General Relativity

Extra Problems and Solutions

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Problems

EP #1- The Permutation Symbol I

The alternating symbol is defined by

$$\varepsilon_{abcd} = \begin{cases} 1 & \text{if abcd is an even permutation of 1234} \\ -1 & \text{if abcd is an odd permutation of 1234} \\ 0 & \text{otherwise} \end{cases}$$

Show that if, T, X, Y, and Z are 4−vectors with

$$T = (1, 0),\ X = (0, \vec{x}),\ Y = (0, \vec{y}),\ and\ Z = (0, \vec{z})$$

then

$$\varepsilon_{abcd}T^aX^bY^cZ^d = \vec{x} \cdot (\vec{y} \times \vec{z})$$

EP #2 - The Permutation Symbol II

Let ε have components ε_{abcd} (defined in EP #1) in every inertial coordinate system

(a) Show that ε is a tensor of rank 4

(b) Write down the values of the components of the contravariant tensor ε^{abcd}

(c) Show that $\varepsilon_{abcd}\varepsilon^{abcd} = -24$ and that $\varepsilon_{abcd}\varepsilon^{abce} = -6\delta^e_d$

EP #3 - Electromagnetic Field Tensor

Let F^{ab} be an electromagnetic field tensor. Write down the components of the dual tensor

$$F^*_{ab} = \frac{1}{2}\varepsilon_{abcd}F^{cd}$$

in terms of the components of the electric and magnetic fields. By constructing the scalars $F_{ab}F^{ab}$ and $F^*_{ab}F^{ab}$, show that $\vec{E} \cdot \vec{B}$ and $\vec{E} \cdot \vec{E} - \vec{B} \cdot \vec{B}$ are invariants.

EP #4 - Vanishing Magnetic Field

An observer has 4−velocity U^a. Show that $U^aU_a = 1$. The observer moves through an electromagnetic field F^{ab}. Show that she sees no magnetic field if $F^{*ab}U_b = 0$, and show that this equation is equivalent to $\vec{B} \cdot \vec{u} = 0$ and $B - \vec{u} \times \vec{E} = 0$. Hence show that there exists a frame in which the magnetic field vanishes at an event if and only if in every frame $\vec{E} \cdot \vec{B} = 0$ and $\vec{E} \cdot \vec{E} - \vec{B} \cdot \vec{B} > 0$ at the event.

EP #5 - Null and Orthogonal

(a) Show that the sum of any two orthogonal (scalar product is zero) spacelike vectors is spacelike.

(b) Show that a timelike vector and a null vector cannot be orthogonal.

EP #6 - Rindler Coordinates

Let $\Lambda_B(\vec{v})$ be a Lorentz boost associated with 3-velocity $\vec{v}$. Consider

$$\Lambda \equiv \Lambda_B(\vec{v}_1) \cdot \Lambda_B(\vec{v}_2) \cdot \Lambda_B(-\vec{v}_1) \cdot \Lambda_B(-\vec{v}_2)$$

where $\vec{v}_1 \cdot \vec{v}_2 = 0$. Assume that $v_1 << 1$, $v_2 << 1$. Show that Λ is a rotation. What is the axis of rotation? What is the angle of rotation?

Let $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$ be inertial coordinates on flat space-time, so the Minkowski metric has components

$$(g_{ab}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Find the metric coefficients $\tilde{g}_{ab}$ in each of the following coordinate systems.

(a) $\tilde{x}^0 = t$, $\tilde{x}^1 = r$, $\tilde{x}^2 = \theta$, $\tilde{x}^3 = z$

(b) $\tilde{x}^0 = t$, $\tilde{x}^1 = r$, $\tilde{x}^2 = \theta$, $\tilde{x}^3 = \varphi$

(c) $\tilde{x}^0 = \tau$, $\tilde{x}^1 = \varphi$, $\tilde{x}^2 = y$, $\tilde{x}^3 = z$

where r, θ in the first case, are plane polar coordinates in the x, y plane, in the second case r, θ, φ are spherical polars coordinates, and in the third, τ, φ are *Rindler coordinates*, defined by $t = \tau \cosh\varphi$, $x = \tau \sinh\varphi$. In each case, say which region of Minkowski space the coordinate system covers. A quick method is to write the metric as $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$ and substitute, for example, $dx = \cos\theta\, dr - r\sin\theta\, d\theta$, and so on. Of course, the penalty is that you must convince yourself that this is legitimate!

EP #7 - $4-$Velocities

In some reference frame, the vector fields $\vec{U}$ and $\vec{D}$ have the components

$$U^\alpha \doteq (1 + t^2, t^2, \sqrt{2}t, 0)$$
$$D^\alpha \doteq (x, 5tx, \sqrt{2}t, 0)$$

where t, x, and y are the usual Cartesian coordinates in the specified reference frame. The scalar ρ has the value $\rho = x^2 + t^2 - y^2$. The relationship "$LHS \doteq RHS$" means "the object on the LHS is represented by the object on the RHS in the specified reference frame".

(a) Show that $\vec{U}$ is suitable as a 4−velocity. Is $\vec{D}$?

(b) Find the spatial velocity $\vec{v}$ of a particle whose 4−velocity is $\vec{U}$, for arbitrary t. Describe the motion in the limits $t = 0$ and $t \to \infty$.

(c) Find $\partial_\beta U$ for all α, β. Show that $U_\alpha \partial_\beta U^\alpha = 0$. There is a clever way to do this, which you are welcome to point out. Please do it the brute force way as well as practice manipulating quantities like this.

(d) Find $\partial_\alpha D^\alpha$

(e) Find $\partial_\beta (U^\alpha D^\beta)$ for all α

(f) Find $U_\alpha \partial_\beta (U^\alpha D^\beta)$. Why is answer so similar to that for (d)?

(g) Calculate $\partial_\alpha \rho$ for all α. Calculate $\partial^\alpha \rho$

(h) Find $\nabla_{\vec{U}} \rho$ and $\nabla_{\vec{D}} \rho$

EP #8 - Projection Operators

Consider a timelike 4−vector $\vec{U}$ and the tensor $P_{\alpha\beta} = \eta_{\alpha\beta} + U_\alpha U_\beta$. Show that this tensor is a projection operator that projects an arbitrary vector $\vec{V}$ into one orthogonal to $\vec{U}$. In other words, show that the vector $\vec{V}_\perp$ whose components are $V_\perp^\alpha = P^\alpha_\beta V^\beta$ is

(a) orthogonal to $\vec{U}$

(b) unaffected by further projections: $V^\alpha_{\perp\perp} = P^\alpha_\beta V^\beta_\perp = V^\alpha_\perp$

(c) Show that $P_{\alpha\beta}$ is the metric for the space of vectors orthogonal to $\vec{U}$:

$$P_{\alpha\beta} V_\perp^\alpha W_\perp^\beta = \vec{V}_\perp \cdot \vec{W}_\perp$$

(d) Show that for an arbitrary non-null vector $\vec{q}$, the projection tensor is given by

$$P_{\alpha\beta}(q^\alpha) = \eta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^\gamma q_\gamma}$$

Do we need a projection operator for null vectors?

EP #9 - Killing Vectors in Flat Space

Find the Killing vectors for flat space $ds^2 = dx_1^2 + dx_2^2 + dx_3^2$, i.e., write out Killings equation in flat space, differentiate it once and then solve the resulting differential equation.

EP #10 - Oblate Spheroidal Coordinates

Let x, y, z be the usual Cartesian coordinates in flat space. Oblate spheroidal coordinates are defined by the relations

$$x = \sqrt{r^2 + c^2} \sin\theta \cos\varphi \ , \ y = \sqrt{r^2 + c^2} \sin\theta \sin\varphi \ , \ z = r\cos\theta$$

where c is a constant.

(a) What is the shape of surfaces of constant r?

(b) What is the metric in r, θ, φ?

(c) What is the Laplacian operator on a scalar field, $\nabla^2\Phi$, in these coordinates?

(d) Show that $\nabla^2\Phi = 0$ is separable in oblate spheroidal coordinates, $\Phi(r, \theta, \varphi) = R(r)\Theta(\theta)\phi(\varphi)$. Find the θ and φ solutions explicitly, and write an equation for $R(r)$

(e) If $r = r_0$ is the surface of a conducting disk that has net charge Q, what is the electrostatic potential exterior to r_0? What is the surface charge density as a function of $\rho = \sqrt{x^2 + y^2}$ in the limit that the spheroid becomes a flat disk, $r_0 \to 0$?

EP #11 - Uniformly Accelerating Observer A coordinate system for a uniformly accelerating observer

Background definitions: A former physics 008 student is now an astronaut. She moves through space with acceleration g in the $x-$direction. In other words, her $4-$acceleration $\vec{a} = d\vec{u}/d\tau$ (where τ is time as measured on the astronauts own clock) only has spatial components in the $x-$direction and is normalized such that $\sqrt{\vec{a} \cdot \vec{a}} = g$.

This astronaut assigns coordinates $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$ as follows:

First, she defines spatial coordinates to be $(\tilde{x}, \tilde{y}, \tilde{z})$, and sets the time coordinate $\tilde{t}$ to be her own proper time. She defines her position to be $(\tilde{x} = g^{-1}, \tilde{y} = 0, \tilde{z} = 0)$ (not a unique choice, but a convenient one). Note that she remains *fixed* with respect to these coordinates - thats the point of coordinates for an accelerated observer!

Second, at $\tilde{t} = 0$, the astronaut chooses $(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$ to coincide with the Euclidean coordinates (t, x, y, z) of the inertial reference frame that momentarily coincides with her motion. In other words, though the astronaut is not inertial, there is an inertial frame that, at $\tilde{t} = 0$, is momentarily at rest with respect to her. This is the frame used to assign $(\tilde{x}, \tilde{y}, \tilde{z})$ at $\tilde{t} = 0$. The clocks of that frame are set such that they are synchronized with her clocks at that moment.

Observers who remain at fixed values of spatial coordinates are called coordinate-stationary observers(CSOs). Note that CSOs are also accelerated observers, though not necessarily accelerating at the same rate as the astronaut. The astronaut requires the CSO worldlines to be orthogonal to hypersurfaces $\tilde{t} = constant$. She also requires that for each $\tilde{t}$ there exists some inertial frame, momentarily at rest with respect to the astronaut, in which all events with $\tilde{t} = constant$ are simultaneous. The accelerated motion of the astronaut can thus be described as movement through a sequence of inertial frames which momentarily coincide with her motion.

It is easy to see that $\tilde{y} = y$ and $\tilde{z} = z$; henceforth we drop these coordinates from the problem.

(a) What is the 4−velocity $\vec{u}$ of the astronaut, as a function of $\tilde{t}$, as measured by CSOs in the initial inertial frame[the frame the uses coordinates (t, x, y, z)]? HINT: by considering the conditions on $\vec{u} \cdot \vec{u}$, $\vec{u} \cdot \vec{a}$, and $\vec{a} \cdot \vec{a}$, you should be able to find simple forms for u^t and u^x. After you have worked out $\vec{u}$, compute $\vec{a}$.

(b) Integrate this 4−velocity to find the position $[T(\tilde{t}), X(\tilde{t})]$ of the astronaut in the coordinates (t, x). Recall that at $t = \tilde{t} = 0$, $X = \tilde{x} = 1/g$. Sketch the astronauts worldline on a spacetime diagram in the coordinates (t, x). You will return to and augment this sketch over the course of this problem, so you may want to to do this on a separate piece of paper.

(c) Find the orthogonal basis vectors $\vec{e}_{\tilde{t}}$ and $\vec{e}_{\tilde{x}}$ describing the momentarily inertial coordinate system at some time $\tilde{t}$. Add these vectors to the sketch of your worldline.

We now *promote* $\tilde{t}$ to a coordinate, i.e., give it meaning not just on the astronauts worldline, but everywhere in spacetime, by requiring that $\tilde{t} = constant$ be a surface of constant time in the Lorentz frame in which the astronaut is instantaneously at rest.

(d) By noting that this *surface* must be parallel to $\vec{e}_{\tilde{x}}$ and that it must pass through the point $[T(\tilde{t}), X(\tilde{t})]$, show that it is defined by the line

$$x = t \cosh g\tilde{t}$$

In other words, it is just a straight line going through the origin with slope $\cosh g\tilde{t}$.

We have now defined the time coordinate $\tilde{t}$ that the astronaut uses to label spacetime. Next, we need to come up with a way to set her spatial coordinates $\tilde{x}$.

(e) Recalling that CSOs must themselves be accelerated observers, argue that their worldlines are hyperbolae, and thus that a CSOs position in (t, x)

must take the form

$$t = \frac{A}{g}\sinh g\tilde{t} \quad , \quad x = \frac{A}{g}\cosh g\tilde{t}$$

From the initial conditions, find A.

(f) Show that the line element $ds^2 = d\vec{x}\cdot d\vec{x}$ in the new coordinates takes the form

$$ds^2 = -dt^2 + dx^2 = -(g\tilde{x})^2 d\tilde{t}^2 + d\tilde{x}^2$$

This is known as the Rindler metric. As the problem illustrates, it is just the flat spacetime of special relativity; but, expressed in coordinates that introduce some features that will be very important in general relativity.

EP #12 - Jumping Seagull

A Newton-Galilean Problem - A seagull sits on the ground. The wind velocity is $\vec{v}$. How high can the gull rise without doing any work? The trick here is

(a) to identify the most convenient reference frame

(b) transform the problem to that frame

(c) solve the problem

(d) transform the result back again so that it is expressed in the original frame

EP #13 - Lagrange Equations for Kepler Orbits

Use Lagrange equations to solve the problem of Kepler planetary orbits in a gravitational field. Work in 3 dimensions in spherical coordinates. Determine the orbital equation $r(\theta)$.

EP #14 - Lagrange Equations for Double Pendulum

Use Lagrange equations to solve the problem of the double pendulum in a gravitational field. The double pendulum is a fixed pivot O, a light rigid rod OA of unit length at an angle α to the vertical, mass m at A, light rigid rod AB of unit length at an angle β to the vertical, mass m at B, constant gravitational field g downwards. Solve the equations in the small angle approximation.

EP #15 - Uniform Relativistic Circular Motion

A particle (in special relativity) moves in uniform circular motion, that is (with $c = 1$),

$$x^\mu = (t, r\cos\omega t, r\sin\omega t, 0)$$

(a) Write down its worldline according to an observer moving with velocity $\vec{v}$ along the $y-$axis. You will need to use the old time t as a parameter. HINT: this follows directly from the Lorentz transformation.

(b) If the particle at rest decays with half-life $\tau_{1/2}$, what is its observed half-life?

(c) Show that the proper acceleration α is given by

$$\alpha = \frac{r\omega^2}{1 - r^2\omega^2}$$

EP #16 - Accelerated Motion

(a) Consider a particle moving along the $x-$axis with velocity u and acceleration a, as measured in frame S. S moves relative to S with velocity v along the same axis. Show that

$$\frac{du}{dt} = \frac{1}{\gamma^3}\frac{1}{(1+u'v)^3}a'$$

(b) Suppose that S is chosen to be the instantaneous rest frame of the particle and $a = 9.806\, m/s^2$. That is, $u' = 0$, $a' = g$, and $u = v$. Using the result from part (a), derive expressions for u (velocity as measured in S) and x as a function of time. Write x as a function of τ, the proper time as measured along the particles worldline, and evaluate for $\tau = 20\, years$. Discuss the significance of this result for space travel. You can use $dt/d\tau = \gamma$ to derive an expression for τ as a function of t.

EP #17 - High Energy Kinematics

(a) In a high energy accelerator, the energy available to create new particles is the energy in the center-of-mass(CM) frame. Consider a proton with momentum $1\, TeV/c$ incident on a target proton at rest. What is the available CM energy?

(b) Next consider a $1\, TeV$ proton heading east colliding with a $1\, TeV$ proton headed west. What is the available CM energy? What momentum would be needed in a fixed-target experiment to obtain the same available energy?

(c) A Λ^0 baryon ($m_\Lambda = 1115.7\, MeV$) decays into a proton ($m_p = 938.3\, MeV$) anda negative pion ($m_\pi = 139.6\, MeV$). What is the momentum of the proton or pion in the CM frame?

(d) The decaying Λ^0 has momentum $28.5\, GeV/c$ in the lab frame. What is the maximum angle between the proton and the pion in the lab?

EP #18 - Tensor Properties

Consider a tensor T_{ij} in three-dimensional Euclidean space. Under an arbitrary rotation of the three-dimensional coordinate space, the tensor is transformed. Show that

(a) If T_{ij} is symmetric and traceless, then the transformed tensor is symmetric and traceless.

(b) If T_{ij} is antisymmetric, then the transformed tensor is symmetric.

State the analogous result for a tensor $T^{\mu\nu}$ in four-dimensional Minkowski space. Define carefully what you mean by a traceless tensor in this case.

EP #19 - Transforming Electromagnetic Fields

Under a Lorentz transformation, a tensor transforms as follows:

$$F'^{\mu\nu} = \Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}F^{\alpha\beta}$$

where Λ is the Lorentz transformation matrix. Consider an inertial frame K at rest, and a second inertial frame K' moving with velocity v along the $x-$direction with respect to K. Using the explicit result for Λ corresponding to the transformation between K and K', determine the electric and magnetic fields in frame K' in terms of the corresponding fields in frame K.

EP #20 - Twins In Relativity

Consider a pair of twins that are born somewhere in spacetime. One of the twins decides to explore the universe. She leaves her twin brother behind and begins to travel in the $x-$direction with constant acceleration $a = 10\ m/s^2$ as measured in her rocket frame. After 10 years according to her watch, she reverses the thrusters and begins to accelerate with a constant acceleration $-a$ for a while.

(a) At what time on her watch should she again reverse her thrusters so she ends up home at rest?

(b) According to her twin brother left behind, what was the most distant point on her trip?

(c) When the sister returns, who is older, and by how much?

EP #21 - Multiple Lorentz Transformations

A Lorentz transformation is the product of a boost with rapidity ζ in the direction $\hat{n}_1$, followed by a boost with rapidity ζ in the direction $\hat{n}_2$, followed by a boost with rapidity ζ in the direction $\hat{n}_3$, where ζ is the same in each case, and the three directions $\hat{n}_1$, $\hat{n}_2$, and $\hat{n}_3$ lie in the same plane separated by 120°. What is the resulting transformation? To lowest order for small ζ, is it a boost or a rotation? At what order does the other (boost or rotation) enter?

EP #22 - Tensor Transformations

(a) Observer O at *rest* sees a symmetric tensor $T^{\mu\nu}$ to be diagonal with components (ρ, p, p, p). What are the components of $T_{\mu\nu}$

(b) Frame O moves with speed v in the $+x$-direction with respect to O. What are the components of $T'^{\mu\nu}$ in frame O? What are the components of $T'_{\mu\nu}$? How can the *rest* frame be identified? Suppose that $p = -\rho$ in the original frame O, what is $T'^{\mu\nu}$ then? Make an insightful observation.

EP #23 - Newtonian Gravity

Poissons formulation of Newtonian gravity is

$$\nabla^2\varphi = 4\pi\rho \quad , \quad \vec{g} = -\nabla\varphi$$

where ρ is the matter density, φ is the gravitational potential and $\vec{g}$ is the acceleration due to gravity. Show that this gives the usual Newtonian formula for a point-like source.

EP #24 - Tides

Tides occur because the force of gravity is slightly different at two nearby points, such as a point at the earths surfaced and at its center.

(a) What is the difference between the gravitational acceleration induced by a mass M (the sun or the moon) evaluated at the center and at a point on the surface of a sphere of radius r (the earth) located a distance R from M (take $r << R$). Write the radial component of this difference at the surface as a function of the angle from the line joining the two objects. How many high/low tides are there in a day?

(b) If the earth were a perfect sphere covered with water, compute or estimate the height difference between high and low tides(ignoring complications such as rotation, friction, viscosity) for spring tides (directions of sun and moon aligned) and neap tides (sun and moon at right angles).

(c) A neutron star is a collapsed object of nuclear density with mass $M = 1.4M_{Sun}$,and radius $R = 10\,km$. In Larry Nivens short story Neutron Star(1966), tidal forces in the neighborhood of the title object prove fatal to the unwary. What is the tidal acceleration across the diameter of a person (say a distance of $1\,m$) at a distance of $100\,km$ from a neutron star?

EP #25 - An Invisible Sphere

A hollow sphere has density ρ, inner radius a and outer radius b. Find the gravitational field in the region $r < a$. Suppose now that the sphere were invisible. Could an observer at the center deduce its existence without leaving the region $r < a$?

EP #26 - Gravitational Fields

(a) Compute the gradient of the gravitational field $\partial g_i/\partial x_j$ (a nine component object) corresponding to a sphere of density ρ and radius R centered at the origin.

(b) Find a mass distribution $\rho(x, y, z)$ on a bounded domain, that is, zero whenever $x^2 + y^2 + z^2 > R^2$ for some positive constant R; uniformly bounded, i.e., $|\rho(x, y, z)| < C$ for some positive constant C independent of position; and for which at least one component of the gradient of the gravitational field is infinite at some point.

EP #27 - Riemann Tensor

Find an expression for

$$\nabla_c \nabla_d T^a_b - \nabla_d \nabla_c T^a_b$$

in terms of the Riemann tensor.

EP #28 - Isometries and Killing Vectors

(a) Define an isometry

(b) Define the Killing vector and show that it satisfies

$$\nabla_a k_b + \nabla_b k_a = 0$$

(c) Show how the Killing vector defines a constant along geodesics

(d) If k_a and l_a are Killing vectors, show that

$$[k, l]_a = k^b \nabla_b l_a - l^b \nabla_b k_a$$

is also a Killing vector. It is useful to recall the symmetry properties of the Riemann tensor $R_{abcd} = -R_{bacd}$ and $R_{abcd} = R_{cdab}$.

EP #29 - On a Paraboloid

A paraboloid in three dimensional Euclidean space

$$ds^2 = dx^2 + dy^2 + dz^2$$

is given by

$$x = u \cos\varphi \quad , \quad y = u \sin\varphi \quad , \quad z = u^2/2$$

where $u \geq 0$ and $0 \leq \varphi \leq 2\pi$.

(a) Show that the metric on the paraboloid is given by

$$ds^2 = (1 + u^2)du^2 + u^2 d\varphi^2$$

(b) Writing $x^1 = u$ and $x^2 = \varphi$ find the Christoffel symbols for this metric.

(c) Solve the equation for parallel transport

$$U^a \nabla_a V^b = 0$$

where

$$U^a = \frac{dx^a}{dt}$$

for the curve $u = u_0$ where u_0 is a positive constant and with initial conditions $V^1 = 1$ and $V^2 = 0$. HINT: the problem is simplified if you take $t = \varphi$. The equation of parallel transport will give you two coupled equations for U^1 and U^2, differentiating the $dU^1/d\varphi$ equation again allows you to decouple the U^1 equation.

EP #30 - A Two-Dimensional World

A certain two-dimensional world is described by the metric

$$ds^2 = \frac{dx^2 + dy^2}{\left[1 + \frac{x^2+y^2}{4a^2}\right]^2}$$

(a) Compute the connection coefficients Γ^i_{jk}

(b) Let $\vec{\xi} = -y\hat{e}_x + x\hat{e}_y$. Show that $\vec{\xi}$ is a solution of Killings equation.

