|\|A,t\rangle$$

or, in words,

the state vector at time= $t+\Delta t$ **is given by**
the time evolution operator $\hat{T}$ **operating on**
the state vector at time = t

Note how the things inside the vector symbol change when I have to indicate that there is new information known about the state (the "**when"** in this case).

In general, that is the only thing that changes..... the labels .... which contain whatever we know(have measured) about the state.

The time evolution operator is also a **linear** operator. It is a unitary operator $\hat{U}$, which means that

if $\hat{U}\hat{U}^{-1} = \hat{I}$ **or** $\hat{U}^{-1}$ **is the inverse of** $\hat{U}$**,**
then the Hermitian conjugate $\hat{U}^{+} = \hat{U}^{-1}$

We will explicitly specify the time evolution operator later.

(D) The Connection with Experiment

So far, none of our assumptions have touched upon the results of measurements.

All we have said so far, in this arena, is that the particular physical state whose state vector is an eigenvector, with eigenvalue α, of the operator associated with some particular measurable property will "**have**" the value α for that property and that a measurement of that property, carried out on a system which happens to be in that state, will produce the result α with certainty (probability = 1) and that if we are not in an eigenvector of the observable being measured, then we can only measure one of its eigenvalues.

But we will need much more than that to deal with experiments.

What if we measure a certain property of a physical system at the moment when the state vector of the system does not happen to be an eigenvector of that property operator? (this will, in fact, be the case most of the time).

What if we measure the color of a hard electron(which is an electron in a superposition of being green and being magenta)?

What happens then?

All of our earlier assumptions are no help here.

We need a **new assumption**.

Suppose we have a system in some state $|\psi\rangle$, and we carry out a measurement of the value of a property (observable) associated with the operator $\hat{B}$.

We assume that the eigenvectors of $\hat{B}$ are the states $|b_i\rangle$ which means that

$$\hat{B}|b_i\rangle = b_i|b_i\rangle \quad \text{for all} \quad i = 1,2,3,4,......$$

where the b_i are the corresponding eigenvalues.

Quantum theory then says, as we have seen, that the outcome of any such a measurement is strictly a matter of "**probability**".

In particular, quantum theory stipulates that the **probability that the outcome** of the measurement of $\hat{B}$ on state $|\psi\rangle$ (not an eigenvector) will yield the result b_i (remember the only possible results are the eigenvalues of $\hat{B}$, no matter what state we are in), is **equal** to

$$|\langle b_i|\psi\rangle|^2 = \text{absolute-value-squared of component}$$

This postulate means the following:

(a) The probability as so defined is always ≤ 1 as it must be, which results from the requirement that all real states be of length = 1. (normalized to 1). This is the reason for the normalization requirement.

(b) If $|\psi\rangle$ = $|b_i\rangle$ (the state we are in corresponds to an eigenvector), then the probability that we will measure b_i is

$$\text{probability} = |\langle b_i|b_i\rangle|^2 = 1$$

This is just the special case when the state we are measuring is an eigenvector of the observable being measured and corresponds to Postulate #2. Postulate #4 agrees with Postulate #2 as it must.

(c) The probability that a green electron will be found during a hardness measurement to soft is 1/2 (this must be true to explain earlier experimental results). This follows as shown below state being measured ---> $|g\rangle = \frac{1}{\sqrt{2}}|h\rangle + \frac{1}{\sqrt{2}}|s\rangle$

and the probability to be soft is given by

$$prob(soft \mid green) = \text{probability to be soft on condition it is green}$$

$$= |\langle s|g\rangle|^2 = \left|\langle s|\left(\frac{1}{\sqrt{2}}|h\rangle + \frac{1}{\sqrt{2}}|s\rangle\right)\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}\left(\langle s|h\rangle + \langle s|s\rangle\right)\right|^2 = \left|\frac{1}{\sqrt{2}}(0+1)\right|^2 = \frac{1}{2}$$

That is why we chose the particular superposition earlier.

(d) Similarly, the probability that a green electron will be found during a hardness measurement to be hard is 1/2.

The probability that a hard electron will be found during a color measurement to be green is 1/2.

The probability that a hard electron will be found during a color measurement to be magenta is 1/2.

and so on.

The new formalism correctly predicts the required (experimental) results for hardness and color measurements.

Now we return to the saga of Swarthmore students and height measurements.

Suppose that we find from measurements of the observable $\hat{h}$ on a sample of 1300 Swarthmore students that the only allowed values (eigenvalues of the $\hat{h}$ operator) on the Swarthmore campus (and maybe everywhere) are

150, 160, 170, 180, 190, 200, 210, 220

or that the only that the only eigenstates of the $\hat{h}$ operator present on the Swarthmore Campus are

$$|150\rangle, |160\rangle, \ldots\ldots\ldots, |220\rangle$$

Suppose our measurements found that

n_{150} = 50 n_{160} = 100 n_{170} = 200 n_{180} = 300

n_{190} = 300 n_{200} = 200 n_{210} = 100 n_{220} = 50 where n_m = number of times we found height m during the experiment.