(c) What is the conserved quantity that corresponds to this symmetry? Show from the geodesic equation that this quantity is indeed conserved.

(d) Compute the Riemann tensor R^{ij}_{kl}, the Ricci tensor R^i_j, and the Ricci scalar R. What is the shape of this world?

EP #31 - Timelike Geodesics

Find the timelike geodesics for the metric

$$ds^2 = \frac{1}{t^2}\left(-dt^2 + dx^2\right)$$

EP #32 - More Geodesics

Consider the 2-dimensional metric

$$ds^2 = a^2\left(d\chi^2 + \sinh^2\chi d\varphi^2\right)$$

(a) Compute the connection coefficients Γ^i_{jk}

(b) Compute all components of the Riemann tensor R^{ij}_{kl}, the Ricci tensor R^i_j, and the Ricci scalar R.

(c) A geodesic starts at $\chi = b$, $\varphi = 0$ with tangent $d\varphi/d\lambda = 1$, $d\chi/d\lambda = 0$. Find the trajectory $\chi(\varphi)$.

(d) A second geodesic starts at $\chi = b + \xi(\xi \ll 1)$, also initially in the ?φ-direction. How does the separation initially increase or decrease along the two curves.

(e) What is the shape of the geodesic trajectory as $a \to \infty$, $\chi \to 0$ with $r = a\chi$ fixed.

EP #33 - Parallel Transport on a Sphere

On the surface of a 2−sphere of radius a

$$ds^2 = a^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

Consider the vector $\vec{A}_0 = \vec{e}_\theta$ at $\theta = \theta_0$, $\varphi = 0$. The vector is parallel transported all the way around the latitude circle $\theta = \theta_0$ (i.e., over the range $0 \leq \varphi \leq 2\pi$ at $\theta = \theta_0$). What is the resulting vector $\vec{A}$? What is its magnitude $\left(\vec{A} \cdot \vec{A}\right)^{1/2}$? HINT: derive differential equations for A^θ and A^φ as functions of φ.

EP #34 - Curvature on a Sphere

(a) Compute all the nonvanishing components of the Riemann tensor R_{ijkl} $((i,j,k,l) \in (\theta,\varphi))$ for the surface of a 2−sphere.

(b) Consider the parallel transport of a tangent vector $\vec{A} = A^\theta \hat{e}_\theta + A^\varphi \hat{e}_\varphi$ on the sphere around an infinitesimal parallelogram of sides $\hat{e}_\theta d\theta$ and $\hat{e}_\varphi d\varphi$. Using the results of part (a), show that to first order in $d\Omega = \sin\theta d\theta d\varphi$, the length of $\vec{A}$ is unchanged, but its direction rotates through an angle equal to $d\Omega$.

(c) Show that, if $\vec{A}$ is parallel transported around the boundary of any simply connected solid angle Ω, its direction rotates through an angle Ω. (*Simply connected* is a topological term meaning that the boundary of the region could be shrunk to a point; it tells us that there are no holes in the manifold or other pathologies). Using the result of part (b) and intuition from proofs of Stokes theorem, this should be an easy calculation. Compare with the result of EP #32.

EP #35 - Riemann Tensor for 1+1 Spacetimes

(a) Compute all the nonvanishing components of the Riemann tensor for the spacetime with line element

$$ds^2 = -e^{2\varphi(x)}dt^2 + e^{2\psi(x)}dx^2$$

(b) For the case $\varphi = \psi = \frac{1}{2}\ln|g(x - x_0)|$ where g and x_0 are constants, show that the spacetime is flat and find a coordinate transformation to globally flat coordinates $(\bar{t}, \bar{x})$ such that $ds^2 = -d\bar{t}^2 + d\bar{x}^2$.

EP #36 - About Vectors Tangent to Geodesics

Let $x^\mu(\tau)$ represent a timelike geodesic curve in spacetime, where τ is the proper time as measured along the curve. Then $u^\mu \equiv dx^\mu/d\tau$ is tangent to the geodesic curve at any point along the curve.

(a) If $g_{\mu\nu}$ is the metric of spacetime, compute the magnitude of the vector u^μ. Do not use units where $c = 1$, but keep any factors of c explicit. Compare your result with the one obtained in flat Minkowski spacetime. HINT: The magnitude of a timelike vector v^μ is given by $(-g_{\mu\nu}v^\mu v^\nu)^{1/2}$.

(b) Consider a contravariant timelike vector v^μ at a point P on the geodesic curve. Move the vector v^μ from the point P to an arbitrary point Q on the geodesic curve via parallel transport. Prove that the magnitude of the vector v^μ at the point Q equals the magnitude of the vector v^μ at point P.

(c) Suppose that at the point P on the geodesic curve, $v^\mu = u^\mu$. Now, parallel transport the vector v^μ along the geodesic curve to arbitrary point Q. Show that $v^\mu = u^\mu$ at the point Q. NOTE: This result implies that a vector tangent to a geodesic at a given point will always remain tangent to the geodesic curve when parallel transported along the geodesic.

EP #37 - Velocity of Light

The Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

As a function of r, what is the coordinate velocity of light in this metric (a) in the radial direction? (b) in the transverse direction? What are the physical consequences of these results.

EP #38 - Orbiting Photons

Consider a photon in orbit in a Schwarzschild geometry. For simplicity, assume that the orbit lies in the equatorial plane (i.e., $\theta = \pi/2$ is constant).

(a) Show that the geodesic equations imply that

$$\bar{E}^2 = \frac{1}{c^2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{\bar{J}^2}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right)$$

where $\bar{E}$ and $\bar{J}$ are constant of the motion and λ is an affine parameter.

(b) Define the effective potential

$$V_{eff} = \frac{\bar{J}^2}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right)$$

The effective potential yields information about the orbits of massive particles. Show that for photons there exists an unstable circular orbit of radius $3r_s/2$, where $r_s = 2GM/c^2$ is the Schwarzschild radius. HINT: Make sure you check for minima and maxima.

(c) Compute the proper time for the photon to complete one revolution of the circular orbit as measured by an observer stationed at $r = 3r_s/2$.

(d) What orbital period does a very distant observer assign to the photon?

(e) The instability of the orbit can be exhibited directly. Show, by perturbing the geodesic in the equatorial plane, that the circular orbit of the photon at $r = 3r_s/2$ is unstable. HINT: in the orbit equation put $r = 3r_s/2 + \eta$, and deduce an equation for η. Keep only the first order terms in $\eta << 1$, and solve the resulting equation.

EP #39 - Light Cones

Consider the 2-dimensional metric

$$ds^2 = -x dw^2 + 2dwdx$$

(a) Calculate the light cone at a point (w, x), i.e., find dw/dx for the light cone. Sketch a (w, x) spacetime diagram showing how the light cones change with x. What can you say about the motion of particles, and in particular, about whether they can cross from positive to negative x and vice versa.

(b) Find a new system of coordinates in which the metric is diagonal.

EP #40 - Circular Orbit

An object moves in a circular orbit at Schwarzschild radius R around a spherically symmetric mass M. Show that the proper time τ is related to coordinate time t by

$$\frac{\tau}{t} = \sqrt{1 - \frac{3M}{R}}$$

HINT: It is helpful to derive a relativistic version of Keplers third law.

EP #41 - Space Garbage

In a convenient coordinate system, the spacetime of the earth is approximately

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 + \frac{2GM}{r}\right) \left[dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right] \\ &= -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 + \frac{2GM}{r}\right) \left[dx^2 + dy^2 + dz^2\right] \end{aligned}$$

where M is the earths mass. In the second version we remapped the spherical coordinates to cartesian coordinates in the usual way:

$$x = r \sin\theta \cos\varphi \quad , \quad y = r \sin\theta \sin\varphi \quad , \quad z = r \cos\theta$$

Note that the Cartesian form of the spacetime metric is conveniently written $g_{\alpha\beta} = \eta_{\alpha\beta} + 2\Phi\hat{I}$ where $\hat{I} = diag(1,1,1,1)$ and $\Phi = GM/r$. We can assume that $\Phi << 1$ throughout this problem.

The space shuttle orbits the earth in a circular $u^r = 0$), equatorial ($\theta = \pi/2$, $u^\theta = 0$) orbit of radius R.

(a) Using the geodesic equation, show that an orbit which begins equatorial remains equatorial: $du^\theta/dt = 0$ if $u^\theta = 0$ and $\theta = \pi/2$ at $t = 0$. HINT: Begin by computing the non-zero connection coefficients; use the fact that $\Phi << 1$ to simplify your answer. We now require that the orbit must remain circular: $du^r/dt = 0$. This has already been done in earlier problems and in the text.

(b) By enforcing this condition with the geodesic equation, derive an expression for the orbital frequency

$$\Omega = \frac{d\varphi/d\tau}{dt/d\tau}$$

Does this result look familiar? This has been done in Problem #41 The next part is most conveniently described in Cartesian coordinates; you may describe the shuttles orbit as

$$x = R\cos\Omega t \ , \ \ y = R\sin\Omega t$$

An astronaut releases a bag of garbage into space, spatially displaced from the shuttle by $\xi^i = x^i_{garbage} - x^i_{shuttle}$.

(c) Using the equations of geodesic deviation, work out differential equations for the evolution of ξ^t, ξ^x, ξ^y, and ξ^z as a function of time. You may neglect terms in $(GM/r)^2$, and treat all orbital velocities as non-relativistic. You will need the Cartesian connection coefficients for this.

(d) Suppose the initial displacement is $\xi^x = \xi^y = 0$, $\xi^z = L$, $d\xi^i/dt = 0$. Further, synchronize the clocks of the garbage and the space shuttle: $\xi^0 = 0$, $\partial_t\xi^0 = 0$. Has the astronaut succeeded in getting rid of the garbage?

EP #42 - Astronauts in Orbit

Consider a spacecraft in a circular orbit in a Scharzschild geometry. As usual, we denote the Schwarzschild coordinates by (ct, r, θ, φ) and assume that the orbit occurs in the plane where $\theta = \pi/2$. We denote two conserved quantities by

$$e = \left[1 - \frac{r_s}{r}\right]\frac{dt}{d\tau} \quad and \quad \ell = r^2\frac{d\varphi}{d\tau}$$

where $r_s = 2GM/c^2$ and τ is the proper time.

(a) Write down the geodesic equation for the variable r. Noting that r is independent of τ for a circular orbit, show that:

$$\frac{\ell}{c} = c\left(\tfrac{1}{2}r_s r\right)^{1/2}\left[1 - \frac{r_s}{r}\right]^{-1}$$

(b) Show that for a timelike geodesic, $g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -c^2$, where $\dot{x}^\mu = dx^\mu/d\tau$. From this result, derive a second relation between ℓ and e for a circular orbit. Then, using the result of part (a) to eliminate e, obtain an expression for $d\tau/d\varphi$ in terms of the radius r of the orbit.

(c) Using the result of part (b), determine the period of the orbit as measured by an observer at rest inside the orbiting spacecraft, as a function of the radius r of the orbit.

(d) Suppose an astronaut leaves the spacecraft and uses a rocket-pack to maintain a fixed position at radial distance r equal to the orbital radius and at fixed $\theta = \pi/2$ and $\varphi = 0$. The astronaut outside then measures the time it takes the spacecraft to make one orbital revolution. Evaluate the period as measured by the outside astronaut. Does the astronaut outside the spacecraft age faster or slower than the astronaut orbiting inside the spacecraft?

EP #43 - Weak Gravity

In weak gravity, the metric of a mass M at rest at the origin is

$$ds^2 = -(1 + 2\varphi)dt^2 + (1 - 2\alpha\varphi)\delta_{ij}dx^i dx^j$$

where α is a constant and $\varphi = -GM/r$.

(a) What is the value of α in general relativity?

(b) Instead of sitting at rest at the origin, the mass M moves in the $+x-$direction with speed v, passing through the origin at time $t = 0$, so that its position as a function of time is $x = vt$. What is the metric in this case?

(c) A photon moves along a trajectory originally in the $+y-$direction with offset b behind the $y-$axis, so that its undeflected trajectory is $x_0 = -b\hat{x} + t\hat{y}$. By what angle is the path of this test particle deflected?

(d) What is change in energy of deflected photon in part (c).

EP #44 - Star with Constant Density

The metric of a star with constant density is

$$ds^2 = -\left(1 - \frac{2\mathrm{M}(r)}{r}\right)c^2dt^2 + \left(1 - \frac{2\mathrm{M}(r)}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2$$

where

$$M(r) = \begin{cases} M(r/R)^3 & 0 < r < R \\ M & R < r \end{cases}$$

is the mass interior to radius r, M is the total mass of the star, and R is the coordinate radius of the surface of the star. Assume $R > 2M$. We consider the orbits of photons where $g_{\mu\nu}u^\mu u^\nu = 0$.

(a) Are there any singularities (coordinate or otherwise) of the metric?

(b) Write the timelike and spacelike Killing vectors for this spacetime. There are actually two spacelike Killing vectors, but we will only need one since the photon orbits are planar. You may set $\theta = \pi/2$. Write out the associated conserved quantities.

(c) Derive an expression for $dr/d\lambda$ where λ is the affine parameter. Put your expression in the form

$$\frac{1}{b^2} = \frac{1}{l^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r)$$

and define b in terms of the constants of motion and W_{eff}.

(d) Sketch W_{eff} and describe the photon orbits. How do these differ from the photon orbits in the standard Schwarzschild geometry?

(e) Calculate the coordinate time t for a photon to travel from the center of the star at $r = 0$ to the surface at $r = R$.

(f) Assume $R >> M$ and find the approximate delay, i.e., the extra time relative to the result from special relativity ($t = R$) to leading order. What is the value for the Sun where $M = 1.5\,km$ and $R = 7.0 \times 10^3\,km$.

EP #45 - In the Schwarzschild Geometry

Consider a spacetime described by the Schwarzschild line element:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2$$

(a) A clock at fixed (r, θ, φ) measures an (infinitesimal) proper time interval, which we denote by dT. Express dT (as a function of r) in terms of the coordinate time interval dt.

(b) A stationary observer at fixed (t, θ, φ) measures an (infinitesimal) radial distance, which we denote by dR. Express dR (as a function of r) in terms of the coordinate radial distance dr.

(c) Consider the geodesic equations for free particle motion in the Schwarzschild geometry. Write out explicitly the equation corresponding to the time component. The equations corresponding to the space components will not be required. The resulting equation can be used to determine $dt/d\tau$ (where τ is the proper time and $t = x^0/c$ is the coordinate time). In particular, show that the quantity

$$k = \left(1 - \frac{2GM}{c^2 r}\right) \frac{dt}{d\tau}$$

is a constant independent of τ. Using the time component of the geodesic equation obtained earlier, compute the values of $\Gamma^0_{\alpha\beta}$ for this geometry. Consider all possible choices of α and β.

(d) Consider a particle falling radially into the center of the Schwarzschild metric, i.e., falling in radially towards $r = 0$. Assume that the particle initially starts from rest infinitely far away from $r = 0$. Since this is force-free motion, the particle follows a geodesic. Using the results of part (c), evaluate the constant k and thereby obtain a unique expression for $dt/d\tau$ that is valid at all points along the radial geodesic path. HINT: What is the value of $dt/d\tau$ at $r \to \infty$ (where the initial velocity of the particle is zero)?

(e) Since $ds^2 = -c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$ it follows that

$$g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -c^2$$

In this problem, $g_{\mu\nu}$ is determined from the Schwarzschild line element. Using these results and the result obtained in part (d) for $dt/d\tau$, compute the particles inward coordinate velocity, $v = dr/dt$, as a function of the coordinate radial distance r. Invert the equation, and integrate from $r = r_0$ to $r = r_s$, where r_0 is some finite coordinate distance such that $r_0 > r_s$ and $r_s = 2GM/c^2$ is the Schwarzschild radius. Show that the elapsed coordinate time is infinite, independent of the choice of the starting radial coordinate r_0, i.e., it takes an infinite coordinate time to reach the Schwarzschild radius. HINT: For radial motion, θ and φ are constant independent of τ. Note that for inward radial motion $dt/d\tau$ is negative.

(f) Compute the velocity dR/dT as measured by a stationary observer at a coordinate radial distance r. Verify that $|dR/dT| \to c$ as $r \to r_s$. HINT: Use the result for dR and dT obtained in parts (a) and (b).

EP #46 - Lightcones and Embedding

A certain spacetime is describe by the metric

$$ds^2 = -(1 - H^2r^2)dt^2 + (1 - H^2r^2)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

(a) Describe the lightcone structure in the (r, t) plane using both equations and a spacetime diagram. Think carefully about the lightcone structure for $r > H^{-1}$ versus $r < H^{-1}$.

(b) Construct an embedding diagram for this spacetime. The following steps will guide you through the process:

  (1) Argue that it is sufficient to consider the 2-dimensional slice

  $$d\Sigma^2 = (1 - H^2r^2)^{-1}dr^2 + r^2d\varphi^2$$

  (2) Pick one of the three common forms for the 3-dimensional flat space line element:

  $$\begin{aligned} ds_{3D}^2 &= dx^2 + dy^2 + dz^2 \\ ds_{3D}^2 &= d\rho^2 + \rho^2d\varphi^2 + dz^2 \\ ds_{3D}^2 &= dw^2 + w^2(d\Theta^2 + \sin^2\Theta d\Phi^2) \end{aligned}$$

  and find the equations that describe the 2-dimensional surface corresponding to the 2-dimensional slice metric above. What is the geometry of this surface?

EP #47 - Time delay to Jupiter

The Solar System is accurately described by the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{c^2r}\right)c^2dt^2 + \left(1 - \frac{2GM}{c^2r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2$$

where M is the mass of the Sun, t the time coordinate, r the radial coordinate, and θ and φ are polar angles.

A radio pulse is sent from the Earth, reflected off a satellite of Jupiter (the satellite is a point), and received on Earth. Jupiter is a distance r_2 from the Sun, the Earth is a distance r_1. Assume that Jupiter is on the other side of the Sun relative to the Earth. Let r_0 be the distance of closest approach of the radio pulse to the Sun. Calculate the gravitational delay in the round-trip time of the radio pulse as a function of r_0, to lowest order in G. Estimate very roughly the magnitude of the effect, given that

$$\begin{aligned} &\text{mass of Sun} \approx 2 \times 10^{33} gm \\ &\text{radius of Sun} \approx 7 \times 10^{10} cm \\ &\text{Sun} - \text{Earth distance} \approx 1.5 \times 10^{13} cm \\ &\text{Sun} - \text{Jupiter distance} \approx 8 \times 10^{13} cm \\ &G \approx 6.67 \times 10^{-8} cm^3/gm - \sec^2 \end{aligned}$$

EP #48 - Geodesic Effect

If in flat spacetime a spacelike vector λ^μ is transported along a timelike geodesic without changing its spatial orientation, then, in Cartesian coordinates, it satisfies $d\lambda^\mu/d\tau = 0$ where τ is the proper time along the geodesic. That is, λ^μ is parallel transported through spacetime along the geodesic. Moreover, if at some point λ^μ is orthogonal to the tangent vector $\dot{x}^\mu = dx^\mu/d\tau$ to the geodesic, then $\eta_{\mu\nu}\lambda^\mu\dot{x}^\nu = 0$, and this relationship is preserved under parallel transport. This orthogonality condition simply means that λ^μ has no temporal component in an instantaneous rest frame of an observer traveling along the geodesic. The corresponding criteria for transporting a spacelike vector λ^μ in this fashion in the curved spacetime of general relativity are, therefore,

$$\frac{d\lambda^\mu}{d\tau} + \Gamma^\mu_{\nu\sigma}\lambda^\nu\dot{x}^\sigma = 0 \quad , \quad g_{\mu\nu}\lambda^\mu\dot{x}^\nu = 0$$

(a) Explain why these are the correct equations.

(b) Consider a spinning particle (perhaps a gyroscope) moving in a gravitational field. No non-gravitational forces are present. Write down and explain the equation which governs the behavior in time of the spin(vector) of the particle.

(c) Consider a slowly rotating thin spherical shell of mass M, radius R and rotation frequency ω. The metric of the field due to this shell can be written as

$$ds^2 = -c^2H(r)dt^2 + \frac{1}{H(r)}\left[dr^2 + r^2d\theta^2 + r^2\sin^2\theta(d\varphi - \Omega dt)^2\right]$$

where $\Omega = 4GM\omega/3Rc^2$ for $r < R$, $\Omega \to 0$ for $r \to \infty$, and

$$H(r) = \begin{cases} 1 - \frac{2GM}{rc^2} & r > R \\ 1 - \frac{2GM}{Rc^2} & r < R \end{cases}$$

This form of the metric is valid if $GM/Rc^2 << 1$. Consider a spinning particle at rest at the center of the sphere($r = 0$). Using the equation from part (b), with what frequency will the spin of the particle precess? What is the precession frequency quantitatively, if ω is the rotational frequency of the Earth and M and R, the mass and radius of the Earth, are $M \approx 6.0\times10^{27}\,gm$ and $R \approx 6.4\times10^3\,km$? A rough estimate is enough.

EP #49 - Kruskal Coordinates

Consider the Schwarzschild metric, which in (t, r, θ, φ) coordinates is

$$ds^2 = -\left(1 - \frac{2M}{r}\right)c^2dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

(a) Show that if we define

$$u' = \left(\frac{r}{2M} - 1\right)^{1/2} e^{(r+t)/4M} \quad , \quad v' = -\left(\frac{r}{2M} - 1\right)^{1/2} e^{(r-t)/4M}$$

the metric in $u', v', \theta, \varphi)$ coordinates (Kruskal coordinates) is

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M} du' dv' + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

(b) Find the locations in the (u,v) plane where this metric has singularities.

(c) What are the possible u,v values for events that can send signals to an event at ($u = u_0$,$v = v_0$)?

(d) What are the possible u,v values for events that can receive signals to an event at ($u = u_0$,$v = v_0$)?

(e) Consider a timelike observer in a cicular orbit at $r = 6M$. How is this described in Kruskal coordinates?

(f) What part of the spacetime cannot send signals to this observer? What part of the spacetime cannot receive signals from this observer?

EP #50 - Perturbing Circular Orbits

A particle is in a circular orbit around a black hole. It is perturbed so that the angular momentum is the same, but the energy is slightly increased so that there is a small velocity component outwards. Describe and sketch the resulting behavior, for initial radii $3M$, $4M$, $5M$, $6M$ and $7M$. HINT: you need to consider both the stability of the circular orbit and whether the particle has sufficient energy to escape to infinity.

EP #51 - Null Geodesics in Strange Metric

Consider the metric

$$ds^2 = -dt^2 + (1 - \lambda r^2)dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

where λ is a positive constant. Consider the null geodesics, and choosing coordinates so that the geodesics lie in the plane $\theta = \pi/2$, show that they satisfy

$$\left(\frac{dr}{d\varphi}\right)^2 = r^2(1 - \lambda r^2)\left(\mu r^2 - 1\right)$$

where μ is a constant. Integrate this and show that the paths of light rays are ellipses.

EP #52 - A Charged Black Hole

The metric for the spacetime around a static spherically symmetric source of mass M and charge Q(in appropriate units) is

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right)$$

This is called the Reissner-Nordstrom metric.

(a) Show that if $Q > M$, this metric is only singular at $r = 0$.

(b) For $Q < M$, the metric in this coordinate system is also singular at $r = r_\pm (r_+ > r_-)$. Find $r_\pm$ in terms of Q, M.

(c) Define a new coordinate u (analogous to Eddington-Finkelstein coordinates) so that the metric in (u, r, θ, φ) coordinates is regular at r_+.

EP #53 - Do Not Touch Anything!

An astronaut in command of a spaceship equipped with a powerful rocket motor enters the horizon $r = r_s$ of a Schwarzschild black hole.

(a) Prove that in proper time no larger than $r_s\pi/2$, the astronaut reaches the singularity at $r = 0$.

(b) Prove that in order to avoid the singularity for as long as possible, the astronaut ought to move in a purely radial direction. HINT: For purely radial motion, with $dr < 0$ and $dt = d\varphi = d\theta = 0$, show that the increment in proper time is

$$d\tau = -\frac{dr}{\sqrt{\frac{r_s}{r} - 1}} \quad , \quad for\ r \le r_s$$

and then integrate this between $r = r_s$ and $r = 0$ to obtain

$$\Delta\tau = \frac{\pi r_s}{2}$$

Finally, check that if dt, $d\varphi$, $d\theta$ are different from zero, then the increment $d\tau$, for a given value of $-dr$, is necessarily smaller than the value given above.

(c) Show that in order to achieve the longest proper time the astronaut must use her rocket motor in the following way: outside the horizon, she must brake her fall so as to arrive at $r = r_s$ with nearly zero radial velocity; inside the horizon she must shut off her motor and fall freely. HINT: show that $\Delta\tau = \pi r_s/2$ corresponds to free fall from $r = r_s$ (do not do anything!).

EP #54 - Escape from Black Hole by Ejecting Mass

A spaceship whose mission is to study the environment around black holes is hovering at the Schwarzschild radius coordinate R outside a spherical black hole of mass M. To escape back to infinity, the crew must eject part of the rest mass of the ship to propel the remaining fraction to escape velocity. What is the largest fraction f of the rest mass that can escape to infinity? What happens to this fraction as R approaches the Schwarzschild radius of the black hole?

EP #55 - Gravitational Wave Stuff

(a) Explain briefly why in Einsteins theory of general relativity it is impossible to have monopole or dipole gravitational radiation.

(b) Suppose two compact stars, each of one solar mass, are in circular orbit around each other with a radius of one solar radius. What is the approximate rate of energy loss due to gravitational radiation from this system? What is the time scale for decay for this orbit? Take

$$solar\ mass = 2 \times 10^{33} gm$$
$$solar\ radius = 7 \times 10^{10} cm$$

EP #56 - Waves from Masses on a Spring

Two equal masses M are at the ends of a massless spring of unstretched length L and spring constant k. The masses started oscillating in line with the spring with an amplitude A so that their center of mass remains fixed.

(a) Calculate the amplitude of gravitational radiation a long distance away from the center of mass of the spring as a function of the angle θ from the axis of the spring to lowest non-vanishing order in A.

(b) Analyze the polarization of the radiation.

(c) Calculate the angular distribution of power radiated in gravitational waves.

EP #57 - Waves from Accelerating Particle

A particle of mass m moves along the $z-$axis according to $z(t) = gt^2/2$ (g is a constant) between times $t = -T$ and $t = +T$ and is otherwise moving with constant speed. Calculate the gravitational wave metric perturbations at a large distance L along the positive z-axis.

EP #58 - Waves from Colliding Battleships

In a desperate attempt to generate gravitational radiation artificially, we take two large battleships of $70,000$ tons each, and we make them collide head-on at $40\,km/h$. Assume that during the collision the battleships decelerate at a constant rate and come to rest in $2.0\,sec$.

(a) Estimate the gravitational energy radiated during the collision. Treat the battleships as point masses.

(b) Could we detect these waves?

EP #59 - Waves from a Cannon

A cannon placed at the origin of coordinates fires a shot of mass $50\,kg$ in a horizontal direction. The barrel of the cannon has a length of $2.0\,m$; the shot has a uniform acceleration while in the barrel and emerges with a muzzle velocity of $300\,m/s$. Calculate the gravitational radiation field generated by the shot at a point P on the $z-$axis at a vertical distance of $20\,m$ above the cannon. What is the maximum value of the wave field? Ignore the gravitational field of the Earth.

EP #60 - Plane Wave Properties

Show that there is a coordinate choice so that the linearized vacuum Einstein equations are

$$\partial^2 h_{ab} = 0$$

where

$$g_{ab} = \eta_{ab} + h_{ab}$$

Find the plane wave solutions to this equation and explain why there are only two polarizations. The transverse trace-free polarization has basis

$$e_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad and \quad e_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

By taking the e_+ polarization, describe the physical effect of a gravitational plane wave.

EP #61 - Robertson-Walker = Minkowski

The Robertson-Walker line element for absolutely empty space, $T_i^j = 0$ and $\Lambda = 0$, is

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1+r^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right)$$

with $a(t) \propto t$. Show that this describes flat space and find the coordinate transformation that brings it to the Minkowski form.

EP #62 - Red Shift in Model Galaxy

Assume that the universe is isotropic and spatially flat. The metric then takes the form

$$ds^2 = -dt^2 + a^2(t)\left(dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right)$$

where r, θ, and φ are co-moving coordinates. By this is meant any galaxy will have constant values of r, θ, φ(peculiar motions of galaxies are neglected). The universe is assumed to be matter-dominated with matter density $\rho(t)$ at time t.

(a) Under this circumstance show that the Einstein equations are

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 \quad and \quad \ddot{a} = -\frac{4\pi G}{3}\rho a$$

(b) From the fact that light propagates along null geodesics, show that the cosmological red shift of spectral lines emitted at time t_e and received at time t_0, defined as

$$Z = \frac{wavelength\ of\ received\ line\ -\ wavelength\ of\ emitted\ line}{wavelength\ of\ emitted\ line}$$

is

$$Z = \frac{a_0}{a_e} - 1$$

where $a_0 = a(t_0)$, $a_e = a(t_e)$.

(c) In the cosmological model under discussion a given galaxy will decrease in angular size with increasing distance from the observer - up to a critical distance. Beyond this the angular size will increase with distance. What is the red shift Z_{crit} corresponding to the minimum in angular size?

EP #63 - Expanding Universe

The metric of the expanding universe has the form

$$ds^2 = dt^2 - R^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

where the possible curvature of space has been neglected. The detailed form of $R(t)$ depends on the matter content of the universe.

(a) A particle of mass m has energy E_0 and momentum p_0 at time t_0; assume $R(t_0) = R_0$. The particle thereafter propagates freely except for the effects of the above metric. Calculate the energy and momentum as a function of time.

(b) Suppose that the early universe contained a gas of non-interacting massless particle (perhaps photons) subject to gravitational effects only. Show that if at time t_0 they were in a thermal distribution at temperature T_0, they remained in a thermal distribution later, but with a temperature that depends on time in a fashion you should determine. HINT: EP60 shows that

$$photon\ frequencies\ change\ like : \tfrac{\nu'}{\nu} = \tfrac{R(t)}{R(t')}$$
$$volumes\ change\ like : \tfrac{V(t')}{V(t)} = \tfrac{R^3(t')}{R^3(t)}$$

(c) Show that, instead, a gas of non-interacting massive particles initially in a thermal distribution would not remain in a thermal distribution under the influence of the expansion of the universe.

(d) Suppose that the early universe contained a non-interacting gas of massless photons and also a non-interacting gas of massive particles of mass m (massive neutrinos to be definite). Suppose that at some early time the photons and neutrinos were both in a thermal distribution with a temperature $kT = mc^2$ (m being the neutrino mass) for both photons and neutrinos. It has been observed that in todays universe the photons are in a thermal distribution with kT about $3 \times 10^{-4}\, eV$. In terms of the neutrino mass, what (roughly) would be the typical velocity and kinetic energy of a neutrino today? Assume $m >> 3 \times 10^{-4}\, eV$.

EP #64 - Homogeneous, Isotropic Universe

Consider a homogeneous, isotropic cosmological model described by the line element

$$ds^2 = -dt^2 + \left(\frac{t}{t_*}\right)\left(dx^2 + dy^2 + dz^2\right)$$

where t_* is a constant.

(a) Is the model open, closed or flat?

(b) Is this a matter-dominated universe? Explain.

(c) Assuming the Friedmann equation holds for this universe, find $\rho(t)$.

EP #65 - Matter-Dominated RW Universe

Suppose that a galaxy is observed to have a red-shift $z = 1$. Assuming a matter-dominated RW cosmology, at what fraction t/t_0 of the present age of the universe did the light leave this galaxy?

EP #66 - Flat Dust Universe

Consider a flat dust universe with zero cosmological constant.

(a) Solve the cosmological equations and derive the time evolution of the scale parameter $a(t)$.

(b) By considering light emitted at time t, and received at the present time t_0, show that the distance to a star of red-shift z is given by

$$s = 3t_0\left(1 - \frac{1}{\sqrt{1+z}}\right)$$

(c) Explain why a flat universe with zero cosmological constant containing a mixture of dust and radiation will eventually be dominated by the dust.

EP #67 - Particle Horizon in Flat Dust Universe

The particle horizon is the radius of the sphere of all particles that could be seen by us. It is the maximum straight line distance that could be travelled by a light ray since the beginning of the universe. Obviously, in a static universe this would be t_0. What is it for a $k = 0$ dust universe?

EP #68 - The Horizon inside a Collapsing Shell

Consider the collapse of a spherical shell of matter of very small thickness and mass M. The shell describes a spherical three-surface in spacetime. Outside the surface, the geometry is the Schwarzschild geometry with this mass. Inside make the following assumptions:

1. The worldline of the shell is known as a function $r(\tau)$ going to zero at some finite proper time.

2. The geometry inside the shell is flat.

3. The geometry of the three-surface of the collapsing shell is the same inside as outside.

(a) Draw two spacetime diagrams: one an Eddington-Finkelstein diagram and the other corresponding to the spacetime inside in a suitable set of coordinates. Draw the worldline of the shell on both diagrams and indicate how points on the inside and outside correspond. Locate the horizon inside the shell as well as outside.

(b) How does the area of the horizon inside the shell change moving along the light rays which generate it?

EP #69 - Two Observers on a Kruskal Diagram

Two observers in two rockets are hovering above a Schwarzschild black hole of mass M. They hover at fixed radius R such that

$$\left(\frac{R}{2M} - 1\right)^{1/2} e^{R/4M} = \frac{1}{2}$$

and fixed angular position. (In fact $R \approx 2.16M$). The first observer leaves this position at $t = 0$ and travels into the black hole on a straight line in a Kruskal diagram until destroyed in the singularity at the point where the singularity crosses $u = 0$. The other observer continues to hover at R.

(a) On a Kruskal diagram sketch the worldlines of the two observers.

(b) Is the observer who goes into the black hole following a timelike worldline?

(c) What is the latest Schwarzschild time after the first observer departs that the other observer can send a light signal which will reach the first before being destroyed in the singularity?

EP #70 - $k = 1$ Robertson-Walker Spacetime

Suppose that the universe is described by a $k = 1$ Robertson-Walker spacetime with metric

$$ds^2 = -dt^2 + R^2(t)dx^2 + \sin^2 x \left(d\theta^2 + \sin^2 \theta d\varphi^2\right)$$

with $R(t) = R_0 t^{2/3}$ at the present epoch. An observer at $t = t_1$ observes a distant galaxy of proper size D perpendicular to the line of sight at $t = t_0$.

(a) What is the observed red shift in terms of R_0, t_0, t_1?

(b) What is the angular diameter of the galaxy, δ, in terms of the red shift?

(c) Show that as the red shift increases δ reaches a minimum for fixed D and then starts to increase.

EP #71 - General Robertson-Walker Spacetime

The Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right)$$

where $\kappa = 0, +1, -1$, according to whether the 3-dimensional space has zero, positive or negative curvature, respectively, gives rise to the first order Einstein field equation

$$\dot{a}^2 + \kappa = \frac{8\pi G}{3}a^2 \quad , \quad \rho a^3 = \text{constant}$$

for a matter-dominated universe of density ρ.

(a) Derive the above field equation.

(b) Calculate the distance $L_r(t)$ from the origin($r = 0$) to a particle with coordinate r at time t, in terms of r, $a(t)$.

Alternatively, we can formulate the theory in purely classical Newtonian terms by ignoring curvature inside a spherical volume of sufficiently small radius, i.e., assume that the space is flat inside the sphere and that any isotropic distribution of matter outside has no effect on curvature inside.

(c) Write down Newtons equation for the acceleration of a particle towards the origin at a distance L away. HINT: Consider a uniform distribution of matter inside a sphere of radius L.

(d) To conserve matter, we must also have $\rho a^3 =$ constant. Combine this with your result in (c) to determine the equations satisfied by the expansion parameter $a(t)$ and compare your answer with the cosmological one.

EP #72 - Spaceship in Robertson-Walker Spacetime

Assume that the geometry of the universe is described by Robertson-Walker metric (c=1)

$$ds^2 = -dt^2 + R^2(t)\left(\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right)$$

A spaceship sets out with velocity v relative to cosmological observers. At a later time when the universe has expanded by a scale factor $(1+z)$, find the velocity v with respect to cosmological observers.

EP #73 - Equation of State

(a) The equation of state is often written in adiabatic form where p is pressure and ρ is density and $0 \leq \gamma \leq 2$ is the adiabatic index with $\gamma = 0$ for dust and $\gamma = 4/3$ for radiation. Calculate $\rho(a)$ for general γ. For $k = 0$, calculate $a(t)$. Find the age of the universe for $k = 0$ and general γ.

(b) In the same notation as (a), find γ so that the expansion rate is constant. With this value of find $a(t)$ for $k = 1$ and $k = -1$.

(c) In the same notation as (a), show

$$\dot{\Omega} = (2 - 3\gamma)H\Omega(1 - \Omega)$$

Define the logarithmic scale factor $s = \log(a)$ and write an equation for $d\Omega/ds$. Notice that this formula gives a clear idea how Ω behaves.

EP #74 - Flat Universe with Period of Inflation

Consider a simplified model of the history of a flat universe involving a period of inflation. The history is split into four periods

1. $0 < t < t_3$ radiation only
2. $t_3 < t < t_2$ vacuum energy dominates with an effective cosmological constant $\Lambda = 3/(4t_3^2)$
3. $t_2 < t < t_1$ a period of radiation dominance
4. $t_1 < t < t_0$ matter domination

(a) Show that in (3) $\rho(t) = \rho_r(t) = 3/(32\pi t^2)$ and in (4) $\rho(t) = \rho_m(t) = 1/(6\pi t^2)$. The functions ρ_r and ρ_m are introduced for later convenience.

(b) Give simple analytic formulas for $a(t)$ which are approximately true in the four epochs.

(c) Show that during the inflationary epoch the universe expands by a factor

$$\frac{a(t_2)}{a(t_3)} = \exp\left(\frac{t_2 - t_3}{2t_3}\right)$$

(d) Show that

$$\frac{\rho_r(t_0)}{\rho_m(t_0)} = \frac{9}{16}\left(\frac{t_1}{t_0}\right)^{2/3}$$

(e) If $t_3 = 10^{-35}$ seconds, $t_2 = 10^{-32}$ seconds, $t_1 = 10^4$ years and $t_0 = 10^{10}$ years, give a sketch of $\log(a)$ versus $\log(t)$ marking any important epochs.

(f) Define what is meant by the particle horizon and calculate how it behaves for this model. Indicate this behavior on the sketch you made. How does inflation solve the horizon problem?

EP #75 - Worm-Hole Metric

Consider the *worm-hole* metric

$$ds^2 = dt^2 - dr^2 - (b^2 + r^2)d\Omega^2$$

Try and work out why this curve is known as a warp-drive.

(a) Find the Christoffel symbols for this geometry.

(b) Find the geodesic equations for this geometry.

EP #76 - Alcubierre Warp-Drive Spacetime

Consider the spacetime known as the Alcubierre Warp-Drive. The coordinates are t, x, y, z and consider a (not necessarily time-like) trajectory given by $x = x_s(t), y = 0, z = 0$. Then the warp-drive spacetime is given by the following metric

$$ds^2 = dt^2 - \left[dx - v_s(t)f(r_s)dt\right]^2 - dy^2 - dz^2$$

where $v_s(t) = dx_s(t)/dt$ is the velocity associated with the curve and $r_s^2 = \left[(x - x_s(t))^2 + y^2 + z^2\right]$ determines the distance of any point from the curve. The function f is smooth and positive with $f(0) = 1$ and vanishes whenever $r_s > R$ for some R. Notice that if we restrict to a curve with constant t, then the the metric is flat and that the metric is flat whenever a spacetime point is sufficiently far away from $x_s(t)$.

(a) Find the null geodesics $ds^2 = 0$ for this spacetime and draw a space-time diagram with some forward and backward light cones along the path $x_s(t)$.

(b) Check that the curve $x_s(t)$ is a geodesic and show that at every point along this curve the 4−velocity of the ship lies within the forward light cone.

(c) Consider the path $x_s(t)$ that connects coordinate time 0 with coordinate time T. How much time elapses for a spaceship traveling along $x_s(t)$?

(d) Calculate the components of a $4-$vectors normal to a surface of constant t.

(e) Show that

$$T_{\alpha\beta}\eta^\alpha\eta^\beta = -\frac{1}{8\pi}\frac{v_s^2\left(y^2+z^2\right)}{2r_s^2}\left(\frac{df}{dr_s}\right)^2$$

This is the energy density measured by observers at rest with respect to the surfaces of constant t. The fact that it is negative means that the warp-drive spacetime cannot be supported by ordinary matter!

EP #77 - General Relativistic Twins

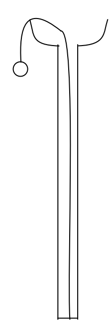
Paul is orbiting a neutron star at a distance of $4GM/c^2$, in circular orbit. Patty, his twin sister, is fired radially outward from the surface with less than escape velocity. Her path crosses Paul's orbit just as Paul comes by, so they synchronize their clocks. On her way back down, Patty again encounters Paul as their trajectories cross, Paul having completed 10 orbits between their meetings. They again compare clocks. How much do their clocks disagree?

EP #78 - Hollow Ball in a Bucket

A hollow plastic ball is held at the bottom of a bucket of water and then released. As it is released, the bucket is dropped over the edge of a cliff. What happens to the ball as the bucket falls?

EP #79 - Einstein's Birthday Present

A version of this device (call an *equivalence principle device* was constructed as a birthday present for Albert Einstein. Simplified, the device consists of a hollow tube with a cup at the top, together with a metal ball and an elastic string as shown below.



When the tube is held vertical, the ball can rest in the cup. The ball is attached to one end of the elastic string, which passes through the hole in the bottom of the cup, and down the hollow center of the tube to the bottom, where the other end is secured. You hold the tube vertical, with your hand at the bottom, the cup at the top, and with the ball out of the cup, suspended on the elastic string. The tension in the string is not quite sufficient to draw the ball back into the cup. The problem is to find an *elegant* way to get the ball back into the cup.

EP #80 - Accelerating Pendulum

A pendulum consists of a light rod and a heavy bob. Initially it is at rest in a vertical stable equilibrium. The upper end is then made to accelerate down a straight line which makes an angle α with the horizontal with constant acceleration f. Show that in the subsequent motion, the pendulum oscillates between the vertical and the horizontal positions if $g = f(\cos\alpha + \sin\alpha)$. This problem is very easy if you apply the equivalence principle and think abou the direction of the apparent gravitational field in an appropriate frame.

EP #81 - What is going on?

For each of the following, either write out the equation with the summation signs included explicitly or say in a few words why the equation is ambiguous or does not make sense.

(i) $x^a = L^a_b M^b_c \hat{(x)}^c$

(ii) $x^a = L^b_c M^c_d \hat{x}^d$

(iii) $\delta^a_b = \delta^a_c \delta^c_d \delta^d_b$

(iv) $\delta^a_b = \delta^a_c \delta^c_c \delta^c_b$

(v) $x^a = L^a_b \hat{x}^b + M^a_b \hat{x}^b$

(vi) $x^a = L^a_b \hat{x}^b + M^a_c \hat{x}^c$

(vii) $x^a = L^a_c \hat{x}^c + M^b_c \hat{x}^c$

EP #82 - Does It Transform Correctly?

Show that if X and Y are vector fields on a manifold, then so is

$$Z^a = X^b \partial_b Y^a - Y^b \partial_b X^a$$

i.e., show that Z transforms correctly under a change of coordinates.

EP #83 - Closed Static Universe

Einstein proposed the following metric as a model for a closed static universe

$$ds^2 = dt^2 - dr^2 - \sin^2 r(d\theta^2 + \sin^2\theta d\phi^2)$$

Find the geodesic equation of the metric from Lagrange's equations and hence write down the Christoffel symbols (take $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$). Show that there are geodesics on which r and θ are constant and equal to $\pi/2$.

EP #84 - Strange Metric

Write down the geodesic equations for the metric

$$ds^2 = dudv + \log(x^2 + y^2)du^2 - dx^2 - dy^2$$

$(0 < x^2 + y^2 < 1)$. Show that $K = x\dot{y} - y\dot{x}$ is a constant of the motion.

By considering an equivalent problem in Newtonian mechanics, show that no geodesic on which $K \neq 0$ can reach $x^2 + y^2 = 0$.

EP #85 - Clocks in Schwarzschild Spacetime

A clock is said to be *at rest* in the Schwarzschild space-time if the r, θ, and ϕ coordinates are constant. Show that the coordinate time and the proper time along the clock's worldline are related by

$$\frac{dt}{d\tau} = \left(1 - \frac{2m}{r}\right)^{-1/2}$$

Note that the worldline is not a geodesic.

Show that along a radial null geodesic, that is, one on which only t and r are varying, that

$$\frac{dt}{dr} = \frac{r}{r - 2m}$$

Two clocks C_1 and C_2 are at rest at (r_1, θ, ϕ) and (r_2, θ, ϕ). A photon is emitted from C_1 at event A and arrives at C_2 at event B. A second photon is emitted from C_1 at event A' and arrives at C_2 at event B'. Show that the coordinate time interval Δt between A and A' is the same as the coordinate time interval between B and B'. Hence show that the time interval $\Delta\tau_1$ between A and A' measured by C_1 is related to the time interval $\Delta\tau_2$ between B and B' measured by C_2 by

$$\Delta\tau_1 \left(1 - \frac{2m}{r_1}\right)^{-1/2} = \Delta\tau_2 \left(1 - \frac{2m}{r_2}\right)^{-1/2}$$

If you wear two watches, one on your wrist and one on your ankle, and you synchronize them at the beginning of the year, by how much is the watch on your wrist faster or slower than the one on your ankle at the end of the year? (Assume that you spend the whole year standing upright without moving. In general units, you must replace m/r by Gm/rc^2).

EP #86 - Particle Motion in Schwarzschild Spacetime

Show that along free particle worldlines in the equatorial plane of the Schwarzschild metric, the quantities

$$J = r^2\dot{\phi} \quad \text{and} \quad E = \left(1 - \frac{2m}{r}\right)\dot{t}$$

are constant. Remember the dot is the derivative with respect to proper time. Explain why the particle cannot escape to infinity if $E < 1$.