A quantum theorist would then claim that the state $|Swarthmore\ student\rangle$ could be represented by

$$
\begin{aligned}
|Swarthmore\ student\rangle = & \sqrt{\frac{1}{26}}|150\rangle + \sqrt{\frac{1}{136}}|160\rangle + \sqrt{\frac{2}{13}}|170\rangle + \sqrt{\frac{3}{13}}|180\rangle \\
& + \sqrt{\frac{3}{13}}|190\rangle + \sqrt{\frac{3}{13}}|200\rangle + \sqrt{\frac{1}{13}}|210\rangle + \sqrt{\frac{1}{26}}|220\rangle
\end{aligned}
$$

This state implies, by Postulate #4, that there is a 3/13 probability that the next student (assume we missed one) will have a height measuring 180 cm.

Does that make sense?

The answer is yes, since our experimental numbers show that 300 students out of 1300 had the height in our set of measurements.

What about each individual student?

How does their state relate to the state |Swarthmore student>?

We cannot say anything definite about the next student in state $|Swarthmore\ student\rangle$**. We can only make probability statements.**

Before we measure the height of the student, the student does not have a height, according to quantum theory!

Our information about the student is only a set of probabilities.

Now you are not bothered if we say this about electrons(or any microscopic object), but saying it about students(or any macroscopic object) is very bothersome because you are certain that the student has a height even if we do not measure it!

That is your view(the standard classical view) about what is real and what is not real.

Quantum theory says that you are wrong in both cases!!

If you believe otherwise, then all quantum effects would not work...but they do!!!

We will eventually devise experiments to show that quantum theory is correct and your classical views about reality are incorrect!!

We must also devise a theory to show why it seems this way for electrons but does not seem to be true for macroscopic objects.

These first 4 postulates are not even the controversial ones!!!

Finally, we state the last, and as we shall see the most controversial postulate.

(E) Collapse

Measurements are always repeatable. This seems very innocuous!

Once a measurement is carried out and a result obtained for some observable, the state of the system must be such as to guarantee(with probability = 1) that if the same measurement is repeated, the exact same result will be obtained.

Since systems evolve in time, this is literally only true if the second measurement follows the first instantaneously or, at least, within such a small time that the system in question does not have a chance to evolve.

What does this mean about the state vector of the measured(after measurement) system?

One view is that something must happen to the state vector when the measurement occurs.

If a measurement of an observable $\hat{O}$ is carried out on a system S, and if the outcome of the measurement is the value o_q (one of the many possible measured values of the operator $\hat{O}$....one of its eigenvalues) then, whatever the state vector of S was before the measurement of $\hat{O}$, the only way to guarantee (with probability = 1) that another measurement of $\hat{O}$ will give the same result is that the state vector of S after the measurement must necessarily be the eigenvector of $\hat{O}$ with eigenvalue o_q. **This must be so according to Postulate #2.**

Thus, in this view, the effect of measuring an observable must necessarily be to "**change**" the state vector of the measured system, to **COLLAPSE** it, to make it "**jump**" from whatever it may have been prior to the measurement into some eigenvector of the measured observable operator.

This is called

collapse of the state vector

or

reduction of the state vector

It says that the state vector changes(discontinuously) during a measurement from representing a range of possibilities (a superposition of all possible states) to a definite state or only one possible outcome.

Which particular eigenvector it gets changed into is determined by the outcome of the measurement and it cannot be known until then!!!!!!. It cannot be predicted!!!!

Since that outcome, according to our earlier assumption, is a matter of probability, it is at this point and at no other point in the discussion, that an element of "**pure chance**" enters into the time evolution of the state vector and **determinism goes out the window.**

Our postulates thus assume that the time evolution of a system is continuous and deterministic between measurements, and discontinuous and probabilitstic(random) during measurements.

It is clear why this postulate is controversial. It will dominate the second half of our discussions.

Those are the principles(postulates) of quantum theory.

They are the most precise mechanism for predicting the outcomes of experiments on physical systems ever devised.

No exceptions to them have ever been discovered.

NOBODY expects any.

We will now spend some time making these principles into operational tools that we can use to carry out quantum theory and make predictions.

We will see, in detail, how quantum theory makes predictions and how the strange results of various experiments are easily accounted for by the theory.

We are about to enter a very strange world where quantum systems behave in rather mysterious and non-intuitive ways.

Remember, however, that any behavior we predict and observe will just be a consequence of the 5 postulates we have just stated.

That is how theoretical physics works!