Show that

$$\dot{r}^2 + \left(1 + \frac{J^2}{r^2}\right)\left(1 - \frac{2m}{r}\right) = E^2$$
$$\ddot{r} + \frac{m}{r^2} - \frac{J^2}{r^3} + 3\frac{mJ^2}{r^4} = 0$$

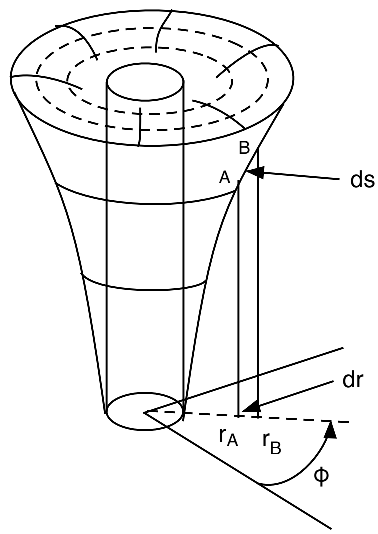
For a circular orbit at radius $r = R$, show that

$$J^2 = \frac{mR^2}{R - 3m} \quad , \quad \frac{d\phi}{dt} = \left(\frac{m}{R^3}\right)^{1/2}$$

Show by letting $r(\tau) = R + \epsilon(\tau)$, with ϵ small, that the circular orbit is stable if and only if $R > 6m$.

EP #87 - Meter Stick Near Black Hole

A standard meter stick lies on the surface shown below(AB). The surface is the two-dimensional riemannian surface defined by the Schwarzschild metric with two coordinates held constant (t = constant, $\theta = \pi/2$ = constant) as viewed(embedded) in three-dimensional euclidean space. The meter stick is oriented in the radial direction. It is then slid inward toward the symmetry axis. The location of its two ends at a given instant in t are reported to a record keeper who plots the the two points shown in the (r, ϕ) plane.



(a) What does the record keeper actually see?

(b) How is the record keeper able to keep track of what is happening physically?

EP #88 - Clocks and Rockets

A rocket of proper length$L$ leaves the earth vertically at speed $4c/5$. A light signal is sent vertically after it which arrives at the rocket's tail at $t = 0$ according to both the rocket and earth based clocks. When does the signal reach the nose of the rocket according to (a) the rocket clocks and (b) according to the earth clocks?

EP #89 - Events in Two Frames

In an inertial frame two events occur simultaneously at a distance of 3 meters apart. In a frame moving with to the inertial or laboratory frame, one event occurs later than the other by 10^{-8} *sec*. By what spatial distance are the two events separated in the moving frame? Solve this problem in two ways: first by finding the Lorentz boost that connects the two frames, and second by making use of the invariance of the spacetime interval between two events.

EP #90 - Geometry in a Curved Space

In a certain spacetime geometry the metric is

$$ds^2 = -(1 - Ar^2)^2 dt62 + (1 - Ar^2)^2 dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

(a) Calculate the proper distance along a radial line from the center $r = 0$ to a coordinate radius $r = R$.

(b) Calculate the area of a sphere of coordinate radius $r = R$.

(c) Calculate the three-volume of a sphere of coordinate radius $r = R$.

(d) Calculate the four-volume of a four-dimensional tube bounded by the sphere of coordinate radius R and two $t =$ constant planes separated by time T.

EP #91 - Rotating Frames

The line element of flat spacetime in a frame (t, x, y, z) that is rotating with an angular velocity Ω about the $z-$axis of an inertial frame is

$$ds^2 = -[1 - \Omega^2(x^2 + y^2)]dt^2 + 2\Omega(ydx - xdy)dt + dx^2 + dy^2 + dz^2$$

(a) Find the geodesic equations for x, y, and z in the rotating frame.

(b) Show that in the non-relativistic limit these reduce to the usual equations of Newtonian mechanics for a free particle in a rotating frame exhibiting the centrifugal force and the Coriolis force.

EP #92 - Negative Mass

Negative mass does not occur in nature. But just as an exercise analyze the behavior of radial light rays in a Schwarzschild geometry with a negative value of mass M. Sketch the Eddington-Finkelstein diagram showing these light rays. Is the negative mass Schwarzschild geometry a black hole?

Solutions

EP #1

No solution yet.

EP #2

No solution yet.

EP #3

No solution yet.

EP #4

No solution yet.

EP #5

(a) If we have two spacelike vectors, $\vec{u}$ and $\vec{v}$, which are orthogonal, then we know the following about the various inner products we can construct:

$$\vec{u} \cdot \vec{u} = u^2 > 0$$
$$\vec{v} \cdot \vec{v} = v^2 > 0$$
$$\vec{u} \cdot \vec{v} = 0$$

Let $\vec{w} = \vec{u} + \vec{v}$. Then

$$\vec{w} \cdot \vec{w} = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) = u^2 + 2\vec{u} \cdot \vec{v} + v^2 > 0$$

So $\vec{w}$ is spacelike, too

(b) If we have a timelike vector, $\vec{u}$, we know we can find a coordinate system in which $\vec{u} \doteq (u^0, 0, 0)$. Further, we can choose this coordinate system such that the null vector $\vec{v} \doteq (v^0, v^0, 0)$. Note that both $u^0 \neq 0$ and $v^0 \neq 0$. Then

$$\vec{u} \cdot \vec{v} = u^\mu v_\mu = -u^0 v^0 \neq 0$$

Because $\vec{u} \cdot \vec{v} \neq 0$, $\vec{u}$ and $\vec{v}$ are not orthogonal.

EP #6

Since $\vec{v}_1 \cdot \vec{v}_2 = 0$, without loss of generality, we can choose a coordinate system which has $\vec{v}_1 \doteq (v_1, 0, 0)$ and $\vec{v}_2 \doteq (0, v_2, 0)$, i.e., given any coordinate system, a rotation and possibly a parity inversion will produce the coordinate system required. Then we have

$$\Lambda_B(\pm\vec{v}_1) = \begin{pmatrix} 1+\frac{v_1^2}{2} & \mp v_1 & 0 & 0 \\ \mp v_1 & 1+\frac{v_1^2}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + O(v_1^3)$$

and

$$\Lambda_B(\pm\vec{v}_2) = \begin{pmatrix} 1+\frac{v_2^2}{2} & \mp v_2 & 0 & 0 \\ \mp v_2 & 1 & 0 & 0 \\ 0 & 0 & 1+\frac{v_2^2}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + O(v_2^3)$$

Multiplying out the matrices, we see that

$$\Lambda_B(\vec{v}_1)\Lambda_B(\vec{v}_2)\Lambda_B(-\vec{v}_1)\Lambda_B(-\vec{v}_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & v_1 v_2 & 0 \\ 0 & -v_1 v_2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

to second-order in v_1 and v_2. We recognize this as a rotation described by the vector $\vec{v}_2 \times \vec{v}_1 = -v_1 v_2 \hat{z}$ in our special coordinate system. (the rotation described by a vector is a right-handed rotation about the axis defined by that vector with angle equal to the vector's magnitude). Because our special coordinate system is related to any arbitrary coordinate system by a rotation and possibly a parity inversion, and $\vec{v}_2 \times \vec{v}_1$ is covariant with respect to these operations, we see that

$$\Lambda_B(\vec{v}_1)\Lambda_B(\vec{v}_2)\Lambda_B(-\vec{v}_1)\Lambda_B(-\vec{v}_2) = R(\vec{v}_2 \times \vec{v}_1)$$

Second part of problem - No solution yet.

EP #7

(a) To be a suitable 4-velocity, a vector must have magnitude -1. We see that $\vec{U}$ does:

$$U^2 = \eta_{\mu\nu}U^\mu U^\nu = -(1+t^2)^2 + t^4 + 2t^2 = -1$$

Unfortunately, $\vec{D}$ does not:

$$D^2 = \eta_{\mu\nu}D^\mu D^\nu = -x^2 + 25t^2x^2 + 2t^2$$

which is not identically -1. However, if we restrict ourselves to the 3-dimensional sub-manifold of spacetime where $-x^2 + 25t^2x^2 + 2t^2 = -1$, then $\vec{D}$ is a suitable 4-velocity. However, it is not a 4-vector velocity for all of spacetime.

(b) A 4-velocity can be written a $(\gamma, \gamma\vec{v})$, so we have

$$v^i = \frac{U^i}{U^0} = \left(\frac{t^2}{1+t^2}, \frac{\sqrt{2}t}{1+t^2}, 0\right)$$

At $t = 0$, we have $\vec{v} = 0$. The velocity will initially increase most rapidly in the $y-$direction, and then, as $t \to \infty$, will asymptote to $\vec{v} = \hat{x}$.

(c) Representing $\partial_\beta U^\alpha$ as a matrix(β indexes rows and α indexes columns), we have

$$\partial_\beta U^\alpha = \begin{pmatrix} 2t & 2t & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Because the components of $\vec{U}$ depend only on t, only terms of the form $\partial_0 U^\alpha$ are nonzero. We find that

$$U_\alpha \partial_\beta U^\alpha = (-2t(1+t^2) + 2t^3 + 2t, 0, 0, 0) = (0, 0, 0, 0)$$

(note that only the first component required any computation). A labor-saving realization is that

$$U_\alpha \partial_\beta U^\alpha = \frac{1}{2}\partial_\beta (U_\alpha U^\alpha) = 0$$

because the norm of $\vec{U}$ is independent of t, x, y, and z.

(d)

$$\partial_\alpha D^\alpha = 5t$$

(e) We have

$$\partial_\beta (U^\alpha D^\beta) = U_\alpha \partial_\beta D^\beta + D^\beta \partial_\beta U_\alpha$$

We can compute this sum by reference to parts (c) and (d). The result is

$$\begin{aligned} \partial_\beta (U^\alpha D^\beta) &= 5t(1+t^2, t^2, \sqrt{2}t, 0) + (x, 5tx, \sqrt{2}t, 0)\begin{pmatrix} 2t & 2t & \sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= (5t^3 + 2tx + 5t, 5t^3 + 2tx, 5\sqrt{2}t^2 + \sqrt{2}x, 0) \end{aligned}$$

(f) Contracting $\vec{U}$ with the result of (e) gives

$$U_\alpha \partial_\beta (U^\alpha D^\beta) = -(1+t^2)(5t^3+2tx+5t)+t^2(5t^3+2tx)+\sqrt{2}t(5\sqrt{2}t^2+\sqrt{2}x) = -5t$$

This must be so because

$$U_\alpha \partial_\beta (U^\alpha D^\beta) = \partial_\beta (U^{U_\alpha \alpha} D^\beta) - U^\alpha D^\beta \partial_\beta U_\alpha$$

The second term on the right is zero(see (c)), and the first reduces to

$$U_\alpha \partial_\beta (U^\alpha D^\beta) = -\partial_\alpha D^\alpha$$

which has been computed in (d).

(g) We have

$$\partial_\mu \rho = (2t, 2x, -2y, 0)$$

Raising the index (contracting with η, we obtain

$$\partial^\mu \rho = (-2t, 2x, -2y, 0)$$

(h) We have

$$\nabla_{\vec{U}} \rho = U^\mu \partial_\mu \rho = 2t(1+t^2) + 2xt^2 - 2y\sqrt{2}t$$

and

$$\nabla_{\vec{D}} \rho = D^\mu \partial_\mu \rho = 2tx + 10tx^2 - 2\sqrt{2}yt$$

EP #8

(a)

$$\begin{aligned} U_\alpha V_\perp^\alpha &= U_\alpha P^\alpha_\beta V^\beta = U^\alpha P_{\alpha\beta} V^\beta \\ &= \eta_{\alpha\beta} U^\alpha V^\beta + U^\alpha U_\alpha U_\beta V^\beta = U^\alpha V_\alpha - U^\beta V_\beta = 0 \end{aligned}$$

(b)

$$P^\alpha_\beta = \eta^{\alpha\gamma} P_{\gamma\beta} = \delta^\alpha_\beta + U^\alpha U_\beta$$

Applying this, we have

$$\begin{aligned} P^\alpha_\beta V^\beta_\perp &= P^\alpha_\beta P^\beta_\gamma V^\gamma \\ &= (\delta^\alpha_\beta \delta^\beta_\gamma + \delta^\alpha_\beta U^\beta U_\gamma + U^\alpha U_\beta \delta^\beta_\gamma + U^\alpha U_\beta U^\beta U_\gamma) V^\gamma \\ &= (\delta^\alpha_\gamma + U^\alpha U_\gamma) V^\gamma = P^\alpha_\gamma V^\gamma = V^\alpha_\perp \end{aligned}$$

(c)

$$P_{\alpha\beta} V^\alpha_\perp W^\beta_\perp = V^\alpha_\perp \eta_{\alpha\gamma} P^\gamma_\beta W^\beta_\perp$$

But, $\vec{W}_\perp$ is unaffected by the projection (see (b)), so we have

$$P_{\alpha\beta} V^\alpha_\perp W^\beta_\perp = V^\alpha_\perp \eta_{\alpha\gamma} W^\gamma_\perp = \vec{V}_\perp \cdot \vec{W}_\perp$$

(d) We must verify the properties in (a) and (b) for the general $P^\alpha_\beta(\vec{q})$; then the property (c) will follow automatically (we never used that $U^\alpha U_\alpha = -1$ in the proof of (c)). First we show that $q^\alpha P_{\alpha\beta}(\vec{q}) = 0$:

$$q^\alpha P_{\alpha\beta}(\vec{q}) = q_\beta - \frac{q^\alpha q_\alpha q_\beta}{q^\gamma q_\gamma} = q_\beta - q_\beta = 0$$

The, we show that $P^\alpha_\beta(\vec{q})P^\beta_\gamma(\vec{q}) = P^\alpha_\gamma(\vec{q})$:

$$\begin{aligned}
P^\alpha_\beta(\vec{q})P^\beta_\gamma(\vec{q}) &= \left(\delta^\alpha_\beta - \frac{q^\alpha q_\beta}{q^\rho q_\rho}\right)\left(\delta^\beta_\gamma - \frac{q^\beta q_\gamma}{q^\sigma q_\sigma}\right) \\
&= \delta^\alpha_\beta \delta^\beta_\gamma - \delta^\alpha_\beta \frac{q^\beta q_\gamma}{q^\sigma q_\sigma} - \delta^\beta_\gamma \frac{q^\alpha q_\beta}{q^\rho q_\rho} + \frac{q^\alpha q_\beta q^\beta q_\gamma}{(q^\rho q_\rho)^2} \\
&= \delta^\alpha_\gamma - \frac{q^\alpha q_\gamma}{(q^\rho q_\rho)^2} = P^\alpha_\gamma(\vec{q})
\end{aligned}$$

EP #9

The Killing's equation in flat space is

$$k_{a,b} + k_{b,a} = 0$$

Clearly three solutions are given by $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Also, if $a = b$ we have $k_{a,a} = 0$ where there is no sum on the a. Differentiating the above equation we find that

$$k^{;a}_{a,b} + k^{;a}_{b,a} = k^{;a}_{b,a} = 0$$

so the k_b are all harmonic functions, ignoring the constant part, already mentioned, this give $k = (a_1 y + a_2 z, a_3 x + a_4 z, a_5 x + a_6 y)$. Now, substitute this back into the Killing's equations, remembering that the three $a = b$ equations have already been solved,

$$a_1 + a_3 = 0 \quad , \quad a_2 + a_5 = 0 \quad , \quad a_4 + a_6 = 0$$

Thus, a basis is given by $(y, -x, 0)$, $(-z, 0, x)$ and $(0, z, -y)$.

EP #10

Consider the oblate spherical coordinates given by:

$$x = \sqrt{r^2 + c^2}\sin\theta\cos\varphi \quad , \quad y = \sqrt{r^2 + c^2}\sin\theta\sin\varphi \quad , \quad z = r\cos\theta$$

(a) If $r = R$, then we have the relation

$$\begin{aligned}
\left(\frac{z}{R}\right)^2 + \left(\frac{x}{\sqrt{r^2+c^2}}\right)^2 + \left(\frac{y}{\sqrt{r^2+c^2}}\right)^2 &= \cos\theta^2 + \sin\theta^2(\cos\varphi^2 + \sin\varphi^2) \\
&= 1
\end{aligned}$$

Thus, surfaces with constant R are ellipsoids that have a *squashed vertical* or an *oblate spheroid*.

(b) In x, y, z coordinates, the metric is given by

$$g_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

or

$$[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Transform to oblate spherical coordinates we have

$$\bar{g}_{ij} = \frac{\partial x^m}{\partial x'^i}\frac{\partial x^n}{\partial x'^j} g_{mn} = (J^T)_i{}^m g_{mn} J^n{}_j$$

where

$$J^m{}_i = \begin{pmatrix} \frac{r\sin\theta\cos\varphi}{\sqrt{r^2+c^2}} & \sqrt{r^2+c^2}\cos\theta\cos\varphi & -\sqrt{r^2+c^2}\sin\theta\sin\varphi \\ \frac{r\sin\theta\sin\varphi}{\sqrt{r^2+c^2}} & \sqrt{r^2+c^2}\cos\theta\sin\varphi & \sqrt{r^2+c^2}\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

Then the metric is

$$[\bar{g}_{ij}] = \begin{pmatrix} \frac{r^2+c^2\cos\theta^2}{r^2+c^2} & 0 & 0 \\ 0 & r^2+c^2\cos\theta^2 & 0 \\ 0 & 0 & (r^2+c^2)\sin\theta^2 \end{pmatrix}$$

or

$$dx^2 + dy^2 + dz^2 = \frac{r^2+c^2\cos\theta^2}{r^2+c^2}dr^2 + (r^2+c^2\cos\theta^2)d\theta^2 + (r^2+c^2)\sin\theta^2 d\varphi^2$$

Either way, not too bad!

(c) The determinant of the metric is

$$g = \det g_{ij} = (r^2+c^2\cos\theta^2)^2\sin\theta^2$$

and using the formula

$$\nabla^2\Phi = \frac{1}{\sqrt{g}}(\sqrt{g}g^{ij}\Phi_{,i})_{,j}$$

the Laplacian is

$$\begin{aligned}\nabla^2\Phi = &\frac{1}{(r^2+c^2\cos\theta^2)\sin\theta}\frac{\partial}{\partial r}\left[(r^2+c^2\cos\theta^2)\sin\theta\left(\frac{r^2+c^2}{r^2+c^2\cos\theta^2}\right)\frac{\partial\Phi}{\partial r}\right] \\ &+ \frac{1}{(r^2+c^2\cos\theta^2)\sin\theta}\frac{\partial}{\partial\theta}\left[(r^2+c^2\cos\theta^2)\sin\theta\left(\frac{1}{r^2+c^2\cos\theta^2}\right)\frac{\partial\Phi}{\partial\theta}\right] \\ &+ \frac{1}{(r^2+c^2\cos\theta^2)\sin\theta}\frac{\partial}{\partial\varphi}\left[(r^2+c^2\cos\theta^2)\sin\theta\left(\frac{1}{(r^2+c^2)\sin\theta^2}\right)\frac{\partial\Phi}{\partial\varphi}\right]\end{aligned}$$

This simplifies to

$$\begin{aligned}\nabla^2\Phi = &\frac{1}{(r^2+c^2\cos\theta^2)}\left[\frac{\partial}{\partial r}\left((r^2+c^2)\frac{\partial\Phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)\right] \\ &+ \frac{1}{(r^2+c^2)\sin\theta^2}\frac{\partial^2\Phi}{\partial\varphi^2}\end{aligned}$$

Alternatively, the Laplacian is given by

$$\nabla_a(\nabla\Phi) = (\nabla\Phi)^a_{;a}$$

(d) As always,

$$\frac{d^2Q}{d\varphi^2} = -m^2 Q \to Q = e^{im\varphi}$$

The θ operator looks a lot like what you get in spherical coordinates, so we try the same Legendre function solution $P_\ell^m(\cos\theta)$,

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \left[\ell(\ell+1) - \frac{m^2}{\sin\theta^2}\right]P = 0$$

Remarkably, these combine to leave

$$\frac{d}{dr}\left((r^2+c^2)\frac{dR}{dr}\right) - \left(\ell(\ell+1) - \frac{m^2c^2}{r^2+c^2}\right)R = 0$$

Thus, the potential separates into $\Phi = R_{\ell m}(r)Y_{\ell m}(\theta,\varphi)$, where the functions satisfy the equations above

(e) If the spheroidal disk is conducting, then Φ must be independent of θ and φ on the surface, so $\ell = m = 0$. For this case,

$$\frac{d}{dr}\left((r^2+c^2)\frac{dR}{dr}\right) = 0 \to (r^2+c^2)\frac{dR}{dr} = K$$

where K is an integration constant. The solution that vanishes as $r \to \infty$ is

$$\Phi = \frac{K}{c}\left(\frac{\pi}{2} - \tan^{-1}\frac{r}{c}\right)$$

We determine K from the behavior as $r \to \infty$, $\Phi \to Q/r$, where Q is the charge on the disk. The potential is then

$$\Phi = \frac{Q}{c}\left(\frac{\pi}{2} - \tan^{-1}\frac{r}{c}\right)$$

The surface charge density is found from the (proper) normal electric field at the surface,

$$\sigma = \frac{1}{4\pi}E_{\hat{r}} = -\frac{1}{4\pi}\sqrt{g^{rr}}\frac{\partial\Phi}{\partial r} = \frac{Q}{4\pi(r^2+c^2)}\sqrt{\frac{r^2+c^2}{r^2+c^2\cos^2\theta}}$$

to be integrated over the proper surface area,

$$d^2a = \sqrt{r^2+c^2\cos^2\theta}\sqrt{r^2+c^2}\sin\theta d\theta d\varphi$$

Thus,

$$dQ = \frac{Q}{4\pi}\sin\theta d\theta d\varphi = \sigma(\rho)\rho d\rho d\varphi$$

and so (adding upper and lower surfaces) the charge density is

$$\sigma(\rho) = \frac{Q}{2\pi} \frac{\sin\theta}{\rho} \frac{d\theta}{d\rho}$$

The polar radius is $\rho^2 = x^2 + y^2 \to c^2 \sin^2\theta$ as $r \to 0$, and so

$$\sigma(\rho) = \frac{Q}{2\pi c^2} \frac{1}{\cos\theta} = \frac{Q}{2\pi c^2} \frac{1}{\sqrt{1 - \frac{\rho^2}{c^2}}}$$

EP #11

(a) We know that $u^2 = -1$, $a^2 = g^2$, and $\vec{u} \cdot \vec{a} = 0$. Writing these in terms of components of $\vec{u}$, we find

$$\begin{aligned} (u^x)^2 &= (u^t)^2 - 1 \\ \frac{du^x}{d\tau}^2 &= \frac{du^t}{d\tau}^2 + g^2 \\ u^t \frac{du^t}{d\tau} &= u^x \frac{du^x}{d\tau} \end{aligned}$$

These equations reduce to

$$\frac{du^x}{d\tau} = g\sqrt{1 + (u^x)^2}$$

which has the solution with initial condition $u^x(0) = 0$ (initially the accelerated axes agree with (t, x), so the motion is purely in the time direction

$$u^x = \sinh(g\tau) = \sinh(g\tilde{t})$$

which implies that

$$u^t = \cosh(g\tilde{t})$$

Given these results, we have

$$a^t = g\sinh(g\tilde{t}) \quad , \quad a^x = g\cosh(g\tilde{t})$$

(b) Integrating, we obtain

$$T(\tilde{t}) = \int_0^{\tilde{t}} \cosh(gs)\, ds = \frac{1}{g}\sinh(g\tilde{t})$$

and

$$X(\tilde{t}) = \int_0^{\tilde{t}} \sinh(gs)\, ds + \frac{1}{g} = \frac{1}{g}\cosh(g\tilde{t})$$

Figure 1 below shows a plot of this trajectory

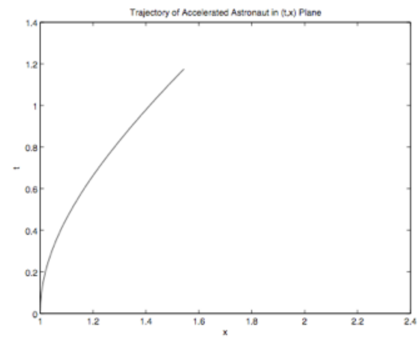


Figure 1: The trajectory of an accelerated observer who accelerates with $a^2 = 1$ and begins at the point $(0, 1)$ in the (t, x) plane.

(c) A good vector for $\vec{e}_{\tilde{t}}$ is $\vec{u}$. We want then to find $\vec{e}_{\tilde{x}}$ such that

$$\vec{e}_{\tilde{t}} \cdot \vec{e}_{\tilde{x}} = 0 \quad \text{and} \quad \vec{e}_{\tilde{x}} \cdot \vec{e}_{\tilde{x}} = 1$$

These equations tell us that we must have

$$e^t_{\tilde{x}} = \sinh{(g\tilde{t})} \quad \text{and} \quad e^x_{\tilde{x}} = \cosh{(g\tilde{t})}$$

Note that $\vec{e}_{\tilde{x}}$ is parallel to the acceleration, $\vec{a}$ (since we are in two dimensions, all vectors perpendicular to $\vec{u}$ ar parallel). Figure 2 below shows these basis vectors attached to the trajectory from part (b).

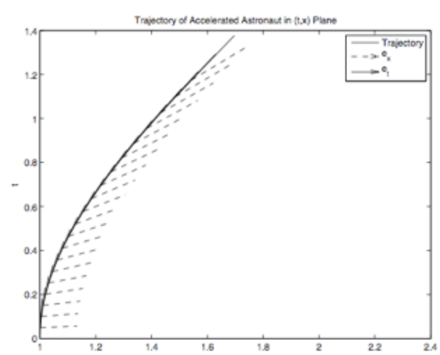


Figure 2: Trajectory and basis vectors for an accelerated observer who begins at the point $(0, 1)$ in the (t, x) plane and accelerates with $a^2 = 1$.

(d) To be parallel to $\vec{e}_{\tilde{x}}$, we require that the surface be a line with slope $m = e^t_{\tilde{x}}/e^x_{\tilde{x}} = \tanh{(gt)}$. The unique line with this slope which passes through the point $(T(\tilde{t}), X(\tilde{t}))$, is given by the equation

$$t = \tanh{(g\tilde{t})}x \quad or \quad x = \coth{(g\tilde{t})}t$$

These are surfaces of constant $\tilde{t}$. Up to this point, the vectors $\vec{e}_{\tilde{t}}$ and $\vec{e}_{\tilde{x}}$ lived only on the trajectory. We have now extended $\vec{e}_{\tilde{t}}$ to the entire spacetime (it is everywhere perpendicular to surfaces of constant $\tilde{t}$, which we just defined). It has now become a vector field.

(e) There was nothing special about our derivation of the trajectory in part (b) except the initial condition that $x = 1/g$ when $t = \tilde{t} = 0$. For an accelerated observer with $x = \tilde{x}$ when $t = \tilde{t} = 0$ (these are the CSOs in question), we have

$$t = \tilde{x}\sinh{(g\tilde{t})} \quad and \quad x = \tilde{x}\coth{(g\tilde{t})}$$

You can verify this by noting that it solves the differential equations in (b) and, when specialized to the initial condition in (b) reproduces the result we derived there). We see that $A = g\tilde{x}$. Now that we have the complete coordinate transforamtion between the barred and un-barred coordinates, we have extended both $\vec{e}_{\tilde{t}}$ and $\vec{e}_{\tilde{x}}$ to be vectors on spacetime.

(f) We have the differentials of the coordinate transformation from $(\tilde{t}, \tilde{x})$ to (t, x) given in (e):

$$\begin{aligned} dt &= \sinh{(g\tilde{t})}d\tilde{x} + g\tilde{x}\cosh{(g\tilde{t})}d\tilde{t} \\ dx &= \cosh{(g\tilde{t})}d\tilde{x} + g\tilde{x}\sinh{(g\tilde{t})}d\tilde{t} \end{aligned}$$

from which

$$ds^2 = -dt^2 + dx^2 = -(g\tilde{x})^2 d\tilde{t}^2 + d\tilde{x}^2$$

EP #12 (Solution due to Erin Martell - Class of 2009)

We consider a seagull on the ground. The wind is blowing with velocity $\vec{v}_{wind} = \vec{v}$ and the seagull is going to rise without doing any work.

In this case, we consider an *ideal seagull.* The seagull will rise according to the lift force, which normally requires a dissipative component. However, we neglect dissipation of energy from the force.

We first consider the case where the wind is in the transverse direction and look in the wind frame. The seagull will initially by moving with a velocity $-\vec{v}_{wind} = -\vec{v}$ in this frame and it will have kinetic energy $mv^2_{wind}/2$. Since the seagull is moving with respect to the air surrounding it, there is a lift force generated. This lift force in the case that the seagull is still moving. We can thus say, in the frame of the wind, using conservation of energy, that the greatest height that the seagull attains may be found by

$$mgh = \frac{1}{2}v_{wind}^2 \rightarrow h = \frac{v_{wind}^2}{2g}$$

Since we have allowed the seagull to experience a pure lift force, no work has been done, and the seagull has risen a distance $v_{wind}^2/2g$.

What if there is a vertical component of gravity, which also contributes to the force on the bird? There are two possible cases:

1. The lift force from an upward vertical component of the wind velocity is sufficient to overcome gravity or is equally strong. In this case, the bird will rise due to the lift force from the wind. If the initial wind velocity were upward, it will accelerate the bird until its velocity matches that of the wind. The bird will no longer be accelerated at that point by the lift force. However, the bird will then begin to fall, which will restore the lift force. Therefore, the bird will again be accelerated to match the velocity of the wind. The bird still has an upward velocity but will slow. This slowing, however, results in an accelerating lift force, so the bird ends up maintaining an upward velocity.

   If the initial wind velocity were downward, the wind velocity will become even greater relative to ... ... as it rises and the bird will continue to accelerate upward, escaping from the gravitational field of the Earth.

2. The lift component from the vertical component of the wind is less strong than gravity. In this case, if the transverse components are strong enough to generate a lift force, the bird will rise. If the initial wind velocity was upward, then the bird will come to match that velocity, at which point it will slow its upward velocity and eventually begin to fall - the transverse components are irrelevant now because the bird is moving with the wind and the lift force from the vertical component is not strong enough to overcome gravity.

   If the initial wind velocity were downward, then the bird will be accelerated upward until its transverse components are the same as the initial velocity. At this point, the bird may fall if the bird's velocity does not generate a strong enough force to overcome gravity, or it may be accelerated upward forever.

EP #13 (Solution due to Ben Good - Class of 2010)

Consider two particles with masses m_1 and m_2 interacting via the gravitational potential. If their positions are given by $\vec{r}_1$ and $\vec{r}_2$, then the Lagrangian is

$$L = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 + \frac{Gm_1m_2}{|\vec{r}_2 - \vec{r}_1|}$$

If we introduce the center of mass (CM) and relative coordinates defined by

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M} \quad , \quad \vec{r} = \vec{r}_2 - \vec{r}_1$$

where $M = m_1 + m_2$, then we can solve for $\vec{r}_1$ and $\vec{r}_2$ to obtain

$$\vec{r}_1 = \vec{R} - \frac{m_2}{M}\vec{r} \quad , \quad \vec{r}_2 = \vec{R} + \frac{m_1}{M}\vec{r}$$

and so we have

$$\begin{aligned}
\dot{\vec{r}}_1^2 &= \dot{\vec{R}}^2 + \left(\frac{m_2}{M}\right)^2 \dot{\vec{r}}^2 - 2\frac{m_2}{M}\dot{\vec{R}} \cdot \dot{\vec{r}} \\
\dot{\vec{r}}_2^2 &= \dot{\vec{R}}^2 + \left(\frac{m_1}{M}\right)^2 \dot{\vec{r}}^2 + 2\frac{m_1}{M}\dot{\vec{R}} \cdot \dot{\vec{r}}
\end{aligned}$$

and so the Lagrangian becomes

$$\begin{aligned}
L = \frac{1}{2}m_1\dot{\vec{R}}^2 + \frac{1}{2}\frac{m_1 m_2^2}{M^2}\dot{\vec{r}}^2 - \frac{m_1 m_2}{M}\dot{\vec{R}} \cdot \dot{\vec{r}} \\
+ \frac{1}{2}m_2\dot{\vec{R}}^2 + \frac{1}{2}\frac{m_2 m_1^2}{M^2}\dot{\vec{r}}^2 + \frac{m_1 m_2}{M}\dot{\vec{R}} \cdot \dot{\vec{r}} + \frac{G m_1 m_2}{|\vec{r}|}
\end{aligned}$$

or

$$L = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 + \frac{G m_1 m_2}{|\vec{r}|}$$

Now, L is cyclic in $\vec{R}$, so we obtain

$$M\dot{\vec{R}} = \text{ constant } = \vec{P} \quad \text{or} \quad \dot{\vec{R}} = \frac{\vec{P}}{M} = \text{ constant}$$

Thus, up to an additive constant,

$$L = \frac{1}{2}\mu\dot{\vec{r}}^2 + \frac{G m_1 m_2}{|\vec{r}|}$$

Now the Lagrangian is rotationally invariant, so we must then have conservation of angular momentum, defined by

$$\vec{r} \times m\dot{\vec{r}} = \vec{r} \times \vec{p} = \text{ constant } = \vec{\ell}$$

Then if we consider $\vec{r}$ at times t and $t + dt$, we have

$$\vec{r}(t + dt) \approx \vec{r}(t) + \dot{\vec{r}}\,dt$$

Using this, we see that the plane swept out by $\vec{r}$ at two successive moments, whose normal is given by

$$\hat{n} = \frac{\vec{r}(t) \times \vec{r}(t + dt)}{|\vec{r}(t) \times \vec{r}(t + dt)|} = \frac{\vec{r} \times \vec{r} + \vec{r} \times \dot{\vec{r}}\,dt}{|\vec{r} \times \vec{r} + \vec{r} \times \dot{\vec{r}}\,dt|} = \frac{\vec{\ell}\frac{dt}{N}}{|\vec{\ell}\frac{dt}{N}|} = \hat{\ell}$$

is constant. Thus, all motion occurs in the plane whose normal is $\vec{\ell}$.

If we define our axes by

$$\hat{x} = \frac{\vec{r}(0)}{|\vec{r}(0)|} \quad , \quad \hat{y} = \frac{\dot{\vec{r}}(0) - (\dot{\vec{r}}(0) \cdot \hat{x})\hat{x}}{|\dot{\vec{r}}(0) - (\dot{\vec{r}}(0) \cdot \hat{x})\hat{x}|} \quad , \quad \hat{z} = \frac{\vec{r}(0)}{|\vec{r}(0)|}$$

then this basis is orthonormal(as we can check) and

$$\begin{aligned} \dot{\vec{r}}^2 &= (\dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}) \cdot (\dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}) \\ &= \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \end{aligned}$$

and

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

Now if we introduce spherical polar coordinates r, θ, φ such that

$$x = r \sin\theta \cos\varphi \quad , \quad y = r \sin\theta \sin\varphi \quad , \quad z = r\cos\theta$$

then we have $|\vec{r}| = r$ and due to our choice of basis vector all motion is in the $x - y$ plane or $\theta = \pi/2$ for all t. Thus, we have

$$x = r\cos\varphi \quad , \quad y = r\sin\varphi \quad , \quad z = 0$$

and

$$\begin{aligned} \dot{\vec{r}}^2 &= \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = (\dot{r}\cos\varphi - r\sin\varphi\dot{\varphi})^2 + (\dot{r}\sin\varphi + r\cos\varphi\dot{\varphi})^2 \\ &= \dot{r}^2\cos^2\varphi + \dot{r}^2\sin^2\varphi + r^2\dot{\varphi}^2\sin^2\varphi + r^2\dot{\varphi}^2\cos^2\varphi \\ &\qquad - 2r\dot{r}\dot{\varphi}\sin\varphi\cos\varphi + 2r\dot{r}\dot{\varphi}\sin\varphi\cos\varphi \\ &= \dot{r}^2 + r^2\dot{\varphi}^2 \end{aligned}$$

Thus, the Lagrangian becomes

$$L = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\varphi}^2 + \frac{GM\mu}{r}$$

and the equations of motion are

$$\begin{aligned} \mu r\dot{\varphi}^2 - \frac{GM\mu}{r^2} &= \mu\ddot{\vec{r}} \\ \mu r\dot{\varphi}^2 &= \text{ constant } = \ell \end{aligned}$$

Thus, we can eliminate $\dot{\varphi}$ to obtain

$$\mu\ddot{\vec{r}} = \frac{\ell^2}{\mu r^3} - \frac{GM\mu}{r^2}$$

Now, if we consider r as a function of φ rather than t and we change the variables to $r = 1/u$, then we obtain

$$\dot{r} = -\frac{1}{u^2}u'\dot{\varphi} \quad \text{ where } u' = \frac{du}{d\varphi}$$

and

$$\ddot{r} = \frac{2}{u^3} u'^2 \dot{\varphi}^2 - \frac{u'}{u^2} \ddot{\varphi} - \frac{\dot{\varphi}}{u^2} u'' \dot{\varphi}$$

where

$$\ddot{\varphi} = +\frac{2\ell}{\mu} u u' \dot{\varphi} = -\frac{2\ell^2 u'}{\mu^2} u^3$$

Thus,

$$\ddot{r} = -\frac{\ell^2 u^2}{\mu^2} u''$$

so that we get (the previous equation)

$$-\frac{\ell^2 u^2}{\mu^2} u'' = \frac{\ell^2}{\mu} u^3 - GM\mu u^2$$

or (assuming $u \neq 0$)

$$u'' = -\left(u - \frac{GM\mu^2}{\ell^2}\right)$$

If we now let

$$u_2 = u - \frac{GM\mu^2}{\ell^2}$$

then we obtain

$$u_2'' = -u_2 notag \tag{1}$$

or

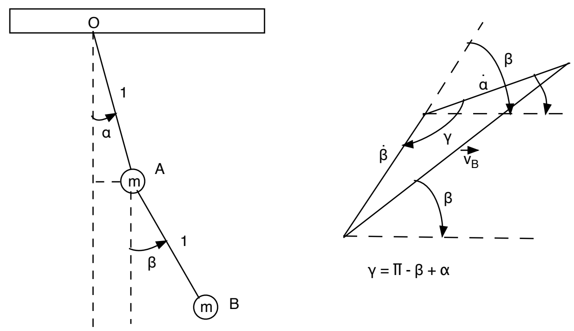
$$u_2 = A\cos(\varphi + \delta)$$

and thus

$$r(\varphi) = \frac{1}{A\cos(\varphi + \delta) + \frac{GM\mu^2}{\ell^2}} = \frac{\frac{\ell^2}{GM\mu^2}}{1 + \varepsilon\cos(\varphi + \delta)}$$

which is the Kepler orbital equation.

EP #14 (Solution due to Markus Kliegel - Class of 2010)



Note we have chosen the string length to be equal to 1 for convenience. We then have referring to the figures above)

$$v_B^2 = \dot{\alpha}^2 + \dot{\beta}^2 + 2\dot{\alpha}\dot{\beta}\cos(\alpha - \beta)$$

The kinetic and potential energies are:

$$T = \frac{1}{2}m\dot{\alpha}^2 + \frac{1}{2}m(\dot{\alpha}^2 + \dot{\beta}^2 + 2\dot{\alpha}\dot{\beta}\cos(\alpha - \beta))$$
$$U = mg(1 - \cos\alpha) + mg(1 - \cos\beta) + 1 - \cos\alpha)$$

Thus, the Lagrangian is

$$L = T - U = m\dot{\alpha}^2 + \frac{1}{2}m\dot{\beta}^2 + m\dot{\alpha}\dot{\beta}\cos(\alpha - \beta) + 2mg\cos\alpha + mg\cos\beta$$

Using

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

where $q = \alpha, \beta$, we then have two Lagrange equations:

$$-2mg\sin\alpha - m\dot{\alpha}\dot{\beta}\sin(\alpha - \beta) = 2m\ddot{\alpha} + m\ddot{\beta}\cos(\alpha - \beta) - m\dot{\beta}(\dot{\alpha} - \dot{\beta})\sin(\alpha - \beta)$$

$$-2mg\sin\beta + m\dot{\alpha}\dot{\beta}\sin(\alpha - \beta) = m\ddot{\beta} + m\ddot{\alpha}\cos(\alpha - \beta) - m\dot{\alpha}(\dot{\alpha} - \dot{\beta})\sin(\alpha - \beta)$$

For small angles we have the two equations:

$$-2mg\alpha = 2m\ddot{\alpha} + m\ddot{\beta}$$

$$-mg\beta = m\ddot{\alpha} + m\ddot{\beta}$$

This corresponds to the equation(matrix)

$$M\ddot{\vec{x}} = -K\vec{x} \rightarrow \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{pmatrix} = -\begin{pmatrix} 2g & 0 \\ 0 & g \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

The standard normal modes methods then give the characteristic frequencies:

$$\omega_1^2 = (2 + \sqrt{2})g \quad , \quad \omega_2^2 = (2 - \sqrt{2})g$$

and the mode behavior

$$\beta = -\sqrt{2}\alpha \quad , \quad \beta = +\sqrt{2}\alpha$$

Thus, the general solution is

$$\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = A\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}\cos\left(\sqrt{(2 + \sqrt{2})g}t - \delta_1\right) + B\cos\left(\sqrt{(2 - \sqrt{2})g}t - \delta_2\right)$$

EP #15

(a) The Lorentz transformation gives $t' = \gamma(t - vy)$, $y' = \gamma(y - vt)$, $x' = x$, $z' = z$. Thus we have

$$\vec{x}\,' = (\gamma(t - vr\sin\omega t), r\cos\omega t, \gamma(r\sin\omega t - vt), 0)$$

where t is now a parameter (not time in the new frame).

(b) We have $d\tau = ds = dt/\gamma$, so the observed half-life in the laboratory frame is $t = \tau/\sqrt{1 = \omega^2 r^2} > \tau$ since the velocity is $r\omega$.

(c) We differentiate to find

$$u^\alpha = \frac{dx^\alpha}{d\tau} = \gamma\frac{dx^\alpha}{dt} = \gamma(1, -r\omega\sin\omega t, r\omega\cos\omega t, 0)$$

where $\gamma = 1/\sqrt{1 - r^2\omega^2}$.

$$a^\alpha = \frac{du^\alpha}{d\tau} = \gamma\frac{du^\alpha}{dt} = \gamma^2(0, -r\omega^2\cos\omega t, -r\omega^2\sin\omega t, 0)$$

$$\alpha = \sqrt{-\vec{a}\cdot\vec{a}} = \gamma^2 r\omega^2 = \frac{r\omega^2}{1 - r^2\omega^2}$$

This calculation is very straightforward because γ does not depend on t.

EP #16

(a) The velocity addition law is

$$u = \frac{u' + v}{1 + u'v}$$

This gives

$$\begin{aligned} du &= \frac{du'}{1 + u'v} - \frac{u' + v}{(1 + u'v)^2} v\,du' \\ &= \frac{du'}{(1 + u'v)^2}(1 + u'v - u'v - v^2) = \frac{1 - v^2}{(1 + u'v)^2} du' \end{aligned}$$

We also have

$$dt = \gamma(dt' + v\,dx') = \gamma\,dt'(1 + u'v)$$

Thus,

$$\frac{du}{dt} = \frac{(1 - v^2)^{3/2}}{(1 + u'v)^3}\frac{du'}{dt'}$$

(b) Using the given conditions, we have

$$\frac{dv}{dt} = (1 - v^2)^{3/2} g$$

which gives

$$\frac{dv}{(1 - v^2)^{3/2}} = g dt$$

Integrating we have

$$\frac{v}{(1 - v^2)^{1/2}} = gt \to v^2(1 - v^2) = (gt)^2 \to v = \frac{gt}{(1 + (gt)^2)^{1/2}}$$

We have

$$dx = \frac{gt}{(1 + (gt)^2)^{1/2}} dt$$

Integrating we get

$$x = \frac{1}{g}\left[(1 + (gt)^2)^{1/2} - 1\right]$$

Check: For $gt \ll 1$, we get

$$x = \frac{1}{g}\left[1 + (gt)^2/2 - 1\right] = \frac{1}{2} g t^2$$

For $gt \gg 1$, we get $x = t$.

Now we have

$$\frac{dt}{d\tau} = \gamma = \frac{1}{(1 - v^2)^{1/2}}$$

Then

$$v = \frac{gt}{(1 + (gt)^2)^{1/2}} \to 1 - v^2 = \frac{1}{1 + (gt)^2} \to \gamma = (1 + (gt)^2)^{1/2}$$

Therefore, integrating we get

$$\frac{dt}{d\tau} = \gamma = (1 + (gt)^2)^{1/2}$$

$$\frac{dt}{(1 + (gt)^2)^{1/2}} = d\tau \to gt = \sinh g\tau$$

Thus,

$$x = \frac{c^2}{g}(\cosh g\tau - 1)$$

Numerically, this corresponds to a travel distance of 8.57×10^8 light-years for $\tau =$ years.

EP #17

(a) The square of the total energy-momentum 4-vector is invariant. In the CM system we have $P^\mu_{CM} = (E_{CM}, 0)$ which gives $P \cdot P = P^2 = E^2_{CM}$ where we have set $c = 1$.

For an incident particle(1) + a fixed target(2) (same mass) we have $P^\mu_1 = (E, \vec{p})$ and $P^\mu_2 = (m, 0)$ and $P^\mu = P^\mu_1 + P^\mu_2 = (E + m, \vec{p})$. Therefore,

$$P \cdot P = P^2 = E^2_{CM} = (E+m)^4 - p^2 = (E^2 - p^2) + 2Em + m^2 = 2mE + 2m^2$$

where we have used $E^2 - p^2 = m^2$. Thus the available CM energy is

$$E_{CM} = \sqrt{2mE + 2m^2} \tag{2}$$

(b) Head-On Collision: We have $P^\mu_1 = (E, \vec{p})$ and $P^\mu_2 = (E, -\vec{p})$. Therefore, $\vec{P}_{Total,Lab} = 0 \rightarrow textLAB = CM$ and thus, in the LAB $P^\mu = P^\mu_1 + P^\mu_2 = (2E, 0)$ and as always in the CM $P^\mu = (E_{CM}, 0)$. Thus by invariance of square of 4-vector we get $E^2_C = 4E^2$ in this case.

Therefore we have

$$\begin{aligned} &\text{For fixed target accelerator: } E_{CM} = \sqrt{2mE + 2m^2} \\ &\text{For colliding beam accelerator: } E_{CM} = 2E \end{aligned}$$

For $E = 1\,TeV$, $m = 1\,GeV$ we get

$$\begin{aligned} &\text{For fixed target accelerator: } E_{CM} = 44.7GeV \\ &\text{For colliding beam accelerator: } E_{CM} = 2\,TeV \end{aligned}$$

This means that in order to have $E_{CM} = 2\,TeV$ for a fixed-target accelerators would require a beam energy $E = E^2_{CM}/2m = 2000\,TeV$!!

(c) In the CM frame the total momentum vanishes. This implies that the proton and pion momenta are equal in magnitude and opposite in direction. This is also the rest frame of the Λ in this case. Therefore $E_{total} = E_p + E_\pi = m_\Lambda$. This gives

$$\sqrt{p^2 + m^2_p} + \sqrt{p^2 + m^2_\pi} = m_\Lambda$$

This gives

$$p^2 = \frac{m^4_\Lambda + m^4_p + m^4_\pi - 2m^2_\Lambda m^2_p - 2m^2_\Lambda m^2_\pi - 2m^2_p m^2_\pi}{4m^2_\Lambda} \rightarrow 100.53\,MeV$$

(d) This part can be solved using energy and momentum conservation in the lab frame, i.e.,

$$E_\Lambda = E_p + E_\pi \quad , \quad \vec{p}_\Lambda = \vec{p}_p + \vec{p}_\pi$$

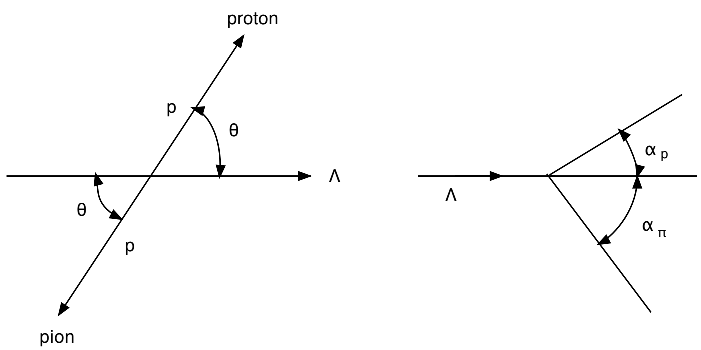
but it is better to apply a *boost* to the solution in part (c).

For the proton:

$$
\begin{aligned}
E_{LAB} &= \gamma(E_p + vp\cos\theta) \\
p_{\parallel} &= \gamma(p\cos\theta + vE_p) \\
p_{\perp} &= p\sin\theta
\end{aligned}
$$

where θ = angle between the proton momentum and the beam direction in CM frame (see diagram below).

For the Λ to have momentum $28.5\,GeV$, the boost has $v = 0.99923 \to \gamma = 25.56$.



For the pion we have

$$
\begin{aligned}
E_{LAB} &= \gamma(E_\pi - vp\cos\theta) \\
p_{\parallel} &= \gamma(-p\cos\theta + vE_\pi) \\
p_{\perp} &= p\sin\theta
\end{aligned}
$$

Therefore (see diagram above) we have

$$
\begin{aligned}
\tan\alpha_p &= \frac{p_\perp}{p_\parallel} = \frac{p\sin\theta}{\gamma(p\cos\theta + vE_p)} \\
\tan\alpha_\pi &= \frac{p_\perp}{p_\parallel} = \frac{p\sin\theta}{\gamma(-p\cos\theta + vE_\pi)}
\end{aligned}
$$

Now

$$
\tan\alpha_p + \alpha_\pi = \frac{\tan\alpha_p + \tan\alpha_\pi}{1 - \tan\alpha_p \tan\alpha_\pi}
$$

Using Mathematica:

f1[x_] := 100.53*Sin[x]/(25.56*(100.53*Cos[x]+0.99923*987.13)) tangent (proton angle)

f2[x_] := 100.53*Sin[x]/(25.56*(-100.53*Cos[x]+0.99923*172.35)) tangent (pion angle) f3[x_] := (f1[x]+f2[x])/(1-f1[x]*f2[x]) tangent(angle between pion and proton)

Solve[f3'[x] == 0,x] find maximum of tangent(angle between pion and proton)

x → 1.0045941941150183 angle in CM giving maximum

f3[1.0045941941150183] = 0.0312544 radians angle between = maximum

angle small so tan(x) ≈ x

maximum angle between proton and pion =1.80 degrees

EP #18

Under rotations we have

$$T'_{ij} = R_{il}R_{jm}T_{lm}$$

(a) If T_{ij} is symmetric, then $T_{ij} = T_{ji}$.

If T_{ij} is traceless, then $TrT \equiv \delta_{ij}T_{ij} = T_{ii} = 0$.

Suppose T_{lm} is traceless and symmetric, then

$$\begin{aligned} TrT' &= \delta_{ij}T'_{ij} = \delta_{ij}R_{il}R_{jm}T_{lm} = R_{jl}R_{jm}T_{lm} \\ &= R^T_{lj}R_{jm}T_{lm} = (R^TR)_{lm}T_{lm} = \delta_{lm}T_{lm} = TrT \end{aligned}$$

Thus, TrT transforms as a scalar. Moreover, if $TrT = 0$, then $TrT' = 0$.

Next, we compute (using $T_{lm} = T_{ml}$ and relabeling dummy indices where needed)

$$\begin{aligned} T'_{ji} &= R_{jl}R_{im}T_{lm} = R_{jl}R_{im}T_{ml} \\ &= R_{im}R_{jl}T_{ml} = R_{il}R_{jm}T_{lm} = T'_{ij} \end{aligned}$$

Hence, the transformed tensor is both traceless and symmetric.

(b) If T_{ij} is antisymmetric, then $T_{ij} = -T_{ji}$. Thus, we compute

$$\begin{aligned} T'_{ji} &= R_{jl}R_{im}T_{lm} = -R_{jl}R_{im}T_{ml} \\ &= -R_{im}R_{jl}T_{ml} = -R_{il}R_{jm}T_{lm} = -T'_{ij} \end{aligned}$$

Hence, the transformed tensor is antisymmetric.

We can repeat the above analysis for second-rank Lorentz tensors, which transform under Lorentz transformations according to

$$T'^{\mu\nu} = \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta T^{\alpha\beta}$$

The trace of a second-rank tensor is now defined as

$$TrT \equiv \eta_{\mu\nu}T^{\mu\nu}$$

It is easy to check that this is a scalar:

$$TrT' = \eta_{\mu\nu}T'^{\mu\nu} = \eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}T^{\alpha\beta}$$

Remembering the defining property for Λ:

$$\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = g_{\alpha\beta}$$

it follows that

$$TrT' = g_{\alpha\beta}T^{\alpha\beta} = TrT$$

In particular, we see that $TrT = 0$ implies that $TrT' = 0$. One can also show that if $T^{\mu\nu} = T^{\nu\mu}$, then $T'^{\mu\nu} = T'^{\nu\mu}$. Similarly, if $T^{\mu\nu} = -T^{\nu\mu}$, then $T'^{\mu\nu} = -T'^{\nu\mu}$. The proofs are similar to the ones above. For example, if $T^{\alpha\beta} = T^{\beta\alpha}$, then

$$\begin{aligned} T'^{\nu\mu} &= \Lambda^{\nu}{}_{\alpha}\Lambda^{\mu}{}_{\beta}T^{\alpha\beta} = \Lambda^{\mu}{}_{\beta}\Lambda^{\nu}{}_{\alpha}T^{\beta\alpha} \\ &= \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}T^{\alpha\beta} = T'^{\mu\nu} \end{aligned}$$

where we have relabeled the dummy indices $\alpha \to \beta$, $\beta \to \alpha$ in the next-to-last step above.

EP #19

In this problem we evaluate

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}F^{\alpha\beta}$$

with

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$ and $\beta = v/c$. It corresponds to a boost in the x^1-direction. In SI units

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

Thus,

$$F'^{10} = \Lambda^{1}{}_{\alpha}\Lambda^{0}{}_{\beta}F^{\alpha\beta} = \Lambda^{1}{}_{0}\Lambda^{0}{}_{1}F^{01} + \Lambda^{1}{}_{1}\Lambda^{0}{}_{0}F^{10}$$

All the other terms in the sum over α and β vanish. Thus,

$$-\frac{E'_x}{c} = (-\beta\gamma)^2 \frac{E_x}{c} + \gamma^2 \left(-\frac{E_x}{c}\right) = -\frac{E_x}{c}\gamma^2(1-\beta^2)$$

Using $\gamma^2 = (1-\beta^2)^{-1} \to \gamma^2(1-\beta^2) = 1$ we have $E'_x = E_x$.

Next,

$$F'^{20} = \Lambda^2{}_\alpha \Lambda^0{}_\beta F^{\alpha\beta} = \Lambda^2{}_2 \Lambda^0{}_0 F^{20} + \Lambda^2{}_2 \Lambda^0{}_1 F^{21}$$

All the other terms in the sum over α and β vanish. Thus,

$$-\frac{E'_y}{c} = -\gamma\frac{E_y}{c} + \beta\gamma B_z$$

Using $v = \beta c$ we have $E'_y = \gamma(E_y - vB_z)$.

Similarly, we get (choosing (30)) that $E'_z = \gamma(E_z + vB_y)$ and (from choosing (32) we get $B'_x = B_x$ and (from choosing (13) we get $B'_y = \gamma(B_y + vE_z/c^2$ and (from choosing (21) we get $B'_z = \gamma(B_z - vE_y/c^2$.

EP #20

(a) There is an obvious symmetry in this problem: if it took her 10 years by her watch to go from rest to her present state, then 10 years of reverse acceleration will bring her to rest, at her farthest point from home. Because of the constant negative acceleration, after reaching her destination at 20 years, she will begin to accelerate towards home again. In 10 more years, when her watch reads 30 years, she will be in the same state as when her watch read 10 years, only going in the opposite direction. Therefore, at 30 years, she should reverse her thrusters again so she arrives home in her home's rest frame.

(b) To do this, we need only solve the equations for the traveling twin's position and time as seen in the stationary twin's frame. We know that her 4-acceleration is normal to her velocity: $a^\xi u_\xi = 0$ everywhere along her trip, and $a^\xi a_\xi = a^2$ is constant. This leads us to conclude that

$$a^t = \frac{du^t}{d\tau} = au^x \quad \text{and} \quad a^x = \frac{du^x}{d\tau} = au^t$$

where τ is the proper time as observed by the traveling twin. This system is quickly solved for an appropriate choice of origin (we consider the twin to begin at $(t = 0, x = 1)$):

$$t = \frac{1}{a}\sinh(a\tau) \quad \text{and} \quad x = \frac{1}{a}\cosh(a\tau)$$

This is valid for the first quarter of the twin's trip - all four legs can be obtained explicitly by gluing together segments built out of the above.

For the purpose of calculating, it is necessary to make $a\tau$ dimensionless. This is done simply by

$$a = 10\, m/sec^2 = 1.053\, year^{-1}$$

So the distance at 10 years is

$$x(10\ years) = \frac{1}{1.053}\cosh 10.53 = 17710\ light\ years$$

The maximum distance traveled by the twin as observed by her (long-deceased) brother is therefore twice this distance, or $max(x) = 35420\ light\ years$.

(c) Well, in the brother's frame, his sister's trip took four legs, each requiring

$$t(10\ years) = \frac{1}{1.053}\sinh 10.53 = 17710\ years$$

which means that $t_{total} = 70838\ years$. In contrast, his sister's time was her proper time, or 40 years. Therefore, the brother who stayed behind is now 70798 years older than his twin sister.

EP #21

The Pauli matrix version of the transformation is

$$\begin{aligned}\Lambda &= (c\hat{I} - s\hat{n}_3\cdot\vec{\sigma})(c\hat{I} - s\hat{n}_2\cdot\vec{\sigma})(c\hat{I} - s\hat{n}_1\cdot\vec{\sigma})\\ &= c^3\hat{I} - c^2 s(\hat{n}_1+\hat{n}_2+\hat{n}_3)\cdot\vec{\sigma}\\ &\qquad + cs^2\left[(\hat{n}_1\cdot\vec{\sigma})(\hat{n}_2\cdot\vec{\sigma}) + (\hat{n}_1\cdot\vec{\sigma})(\hat{n}_3\cdot\vec{\sigma}) + (\hat{n}_2\cdot\vec{\sigma})(\hat{n}_3\cdot\vec{\sigma})\right]\\ &\qquad - s^2(\hat{n}_1\cdot\vec{\sigma})(\hat{n}_2\cdot\vec{\sigma})(\hat{n}_3\cdot\vec{\sigma})\end{aligned}$$

where $c = \cosh\zeta/2$ and $s = \sinh\zeta/2$. This can be reduced using the relation

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b}\hat{I} + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$$

Geometrically, $\hat{n}_1+\hat{n}_2+\hat{n}_3 = 0$, $\hat{n}_1\cdot\hat{n}_2 = \hat{n}_2\cdot\hat{n}_3 = \hat{n}_3\cdot\hat{n}_1 = -1/2$ and $\hat{n}_1\times\hat{n}_2 = \hat{n}_2\times\hat{n}_3 = \hat{n}_3\times\hat{n}_1 = (\sqrt{3}/2)\hat{n}'$ where $\hat{n}'$ is perpendicular to the plane. The net transformation is then

$$\Lambda = \left(c^3 - \frac{3}{2}cs^2\right)\hat{I} + ics^2\frac{\sqrt{3}}{2}(\hat{n}'\cdot\vec{\sigma}) - s^3(\hat{n}_2\cdot\vec{\sigma})$$

For small ζ, we get

$$\Lambda \approx \left(1 - \frac{3}{128}\zeta^4\right)\hat{I} + \frac{i}{2}\frac{\sqrt{3}\zeta^2}{2}(\hat{n}'\cdot\vec{\sigma}) - s^3(\hat{n}_2\cdot\vec{\sigma})$$

Then lowest non-trivial contribution is imaginary, a rotation about an axis normal to the plane by angle $\theta = -\sqrt{3}\zeta^2/4$. There is also a residual boost in the plane, in the direction of $\hat{n}_2$, with rapidity $\zeta^3/8$.

For arbitrary ζ, we can write

$$\Lambda = \cosh\frac{z}{2}\hat{I} - \sinh\frac{z}{2}(\hat{n}\cdot\vec{\sigma})$$

where

$$\cosh\frac{z}{2} = \cosh^3\frac{\zeta}{2} - \frac{3}{2}\cosh\frac{\zeta}{2} \quad , \quad \sinh\frac{z}{2} = \frac{1}{4}\sinh^4\frac{\zeta}{2}\left(\cosh^2\frac{\zeta}{2} - 4\right)$$

The coefficient of $\hat{I}$ is real, and so z is either purely real of purely imaginary. The transformation is more rotation, z is imaginary, for $\cosh(\zeta/2) < 2$, or $\zeta < 2.63392$, and is more boost for $\zeta > 2.63392$.

EP #22

(a) Indices are lowered using the metric, $T_{\mu\nu} = \eta_{\mu\rho}\eta_{\nu\sigma}T^{\rho\sigma}$:

$$T_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

The components are the same.

(b) This time, $T'^{\mu\nu} = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma T^{\rho\sigma}$, or $T' = \Lambda T\Lambda^T$,

$$T'^{\mu\nu} = \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \begin{pmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T'^{\mu\nu} = \begin{pmatrix} \gamma^2(\rho + pv^2) & \gamma^2 v(\rho + p) & 0 & 0 \\ \gamma^2 v(\rho + p) & \gamma^2(p + \rho v^2) & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

$$T'_{\mu\nu} = \begin{pmatrix} \gamma^2(\rho + pv^2) & -\gamma^2 v(\rho + p) & 0 & 0 \\ -\gamma^2 v(\rho + p) & \gamma^2(p + \rho v^2) & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

The rest frame is the frame where there is no flow, $T^{0i} = 0$, or the frame where T^{ij} is isotropic.

If $p = -\rho$, both $T^{\mu\nu}$ and $T_{\mu\nu}$ are diagonal, with components $(\rho, -\rho, -\rho, -\rho)$. This form is invariant under any Lorentz transformation, because it is $T = -\rho\eta$, and Lortentz transformation are those transformations for which $\Lambda\eta\Lambda^T = \eta$. In this case, there is no unique rest frame. The transformations also follow easily from $T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p\eta^{\mu\nu}$.

EP #23

The normal Newtonian formula is that the acceleration due to gravity in the field of point-like source of strength M is

$$\vec{g} = -\frac{M}{r^3}\vec{r}$$

Now, away from $r = 0$ we have

$$-\nabla \cdot \vec{g} = \nabla^2 \varphi = -M \nabla \frac{\vec{r}}{r^3}$$

and

$$\frac{\partial}{\partial x}\frac{x}{r^3} = \frac{1}{r^3} - \frac{3x^2}{r^5}$$

so

$$\frac{\partial}{\partial x}\frac{x}{r^3} + \frac{\partial}{\partial y}\frac{y}{r^3} + \frac{\partial}{\partial z}\frac{z}{r^3} = 0$$

and thus $\rho = 0$ except at $r = 0$. In fact, for a point source, we expect $\rho = \delta(x)\delta(y)\delta(z)M$, so, if S is a 2-sphere around the origin and B the ball it contains, we expect

$$\int\int\int_B \rho \, dxdydz = M$$

Now,

$$\int\int\int_B \nabla^2 \varphi \, dxdydz = -\int\int\int_B \nabla \cdot \vec{g} \, dxdydz = -\int\int_S \vec{g} \cdot \frac{\vec{r}}{r} r^2 \, d\Omega$$

by Gauss theorem. Hence

$$\int\int\int_B \nabla^2 \varphi \, dxdydz = \int\int_S M \, d\Omega = 4\pi M$$

as required.

EP #24

(a) Let $\vec{R}$ be the position of the center of the sphere relative to the sun/moon and $\vec{r}$ be the position of a point on the surface relative to the center. The difference between the acceleration at the surface and at the center is

$$\nabla^2 g = -\frac{GM(\vec{R} + \vec{r})}{|\vec{R} + \vec{r}|^3} - \left(-\frac{GM\vec{R}}{R^3}\right) = \frac{GM}{R^3}\left[3\vec{R}(\vec{r} \cdot \vec{R}) - \vec{r}\right]$$

the last for $r \ll R$. This is a dipole angular distribution. The radial component is

$$\nabla^2 g \cdot \hat{r} = \left(\frac{2GMr}{R^3}\right)\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

This is a stretching force along the line joining centers and a squeeze in the perpendicular directions: there are two bulges, pointing towards and away from the sun/moon, or two high tides per day.

(b) The water surface will deform until the force induced by the deformation balances the external tidal force. Let the deformed surface have $r = r_\oplus + h(\theta)$. Expanded in multipoles(with azimuthal symmetry only $m = 0$ appears), the exterior potential is

$$\Phi = -G \sum_{\ell=0}^{\infty} C_\ell \frac{P_\ell(\cos\theta)}{r^{\ell+1}}$$

The multipole moments are $C_\ell = \int d^3r' \rho r'^\ell P_\ell(\cos\theta')$. We expect that the distortion will have the dipole shape of the tidal force $h(\theta) = hP_2(\cos\theta)$. (Note that to lowest order in h, this distortion preserves volume). Taking ρ to be constant, the two largest C_ℓ's are $C_0 = 4\pi r_\oplus^3 \rho/3 = M_\oplus$ and $C_2 = 3M_\oplus r_\oplus h/5$. If the displaced water has density ρ_1 and the average density of the earth is rho_0, this becomes $C_2 = 3M_\oplus r_\oplus h(\rho_1/\rho_0)/5$. If pressure plays a negligible role in supporting the fluid, the gravitational forces must balance. Equating the radial forces, the amplitude h is

$$h = \frac{10}{9} \frac{M}{M_\oplus} \frac{r_\oplus^4}{R^3} \frac{\rho_0}{\rho_1}$$

For $\rho_0 = \rho_1$, the distortion from the sun alone has an amplitude $h = 19.3\,cm$; from the moon alone $h = 42.3\,cm$. With $\rho_0/\rho_1 = 5.52$, these are $107\,cm$ and $233\,cm$. Aligned (*spring tides*), the total distortion is $5.1\,m$, or $17\,feet$. At right angles(*neap tides*), they partially cancel (recall, at $\cos\theta = 0, P_2 = -1/2$), leaving $h = 1.8\,m$, or $5.9\,feet$. All those effects left out (rotation, friction, viscosity, etc) make the observed tides somewhat less than these numbers.

A *back of the envelope* or dimensional analysis estimate, $\Delta g/g = \Delta r/r$, gives this result except for the factor 10/9 !!

(c) The numbers give a relative acceleration over a distance of 1 meter (between your head and your toes) of

$$\Delta g = \frac{2GMr}{R^3} = \frac{2(6.6742 \times 10^{-8}\,cm^3 g^{-1} s^{-2})(2.786 \times 10^{33}\,g)(100\,cm)}{(10^7\,cm)^3}$$
$$= 3.7 \times 10^7\,cm/s^2 = 38000\,g$$

This is grim enough, but in the Niven story the protagonist ventures to $R = 12\,mi \approx 20\,km$, where $\Delta g \approx 4 \times 10^6\,g$.

The radius of the neutron star does not enter into this, but note that the neutron star is not much larger than its Schwarzschild radius; we expect that relativistic effects must be taken into account for a full understanding of neutron stars.

EP #25

Gauss' law (derived from spherical symmetry and the divergence theorem) states that the gravitational field is given by $|\vec{g}| = GM/r^2$ where M is the mass interior to the given radius (zero in this case). The observer cannot do any local measurements to infer the existence of the hollow sphere, but can observe its effect on distant objects, hence a blue shift of incoming light.

EP #26

(a) Use Gauss's law to obtain

$$\vec{g} = \begin{cases} -\frac{4}{3}\pi\rho G\vec{r} & r < R \\ -\frac{4}{3}\pi\rho G R^3 \frac{\vec{r}}{r^3} & r > R \end{cases} \tag{3}$$

where $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $r = \sqrt{x^2 + y^2 + z^2}$. Differentiating we find

$$\frac{\partial g_i}{\partial x_j} = \begin{cases} -\frac{4}{3}\pi\rho G\delta_{ij} & r < R \\ \frac{4}{3}\pi\rho G R^3 \frac{3x_i x_j - r^2\delta_{ij}}{r^5} & r > R \end{cases} \tag{4}$$

(b) We note that the answer in (a) is scale invariant, i.e., we get the same answer if we divide both R and r by two. Thus, the mass distribution given by an infinite number of spheres $j = 0, 1, 2, ...$ of fixed density ρ, centered at $x_j = 2^{-j}$ and of radius $R_j = 2^{-j-2}$ leads to an infinite gravity gradient at the origin. This singularity is relatively weak (logarithmic with distance from the origin), so it is probably not of physical importance. It is possible to create similar *curvature singularities* in GR, again of minor physical interest. Much stronger singularities are created in black holes, but we can never observe them, as we will see later, due to the *cosmic censorship conjecture.*

EP #27

So, the thing to remember here is that $\nabla_d T^a_{\ b}$ is a three-indexed tensor and must be differentiated accordingly. Hence

$$\nabla_c(\nabla_d T^a_{\ b}) = \partial_c(\nabla_d T^a_{\ b}) + \Gamma^a_{ce}(\nabla_d T^e_{\ b}) - \Gamma^e_{cd}(\nabla_e T^a_{\ b}) - \Gamma^e_{cb}(\nabla_d T^a_{\ e})$$

We also know that

$$\nabla_d T^a_{\ b} = \partial_d T^a_{\ b} + \Gamma^a_{ed} T^e_{\ b} - \Gamma^e_{db} T^a_{\ e}$$

Expanding out the whole lot

$$\nabla_c(\nabla_d T^a_{\ b}) = (\Gamma^a_{ed,c} + \Gamma^a_{cf}\Gamma^f_{ed})T^e_{\ b} + (-\Gamma^e_{db,c} + \Gamma^f_{cb}\Gamma^e_{df})T^a_{\ e}$$
$$+ \text{ terms symmetric in } c \text{ and } d$$

where we have not written out the terms symmetric in c and d because we know they will cancel when we subtract $\nabla_d(\nabla_c T^a_b)$. Now

$$\begin{aligned}\nabla_c(\nabla_d T^a_b) - \nabla_d(\nabla_e T^a_b) &= (\Gamma^a_{ed,c} - \Gamma^a_{ec,d} + \Gamma^a_{cf}\Gamma^f_{ed} - \Gamma^a_{df}\Gamma^f_{ec})T^e_b \\ &\quad + (\Gamma^e_{cb,d} - \Gamma^e_{db,c} + \Gamma^f_{cb}\Gamma^e_{df} - \Gamma^f_{db}\Gamma^e_{cf})T^a_e \\ &= R_{ecd}{}^a T^e_b + R_{cdb}{}^e T^a_e\end{aligned}$$

EP #28

(a) (In Text) Define an isometry.

(b) (in Text) Define Killing vector and show it satisfies $\nabla_a k_b + \nabla_b k_a = 0$.

(c) Consider $C = k^a U_a$. Along a geodesic we have

$$U^a \nabla_a C = U^a \nabla_a (k^b U_b) = (U^a \nabla_a k^b) U_b + U^a \nabla_a U_b k^b$$

Now the second term is zero by the geodesic equation and the first term can be made equal to the geodesic equation by symmetrizing

$$(U^a \nabla_a k^b) U_b = U^a U^b (\nabla_a k_b) = \frac{1}{2} U^a U^b (\nabla_a k_b + \nabla_b k_a) = 0$$

Thus, the Killing vector has defined a constant along a geodesic.

(d) Next, consider substituting the commutator into the Killing equation

$$\begin{aligned}\nabla_a [k,l]_b + \nabla_b [k,l]_a &= \nabla_a (k^c \nabla_c l_b - l^c \nabla_c k_b) + \nabla_b (k^c \nabla_c l_a - l^c \nabla_c k_a) \\ &= \nabla_a k^c \nabla_c l_b + k^c \nabla_a \nabla_c l_b - \nabla_a l^c \nabla_c k_b - l^c \nabla_a \nabla_c k_b + (a \leftrightarrow b)\end{aligned}$$

Now, the big trick is to swap the summed index off the nabla onto the Killing vector using the Killing equation

$$\nabla_a [k,l]_b + \nabla_b [k,l]_a = -\nabla_a k^c \nabla_b l_c - k^c \nabla_c \nabla_c l_b + \nabla_a l^c \nabla_b k_c + l^c \nabla_a \nabla_c k_b + (a \leftrightarrow b)$$

We also swap the double nabla terms using the definition of the Riemann tensor

$$\begin{aligned}\nabla_a [k,l]_b + \nabla_b [k,l]_a &= -\nabla_a k^c \nabla_b l_c - k^c \nabla_c \nabla_a l_b - k^c R_{acbd} l^d \\ &\quad + \nabla_a l^c \nabla_b k_c + l^c R_{acbd} k^d + l^c \nabla_c \nabla_a k_b + (a \leftrightarrow b)\end{aligned}$$

Now, add in the $(a \leftrightarrow b)$ part, the bits with the nabla's separate cancel using the Killing equation, the double nabla terms are also zero by the Killing equation and we are left with

$$\begin{aligned}\nabla_a [k,l]_b + \nabla_b [k,l]_a &= -k^c R_{acbd} l^d + l^c R_{acbd} k^d - k^c R_{bcad} l^d + l^c R_{bcad} k^d \\ &= k^c l^d (R_{acbd} + R_{bcad} + R_{adbc} + R_{bdac}) \\ &= k^c l^d (-R_{acbd} + R_{adbc} - R_{adbc} + R_{acbd}) = 0\end{aligned}$$

Thus, $[k,l]_a$ is a Killing vector.

EP #29

(a) To get the metric do the usual chain rule on the coordinate change:

$$
\begin{aligned}
x &= u\cos\phi \quad , \quad dx = \cos\phi du - u\sin\phi d\phi \\
y &= u\sin\phi \quad , \quad dy = \sin\phi du + u\cos\phi d\phi \\
z &= \frac{u^2}{2} \quad , \quad dz = udu
\end{aligned}
$$

and then substitute into $ds^2 = dx^2 + dy^2 + dz^2$ and use the Pythagorous theorem.

$$
\begin{aligned}
ds^2 &= (\cos\phi du - u\sin\phi d\phi)^2 + (\sin\phi du + u\cos\phi d\phi)^2 + (udu)^2 \\
&= \cos^2\phi du^2 + u^2\sin^2\phi d\phi^2 + \sin^2\phi du^2 + u^2\cos^2\phi d\phi^2 + u^2 du^2 \\
&= (1+u^2)du^2 + u^2 d\phi^2
\end{aligned}
$$

Now let $x^1 = u$ and $x^2 = \phi$ and we get

$$
[g_{ab}] = \begin{pmatrix} 1+u^2 & 0 \\ 0 & u^2 \end{pmatrix}
$$

(b) Now for the connection coefficients you just use the formula and the inverse metric. Therefore,

$$
[g^{ab}] = \begin{pmatrix} \frac{1}{1+u^2} & 0 \\ 0 & \frac{1}{u^2} \end{pmatrix}
$$

and

$$
\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc})
$$

This gives

$$
\begin{aligned}
\Gamma^1_{11} &= -\Gamma^1_{22} = -\frac{u}{1+u^2} \\
\Gamma^1_{12} &= \Gamma^2_{21} = \frac{1}{u} \\
\Gamma^2_{22} &= \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{11} = 0
\end{aligned}
$$

(c) Now, in the parallel transport equation the curve has constant u so it is given by (u_0, ϕ) and hence assuming that $t = \phi$

$$
(U^1, U^2) = \left(\frac{du}{dt}, \frac{d\phi}{dt}\right) = \left(\frac{du_0}{d\phi}, \frac{d\phi}{d\phi}\right) = (0, 1)
$$

and the equation of parallel transport is

$$
U^a \nabla_a V^b = U^1 \nabla_1 V^b + u^2 \nabla_2 V^b = 0 \rightarrow \nabla_2 V^b = 0
$$

or

$$\nabla_2 V^b = \frac{dV^b}{dx^2} + \Gamma^b_{2c} V^c = \frac{dV^b}{d\phi} + \Gamma^b_{21} V^1 + \Gamma^b_{22} V^2 = 0$$

From (b) we have the Γ's so this yields the differential equations

$$\frac{dV^1}{d\phi} - \frac{u}{1+u^2} V^2 = 0$$
$$\frac{dV^2}{d\phi} + \frac{1}{u} V^1 = 0$$

We differentiate the first equation to obtain

$$\frac{d^2V^1}{d\phi^2} - \frac{u}{1+u^2} \frac{dV^2}{d\phi} = 0$$

or

$$\frac{d^2V^1}{d\phi^2} + \frac{1}{1+u_0^2} V^1 = 0$$

Therefore,

$$V^1 = A\cos k\phi + B\sin k\phi \quad \text{where} \quad k = \sqrt{\frac{1}{1+u_0^2}}$$

With the initial condition we have $V^1(0) = A + 0 = 1 \to A = 1$. Then we can find V^2.

$$V^2 = \frac{1+u_0^2}{u_0} \frac{dV^1}{d\phi} = \frac{1+u_0^2}{u_0} k(-\sin k\phi + B\cos k\phi)$$

and using the initial conditions

$$V^2(0) = \frac{1+u_0^2}{u_0} kB = 0 \to B = 0$$

Therefore,

$$V^1 = \cos k\phi \quad , \quad V^2 = -\frac{1}{ku_0}\sin k\phi$$

Notice that $V^a(2\pi) = V^a(0)$.

EP #30

We have, (using $x^1 = x$ and $x^2 = y$ in calculations),

$$[g_{ij}] = \frac{1}{\left[1 + \frac{x^2+y^2}{4a^2}\right]^2} \delta_{ij}$$

(a) Then we can calculate the Γ^a_{bc} using

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc})$$

Tedious algebra (use fact that the metric tensor is diagonal)

$$\Gamma^1_{11} = \Gamma^2_{21} = \Gamma^2_{12} = -\Gamma^1_{22} = -\frac{x}{2a^2\left[1+\frac{x^2+y^2}{4a^2}\right]}$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = -\Gamma^2_{22} = -\frac{y}{2a^2\left[1+\frac{x^2+y^2}{4a^2}\right]}$$

(b) We define vector $\vec{\xi}$ such that $\xi^1 = -y = -x^2$ and $\xi^2 = x = x^1$. Killing's equation is then

$$\nabla_a \xi_b + \nabla_b \xi_a = 0$$

Now,

$$\xi_a = g_{ab}\xi^b = \frac{1}{\left[1+\frac{x^2+y^2}{4a^2}\right]^2}\xi^a$$

Thus,

$$\xi_1 = -\frac{y}{\left[1+\frac{x^2+y^2}{4a^2}\right]^2} \quad , \quad \xi_2 = \frac{x}{\left[1+\frac{x^2+y^2}{4a^2}\right]^2}$$

and using

$$\nabla_a \xi_b = \partial_a \xi_b - \xi_c \Gamma^c_{ab}$$

Killing's equation becomes

$$\partial_a \xi_b + \partial_b \xi_a - 2\xi_c \Gamma^c_{ba} = 0$$

Tedious calculations then show that $\vec{\xi}$ satisfies Killing's equation.

(c) If we consider a vector field $V^a(x^b)$ and consider its time derivative along a geodesic, then

$$dV^a = \nabla_b V^a dx^b$$

so that

$$\frac{dV^a}{dt} = \nabla_b V^a \frac{dx^b}{dt}$$

Thus, if V^a is a velocity 4-vector along a geodesic, i.e., $V^a \nabla_a V^b = 0$ and ξ^b is a Killing vector, i.e., $\nabla_a \xi_b + \nabla_b \xi_a = 0$, then

$$\begin{aligned}
\frac{d}{dt}(V^a \xi_a) &= \frac{dV^a}{dt}\xi_a + V^a \frac{d\xi_a}{dt} \\
&= (V^b \nabla_b V^a)\xi_a + V^a (V^b \nabla_b g_{aq} \xi^q) \\
&= V^a V^b \nabla_b \xi_a = \frac{1}{2} V^a V^b \nabla_a \xi_b + \frac{1}{2} V^a V^b \nabla_b \xi_a \\
&= \frac{1}{2} V^a V^b (\nabla_a \xi_b + \nabla_b \xi_a) = 0
\end{aligned}$$

Thus, $V^a\xi_a$ is a conserved quantity along any geodesic. Now

$$\begin{aligned}
V^a\xi_a &= g_{aq}V^a\xi^q = g_{11}V^1\xi^1 + g_{22}V^2\xi^2 \\
&= g_{11}(V^1\xi^1 + V^2\xi^2) = g_{11}(\dot{(x)}(-y) + \dot{y}x) \\
&= \frac{x\dot{y} - y\dot{x}}{\left[1 + \frac{x^2+y^2}{4a^2}\right]^2}
\end{aligned}$$

Note that in flat space, $V^a\xi_a = x\dot{y} - y\dot{x}$ is the magnitude of the angular momentum vector, so we suppose that $V^a\xi_a$ is related to the angular momentum vector for this curved space. This makes sense because $(-y, x)$ is the generator of rotations and we already know that there is a deep connection between rotational symmetry and angular momentum.

(d) We now compute R^{ij}_{kl} using

$$R_{abc}{}^d = \partial_a\Gamma^d_{bc} - \partial_b\Gamma^d_{ac} - \Gamma^d_{be}\Gamma^e_{ac} + \Gamma^d_{ae}\Gamma^e_{bc}$$

After tedious algebra we get (easy to see that only $a \neq b$, $c \neq d$ can be nonzero)

$$R_{121}{}^2 = -\frac{1}{a^2\left[1 + \frac{x^2+y^2}{4a^2}\right]^2} = -R_{122}{}^1 = -R_{211}{}^2 = R_{212}{}^1$$

Therefore

$$\begin{aligned}
R^{12}_{12} &= g^{1a}R_{12a}{}^2 = g^{11}R_{121}{}^2 = -\frac{1}{a^2} \\
R^{12}_{21} &= g^{22}R_{122}{}^1 = -g^{11}R_{121}{}^2 = \frac{1}{a^2} \\
R^{21}_{21} &= g^{22}R_{212}{}^1 = g^{11}R_{121}{}^2 = -\frac{1}{a^2} \\
R^{21}_{12} &= g^{11}R_{211}{}^2 = -g^{11}R_{121}{}^2 = \frac{1}{a^2}
\end{aligned}$$

The others are zero. Now $R_{ac} = R_{abc}{}^b = R_{a1c}{}^1 + R_{a2c}{}^2$. If $a = c$, then $R_{aa} = R_{a1a}{}^1 + R_{a2a}{}^2 = R_{212}{}^1 = R_{121}{}^2$ because either $a = 1$ or $a = 2$.

Thus, $R_{11} = R_{22}$ and

$$R^1_1 = g^{1a}R_{1a} = g^{11}R_{11} = g^{11}R_{212}{}^1 = -\frac{1}{a^2} = R^2_2$$

If $a \neq c$, then $R_{ac} = R_{a1c}{}^1 + R_{a2c}{}^2 = 0$ since in both terms either $c = 1$ or $a = 1$ or $c = 2$ or $a = 2$. Thus

$$R^1_1 = R^2_2 = -\frac{1}{a^2} \quad , \quad R^1_2 = R^2_1 = 0$$

Finally,

$$R = R^i_i = R^1_1 + R^2_2 = -\frac{2}{a^2}$$

which is constant everywhere. However, strangely enough, the definition I used for $R_{abc}{}^d$ is the negative of the standard definition in other texts(see next problem solution), so in this case $R = 2/a^2$ and it this world is the 2-dimensional surface of a sphere! Formally, we have shown that the sphere is *conformally flat.*

EP #31

For a timelike geodesic we have

$$ds^2 = -d\tau^2 = -\frac{1}{t^2}dt^2 + \frac{1}{t^2}dx^2$$

for real proper time $d\tau$. Using dots for differentiation with respect to τ this gives

$$-1 = -\frac{1}{t^2}(\dot{x}^2 - \dot{t}^2)$$

Now the corresponding Lagrangian $L = g_{ab}\dot{x}^a\dot{x}^b$ is

$$L = -\frac{1}{t^2}(\dot{x}^2 - \dot{t}^2)$$

and since this is independent of x, one of the Euler-Lagrange equations is a conservation equation:

$$\frac{d}{d\tau}\frac{2\dot{x}}{t^2} = 0$$

Integrating, this means that $\dot{x} = ct^2$ for some constant c. Substituting back into

$$-1 = -\frac{1}{t^2}(\dot{x}^2 - \dot{t}^2)$$

we get

$$\dot{t}^2 = t^2(1 + ct^2)$$

or

$$d\tau = \frac{dt}{t\sqrt{1 + ct^2}}$$

Integrating we have

$$e^{2\tau} = \frac{\sqrt{1 + ct^2} - 1}{\sqrt{1 + ct^2} + 1}$$

where the integration constant has been set to zero - which says that $t = 0$ corresponds to $\tau = -\infty$. Solving for $\sqrt{1 + ct^2}$ gives

$$\sqrt{1 + ct^2} = -\frac{e^{\tau} + e^{-\tau}}{e^{\tau} - e^{-\tau}}$$

or

$$t = \pm\frac{1}{c\sinh\tau}$$

EP #32

(a) The metric is diagonal, and the only nonvanishing derivative is $g_{\phi\phi,\chi} = 2a^2 \sinh\chi \cosh\chi$. Therefore, the connections are

$$\begin{aligned}
\Gamma^{\chi}_{\chi\chi} &= \Gamma^{\phi}_{\chi\chi} = \Gamma^{\chi}_{\phi\chi} = \Gamma^{\phi}_{\phi\phi} = 0 \\
\Gamma^{\chi}_{\phi\phi} &= -\frac{1}{2a^2}(2a^2 \sinh\chi \cosh\chi) = -\sinh\chi\cosh\chi \\
\Gamma^{\phi}_{\chi\phi} &= \frac{1}{2a^2\sinh^2\chi}(2a^2\sinh\chi\cosh\chi) = \frac{\cosh\chi}{\sinh\chi}
\end{aligned}$$

These are the same as for a sphere with trig functions replaced by the corresponding hyperbolic functions.

(b) The symmetries of the Riemann tensor leave only one nonvanishing component $R^{\chi\phi}_{\chi\phi}$. Thus, we have

$$\begin{aligned}
R^{\chi}{}_{\phi\chi\phi} &= \Gamma^{\chi}{}_{\phi\phi,\chi} - \Gamma^{\chi}{}_{\phi\chi,\phi} + \Gamma^{\chi}{}_{\chi j}\Gamma^{j}{}_{\phi\phi} - \Gamma^{\chi}{}_{\phi j}\Gamma^{j}{}_{\chi\phi} = \Gamma^{\chi}{}_{\phi\phi,\chi} - \Gamma^{\chi}{}_{\phi\phi}\Gamma^{\phi}{}_{\chi\phi} \\
&= (-\sinh\chi\cosh\chi)_{,\chi} - (-\sinh\chi\cosh\chi)\frac{\cosh\chi}{\sinh\chi} \\
&= (-\cosh^2\chi - \sinh^2\chi) + \cosh^2\chi = -\sinh^2\chi
\end{aligned}$$

Raise the second index to put it in the proper form, i.e.,

$$R^{\chi\phi}{}_{\chi\phi} = -\frac{1}{a^2}$$

The nonvanishing componentns of the Ricci tensor are

$$R^{\chi}_{\chi} = R^{\chi\phi}{}_{\chi\phi} = -\frac{1}{a^2} \quad , \quad R^{\phi}_{\phi} = R^{\phi\chi}{}_{\phi\chi} = -\frac{1}{a^2}$$

and the scalar curvature is

$$R = R^{\chi}_{\chi} + R^{\phi}_{\phi} = -\frac{2}{a^2}$$

This is a space with constant negative curvature.

(c) The geodesic equation gives

$$\frac{d^2\chi}{d\lambda^2} - \sinh\chi\cosh\chi\left(\frac{d\phi}{d\lambda}\right)^2 = 0 \quad , \quad \frac{d^2\phi}{d\lambda^2} + 2\frac{\cosh\chi}{\sinh\chi}\frac{d\chi}{d\lambda}\frac{d\phi}{d\lambda} = 0$$

The second equation is equivalent to

$$\frac{d}{d\lambda}\left(\sinh^2\chi\frac{d\phi}{d\lambda}\right) = \sinh^2\chi\left(\frac{d^2\phi}{d\lambda^2} + 2\frac{\cosh\chi}{\sinh\chi}\frac{d\chi}{d\lambda}\frac{d\phi}{d\lambda}\right) = 0$$

The first quantity in parentheses is a constant, which can also be obtained by noting that since the metric does not depend on ϕ, the momentum p_ϕ is constant. Name this constant β.

$$\sinh^2\chi\frac{d\phi}{d\lambda} = \beta$$

Now, use this to replace $d\phi/d\lambda$ in the first equation,

$$\frac{d^2\chi}{d\lambda^2} - \beta^2 \frac{\cosh\chi}{\sinh^3\chi} = 0$$

This is equivalent to

$$\frac{d}{d\lambda}\left[\left(\frac{d\chi}{d\lambda}\right)^2 + \frac{\beta^2}{\sinh^2\chi}\right] = 2\frac{d\chi}{d\lambda}\left[\frac{d^2\chi}{d\lambda^2} - \sinh\chi\cosh\chi\left(\frac{d\phi}{d\lambda}\right)^2\right] = 0$$

The first quantity in brackets is constant; name this constant α:

$$\left(\frac{d\chi}{d\lambda}\right)^2 + \frac{\beta^2}{\sinh^2\chi} = \alpha$$

Now, change the variable from the parameter λ to the coordinate ϕ:

$$\frac{d\chi}{d\lambda} = \frac{d\chi}{d\phi}\frac{d\phi}{d\lambda} = \frac{d\chi}{d\phi}\frac{\beta^2}{\sinh^2\phi}$$

$$\frac{\beta^2}{\sinh^4\chi}\left(\frac{d\chi}{d\phi}\right)^2 + \frac{\beta^2}{\sinh^2\chi} = \alpha$$

Using the hyperbolic identities

$$\frac{d\coth\chi}{d\chi} = -\frac{1}{\sinh^2\chi} \quad , \quad \coth^2\chi - 1 = \frac{1}{\sinh^2\chi}$$

this becomes

$$\left(\frac{d\coth\chi}{d\chi}\right)^2 + \coth^2\chi = 1 + \frac{\alpha}{\beta^2}$$

The solution of this equation consistent with the boundary condition is

$$\coth\chi = \coth b\cos\phi \quad , \quad \chi = \frac{1}{2}\log\left(\frac{\coth b\cos\phi + 1}{\coth b\cos\phi - 1}\right)$$

The function $\coth b$ is always greater than 1, so the logarithm is well defined at $\phi = 0$. Not all values of ϕ are realized; $\chi \to \pm\infty$ as $\cos\phi \to \tanh b$.

(d) The equation of geodesic deviation with tangent $u^i = dx^i/d\lambda$ states

$$\nabla_u\nabla_u\xi^i = \frac{D^2\xi^i}{D\lambda^2} = R^i{}_{jkl}u^j u^k \xi^l$$

With $u^i = (0, 1)$ and using the results of (b) for the Riemann tensor we have

$$\frac{D^2\xi^\chi}{D\lambda^2} = R^\chi{}_{\phi\phi\chi}\xi^\chi = +\sinh^b\xi^\chi \quad , \quad \frac{D^2\xi^\phi}{D\lambda^2} = R^\phi{}_{\phi\phi\phi}\xi^\phi = 0$$

The angular separation ξ^ϕ remains zero, and the initial *radial* separation ξ^χ increases exponentially, and more quickly the larger b is.

(e) The proper distance from the origin along a ray of fixed ϕ is $r = a\chi$. For $\chi = r/a$ with $a \to \infty$, the geodesic becomes

$$\coth \frac{r}{a} = \coth \frac{b}{a} \cos\phi \to \frac{a}{r} = \frac{a}{b} \cos\phi$$

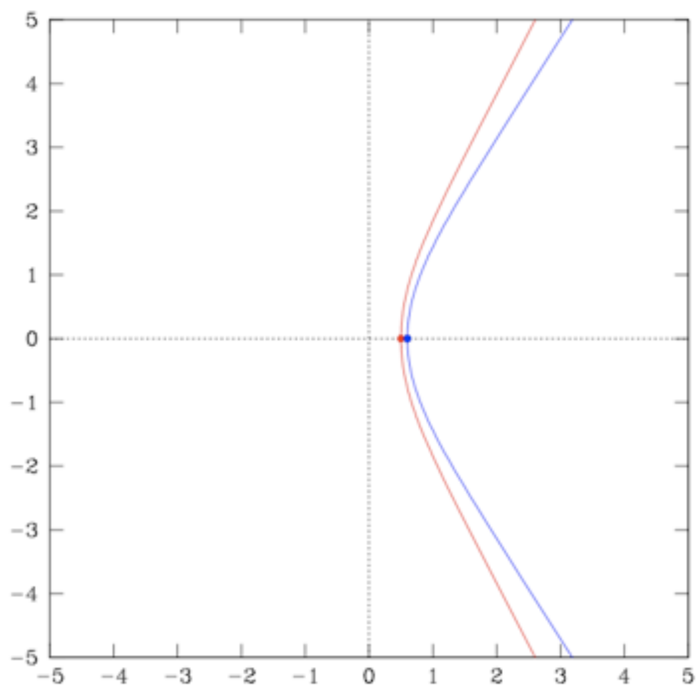
or $r\cos\phi = b$. As the space becomes flat, the geodesic becomes the straight line $x = b$.

Another view of what happens comes from looking at the metric,

$$ds^2 \to a^2(d\chi^2 + \chi^2 d\phi^2) = dr^2 + r^2 d\phi^2$$

where $r = a\chi$: flat space in polar coordinates.

The figure below shows the shape of two geodesics in a polar coordinate system with radial coordinate χ and angular coordinate ϕ. The inner curve has $b = 0.5a$ and the outer curve has $b = 0.6a$.



EP #33

On the surface of a 2-sphere of radius a we have

$$ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2$$

Thus,

$$g_{\theta\theta} = a^2 \quad , \quad g_{\phi\phi} = a^2 \sin^2\theta \quad , \quad g_{\theta\phi} = g_{\phi\theta} = 0$$

so we then have

$$g^{\theta\theta} = \frac{1}{a^2} \quad , \quad g^{\phi\phi} = \frac{1}{a^2 \sin^2\theta} \quad , \quad g^{\theta\phi} = g^{\phi\theta} = 0$$

Now we must calculate the Γ^a_{bc} where

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\partial_c g_{db} + \partial_b g_{dc} - \partial_d g_{bc})$$

Since g is diagonal, wer must have $d = a$ or

$$\Gamma^a_{bc} = \frac{1}{2} g^{aa}(\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{bc})$$

Some algebra then gives $\Gamma^\phi_{\phi\phi} = 0 = \Gamma^\theta_{\theta\theta}$, $\Gamma^\theta_{\theta\phi} = 0 = \Gamma^\theta_{\phi\theta}$, $\Gamma^\phi_{\theta\phi} = \cot\theta = \Gamma^\phi_{\phi\theta}$ and $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$.

Now consider the vector $\vec{A} = \vec{e}_\theta$ at $\theta = \theta_0$, $\phi = 0$, which say that $A^{|theta} = 1$, $A^\phi = 0$.

Now, if this vector is parallel transported around the curve $\theta = \theta_0$, $\phi =$ free, then we take $\tau = \phi$ and the velocity is

$$U^a = \frac{dx^a}{d\tau} \to U^\theta = \frac{d\theta_0}{d\phi} = 0 \quad , \quad U^\phi = \frac{d\phi}{d\phi} = 1$$

The equation of parallel transport is

$$U^a \nabla_a A^b = 0$$

For $b = \theta$: we have

$$\nabla_\phi A^\theta = \partial_\phi A^\theta + \Gamma^\theta_{\phi\phi} A^\phi + \Gamma^\theta_{\phi\theta} A^\theta = 0$$

For $b = \phi$: we have

$$\nabla_\phi A^\phi = \partial_\phi A^\phi + \Gamma^\phi_{\phi\phi} A^\phi + \Gamma^\phi_{\phi\theta} A^\phi = 0$$

Using the results for the connections calculated above we then have two coupled ODEs

$$\frac{dA\theta}{d\phi} - \sin\theta_0 \cos\theta_0 A^\phi = 0$$
$$\frac{dA\phi}{d\phi} + \cot\theta_0 A^\theta = 0$$

If we differentiate the second equation and then use the first equation we have

$$\frac{d^2 A\phi}{d\phi^2} + \cot\theta_0 \frac{dA\theta}{d\phi} = 0 \to \frac{d^2 A\phi}{d\phi^2} + \cos^2\theta_0 A^\phi = 0$$

Therefore,

$$A^\phi = Q\sin k\phi + R\cos k\phi \quad \text{where} \quad k = \cos\theta_0$$

The initial condition, $A^\phi(0) = 0$ then says that $A^\phi = 0 = R$ so that $A^\phi = Q\sin k\phi$.

Then we have for A^θ:

$$A^\theta = -\tan\theta_0 \frac{dA\phi}{d\phi} = -\sin\theta_0 Q\cos k\phi$$

The other initial condition $A^\theta(0) = 1$ then gives $Q = -1/\sin\theta_0$ and we finally obtain the solutions

$$A^\theta(\phi) = \cos(\cos\theta_0\phi) \quad , \quad A^\phi(\phi) = -\frac{1}{\sin\theta_0}\sin(\cos\theta_0\phi)$$

The magnitude of A is given by

$$A = \sqrt{A^a A_a} = \sqrt{A^\theta A_\theta + A^\phi A_\phi}$$

where

$$\begin{aligned} A_\theta &= g_{\theta\theta}A^\theta = a^2 A^\theta = a^2 \sin\theta_0 \cos(\cos\theta_0\phi) \\ A_\phi &= g_{\phi\phi}A^\phi = -a^2\sin^2\theta_0 A^\phi = -a^2\sin\theta_0\sin(\cos\theta_0\phi) \end{aligned}$$

Thus,

$$A = a\sqrt{\cos^2\cos(\theta_0\phi) + \sin^2\cos(\theta_0\phi)} = a$$

EP #34

(a) On the surface of a 2-sphere of radius a we have

$$ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2$$

We want to calculate $R^i{}_{jkl}$ for the 2-sphere where

$$R^i{}_{jkl} = \partial_k\Gamma^i_{jl} - \partial_l\Gamma^i_{jk} + \Gamma^i_{mk}\Gamma^m_{jl} - \Gamma^i_{ml}\Gamma^m_{jk}$$

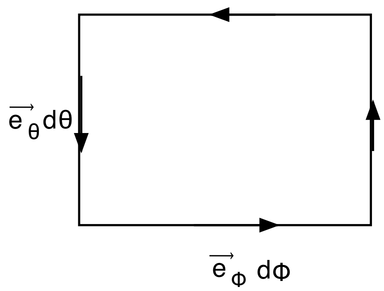
Now, if $i = j$ or $k = l$, then $R^i{}_{jkl} = 0$. Therefore we consider $i = \theta$, $j = \phi$, $k = \theta$, $l = \phi$:

$$\begin{aligned} R^\theta{}_{\phi\theta\phi} &= \partial_\theta\Gamma^\theta_{\phi\phi} - \partial_\phi\Gamma^\theta_{\phi\theta} + \Gamma^\theta_{m\theta}\Gamma^m_{\phi\phi} - \Gamma^\theta_{m\phi}\Gamma^m_{\phi\theta} \\ &= \partial_\theta(-\sin\theta\cos\theta) - 0 + 0 - (-\sin\theta\cos\theta)(\cot\theta) \\ &= -\cos^2\theta + \sin^2\theta + \cos^2\theta = \sin^2\theta = -R^\theta{}_{\phi\phi\theta} \end{aligned}$$

We then have

$$\begin{aligned} R^{\phi}_{\ \theta\theta\phi} &= g^{\phi\phi} R_{\phi\theta\theta\phi} = -g^{\phi\phi} R_{\theta\phi\theta\phi} \\ &= -g^{\phi\phi} g_{\theta\theta} R^{\theta}_{\ \phi\theta\phi} = -\frac{1}{a^2 \sin^2\theta} a^2 \sin^2\theta = -1 = -R^{\phi}_{\ \theta\phi\theta} \end{aligned}$$

(b) Now consider a vector $\vec{A} = A^{\theta}\vec{e}_{\theta} + A^{\phi}\vec{e}_{\phi}$ parallel transported around an infinitesimal parallelogram with sides $\vec{e}_{\theta} d\theta$ and $\vec{e}_{\phi} d\phi$, as shown below, then



$$\Delta A^i = -\frac{1}{2} R^i_{\ jkl} A_0^j S^{lk}$$

where

$$S^{lk} = \frac{1}{2} \oint_L (q_l dq^k - q^k dq^l) \rightarrow S^{\phi\phi} = S^{\theta\theta} = 0$$

and $\vec{A}_0$ is the value of the vector at the center of the parallelogram. We then have

$$\begin{aligned} \Delta A^{\theta} &= -\frac{1}{2} R^{\theta}_{\ jkl} A_0^j S^{lk} \quad \text{(terms are 0 unless } j = \phi) \\ &= -\frac{1}{2} R^{\theta}_{\ \phi kl} A_0^{\phi} S^{lk} \quad \text{(terms are 0 when } l = k) \\ &= -\frac{1}{2} R^{\theta}_{\ \phi\theta\phi} A_0^{\phi} S^{\phi\theta} - \frac{1}{2} R^{\theta}_{\ \phi\phi\theta} A_0^{\phi} S^{\theta\phi} \\ &= -R^{\theta}_{\ \phi\theta\phi} A_0^{\phi} S^{\phi\theta} \end{aligned}$$

and

$$\begin{aligned} \Delta A^{\phi} &= -\frac{1}{2} R^{\phi}_{\ jkl} A_0^j S^{lk} \\ &= -\frac{1}{2} R^{\phi}_{\ \theta kl} A_0^{\theta} S^{lk} \\ &= -\frac{1}{2} R^{\phi}_{\ \theta\phi\theta} A_0^{\theta} S^{\theta\phi} - \frac{1}{2} R^{\phi}_{\ \theta\theta\phi} A_0^{\theta} S^{\phi\theta} \\ &= R^{\phi}_{\ \theta\phi\theta} A_0^{\theta} S^{\phi\theta} \end{aligned}$$

Now since the loop is infinitesimal, we have $vecA \approx \vec{A}_0$, so that

$$\Delta A^\theta = (-R^\theta{}_{\phi\theta\phi} S^{\phi\theta}) A^\phi$$
$$\Delta A^\phi = (R^\phi{}_{\theta\phi\theta} S^{\phi\theta}) A^\theta$$

where $R^\theta{}_{\phi\theta\phi} = \sin^2\theta$, $R^\phi{}_{\theta\phi\theta} = 1$. Thus,

$$\Delta A^\theta = -\sin^2\theta S^{\phi\theta} A^\phi$$
$$\Delta A^\phi = S^{\phi\theta} A^\theta$$

and so

$$A'^\theta = A^\theta + \Delta A^\theta = A^\theta - \sin^2\theta S^{\phi\theta} A^\phi$$
$$A'^\phi = A^\phi + \Delta A^\phi = A^\phi + S^{\phi\theta} A^\theta$$

Now, if we want to treat ϕ and θ like cartesian coordinates, we must give them units of length - rA^θ, $r\sin\theta A^\phi$ - then (if $\phi \sim x$, $\theta \sim -y$), we have

$$\begin{aligned} \begin{pmatrix} r\sin\theta A'^\phi \\ -rA'^\theta \end{pmatrix} &= \begin{pmatrix} r\sin\theta A^\phi + r\sin\theta S^{\phi\theta} A^\theta \\ -rA^\theta + r\sin^\theta S^{\phi\theta} A^\phi \end{pmatrix} \\ &= \begin{pmatrix} 1 & -r\sin\theta S^{\phi\theta} \\ +\sin\theta S^{\phi\theta} & 1 \end{pmatrix} \begin{pmatrix} r\sin\theta A^\phi \\ -rA^\theta \end{pmatrix} \end{aligned}$$

Thus, this is like a rotation with infinitesimal angle $\xi = +\sin\theta S^{\phi\theta}$ ($S^{\phi\theta}$ will be determined later). For the length of the new vector we have

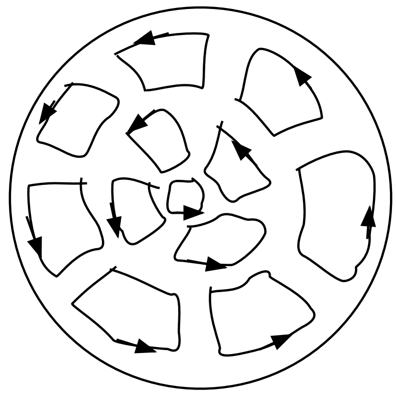
$$|A'| = \sqrt{A'^a A'_a} = \sqrt{A^a A_a + 2A^a \Delta A_a + \Delta A^a \Delta A_a}$$

Now

$$\begin{aligned} A^a \Delta A_a = A^\phi \Delta A_\phi + A^\theta \Delta A_\theta &= g_{\phi\phi} A^\phi \Delta A^\phi + g_{\theta\theta} A^\theta \Delta A^\theta \\ &= a^2 \sin^\theta A^\phi (S^{\phi\theta} A^\theta) + a^2 A^\theta (-\sin^2\theta S^{\phi\theta} A^\phi) = 0 \end{aligned}$$

Thus, to first order in $S^{\phi\theta}$, we have $|A'| \approx |A|$.

(c) If we take it as given that the ξ from earlier is $\xi = \sin\theta d\theta d\phi = d\Omega$(see derivation below), we can approach the following problem: Imagine parallel transporting a vector around a simply connected solid angle Ω. We can think of this as equivalent to parallel transporting this around lots of infinitesimal solid angles as shown below because their path integral is equivalent to the path integral around the edge (a la Stoke's theorem).



Thus, to first order, the change in length is 0 and the change in angle is

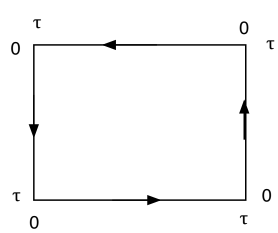
$$\int d\Omega = \Omega$$

in the counterclockwise direction.

Evaluation of $S^{\phi\theta}$: Now

$$S^{\phi\theta} = \frac{1}{2}\oint_L (q_\phi dq^\theta - q^\theta dq^\phi)$$

Consider the path shown below.



We then have

$$S^{\phi\theta} = \frac{1}{2}\int_{top} (q_\phi dq^\theta - q^\theta dq^\phi) + ...$$

For the top path we have $q^\phi = d\phi - \tau d\phi$, $q^\theta = d\theta$. Therefore $dq^\theta = 0$, $dq^\phi = -d\phi d\tau$ and thus

$$\frac{1}{2}\int_{top} (q_\phi dq^\theta - q^\theta dq^\phi) = \frac{1}{2}\int_0^1 (-d\theta(-d\phi d\tau)) = \frac{1}{2}d\theta d\phi$$

For the bottom we have $q^\theta = 0$, ${}^\phi = \tau d\phi$ and thus

$$\frac{1}{2}\int_{bottom} .... = 0$$

On the left we have $q^\theta = d\theta - \tau d\theta$, $d\phi = 0$ and thus

$$\frac{1}{2}\int_{left} .... = 0$$

Finally, on the right we have $q^\theta = \tau d\theta$, $q^\phi = d\phi$ and thus

$$\frac{1}{2}\int_{right}(q_\phi dq^\theta - q^\theta dq^\phi) = \frac{1}{2}\int_0^1 (d\theta d\phi d\tau)) = \frac{1}{2}d\theta d\phi$$

Therefore, $S^{\phi\theta} = d\theta d\phi$ as required.

EP #35

We have the metric given by

$$ds^2 = -e^{2\phi(x)}dt^2 + e^{2\psi(x)}dx^2$$

(a) We set $x^0 = t$ and $x^1 = x$. The metric tensor is given by $ds^2 = g_{ab}dx^a dx^b$ so therefore the covariant components of the metric tensor are

$$[g_{ab}] = \begin{pmatrix} -e^{2\phi(x)} & 0 \\ 0 & e^{2\psi(x)} \end{pmatrix}$$

and the contravariant components are

$$[g^{ab}] = \begin{pmatrix} -e^{-2\phi(x)} & 0 \\ 0 & e^{-2\psi(x)} \end{pmatrix}$$

We note that the metric features dependence only on x^1 and is diagonal. Hence any nonzero element of the affine connection must have either one 1 or two 0's or three 1's if it is to have nonzero components. The affine connection is given by

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}[\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}]$$

First consider $a = 0$. We have $\Gamma^0_{00} = \Gamma^0_{00} = 0$ since there is no dependence on x^0. We also have $\Gamma^0_{10} = \Gamma^0_{01} = \phi'(x)$.

Now consider the $a = 1$ terms. We have $\Gamma^1_{01} = \Gamma^1_{10} = 0$, $\Gamma^1_{11} = \psi'(x)$ and $\Gamma^1_{00} = e^{2(\phi(x)-\psi(x))}\phi'(x)$.

We now compute the components of the Riemann tensor using

$$R^i{}_{jkl} = (\partial_l\Gamma^i_{jk} + \Gamma^i_{lm}\Gamma^m_{jk}) - (\partial_k\Gamma^i_{jl} + \Gamma^i_{km}\Gamma^m_{jl})$$

Since the Riemann tensor is symmetric in its first and last two indices, we must have

$$\begin{aligned} R^0{}_{000} &= R^1{}_{000} = R^0{}_{100} = R^0{}_{010} = R^0{}_{001} = R^0{}_{011} \\ &= R^1{}_{100} = R^0{}_{111} = R^1{}_{011} = R^1{}_{110} = R^1{}_{101} = R^1{}_{111} = 0 \end{aligned}$$

This leaves only four terms of the Riemann tensor, of which we can conclude only one is independent. We have

$$\begin{aligned}
R^0{}_{110} &= -R^0{}_{101} = R^1{}_{001} = -R^1{}_{010} \\
&= (\partial_1 \Gamma^0_{10} + \Gamma^0_{10}\Gamma^0_{10} + \Gamma^0_{11}\Gamma^1_{10}) - (\partial_0 \Gamma^0_{11} + \Gamma^0_{00}\Gamma^0_{11} + \Gamma^0_{01}\Gamma^1_{11}) \\
&= \phi''(x) + \phi'^{\,2}(x) - \phi'(x)\psi'(x)
\end{aligned}$$

(b) We now assume that $\phi = -\psi = \frac{1}{2}\ln|g(x-x_0)|$ where g and x_0 are constants. We start by calculating some derivatives

$$\phi'(x) = \frac{1}{2(x-x_0)} = -\psi'(x) \to \phi''(x) = -\frac{1}{2(x-x_0)^2}$$

Using these results we have $R^0{}_{110} = 0$.

Since every element of the Riemann tensor is zero, we can conclude that the space is globally flat.

We now look for the coordinate transformation . The tensor transformation law is

$$\bar{g}_{ab} = \frac{\partial \bar{x}^c}{\partial x^a}\frac{\partial \bar{x}^d}{\partial x^b} g_{cd}$$

Since the metric is diagonal we have

$$\begin{aligned}
\bar{g}_{00} &= \frac{\partial \bar{x}^0}{\partial x^0}\frac{\partial \bar{x}^0}{\partial x^0} g_{00} + \frac{\partial \bar{x}^1}{\partial x^0}\frac{\partial \bar{x}^1}{\partial x^0} g_{11} = -1 \\
\bar{g}_{11} &= \frac{\partial \bar{x}^0}{\partial x^1}\frac{\partial \bar{x}^0}{\partial x^1} g_{00} + \frac{\partial \bar{x}^1}{\partial x^1}\frac{\partial \bar{x}^1}{\partial x^1} g_{11} = +1 \\
\bar{g}_{01} &= \frac{\partial \bar{x}^0}{\partial x^0}\frac{\partial \bar{x}^0}{\partial x^1} g_{00} + \frac{\partial \bar{x}^1}{\partial x^0}\frac{\partial \bar{x}^1}{\partial x^1} g_{11} = 0 \\
\bar{g}_{10} &= \frac{\partial \bar{x}^0}{\partial x^1}\frac{\partial \bar{x}^0}{\partial x^0} g_{00} + \frac{\partial \bar{x}^1}{\partial x^1}\frac{\partial \bar{x}^1}{\partial x^0} g_{11} = 0
\end{aligned}$$

Now $g_{00} = -g(x-x_0)$ and $g_{11} = 1/g(x-x_0)$. Therefore we have

$$\begin{aligned}
&-\left(\frac{\partial \bar{t}}{\partial t}\right)^2 g(x-x_0) + \left(\frac{\partial \bar{x}}{\partial t}\right)^2 \frac{1}{g(x-x_0)} = -1 \\
&-\left(\frac{\partial \bar{t}}{\partial x}\right)^2 g(x-x_0) + \left(\frac{\partial \bar{x}}{\partial x}\right)^2 \frac{1}{g(x-x_0)} = +1 \\
&-\frac{\partial \bar{t}}{\partial t}\frac{\partial \bar{t}}{\partial x} g(x-x_0) + \frac{\partial \bar{x}}{\partial t}\frac{\partial \bar{x}}{\partial x}\frac{1}{g(x-x_0)} = 0 \\
&-\frac{\partial \bar{t}}{\partial x}\frac{\partial \bar{t}}{\partial t} g(x-x_0) + \frac{\partial \bar{x}}{\partial x}\frac{\partial \bar{x}}{\partial t}\frac{1}{g(x-x_0)} = 0
\end{aligned}$$

Let

$$\frac{\partial \bar{t}}{\partial t} = A\ , \quad \frac{\partial \bar{t}}{\partial x} = B\ , \quad \frac{\partial \bar{x}}{\partial t} = C\ , \quad \frac{\partial \bar{x}}{\partial x} = D$$

We then have the equations

$$
\begin{aligned}
&-A^2g^2(x-x_0)+C^2=-g(x-x_0)\\
&-B^2g^2(x-x_0)+D^2=g(x-x_0)\\
&-ABg^2(x-x_0)+CD=g(x-x_0)
\end{aligned}
$$

For convenience choose A=1. We then have

$$
\begin{aligned}
&-g^2(x-x_0)+C^2=-g(x-x_0)\\
&-B^2g^2(x-x_0)+D^2=g(x-x_0)\\
&-Bg^2(x-x_0)+CD=g(x-x_0)
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&C^2=g^2(x-x_0)-g(x-x_0)\\
&-B^2g^2(x-x_0)+D^2=g(x-x_0)\\
&-Bg^2(x-x_0)+(g^2(x-x_0)-g(x-x_0))D=g(x-x_0)
\end{aligned}
$$

The last equation gives

$$B=\left(1-\frac{1}{g(x-x_0)}\right)D-\frac{1}{g(x-x_0)}$$

so that we have for D

$$\left(\left(1-\frac{1}{g(x-x_0)}\right)D-\frac{1}{g(x-x_0)}\right)^2 g^2(x-x_0)+D^2=g(x-x_0)$$

Solve last equation for D, which then gives B and C.

EP #36

(a) We have $u^\mu = dx^\mu/d\tau$ and

$$g_{\mu\nu}u^\mu u^\nu = g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

But $g_{\mu\nu}dx^\mu dx^\nu = ds^2 = -c^2d\tau^2$. Therefore, $-g_{\mu\nu}u^\mu u^\nu = c^2$ and $c = \sqrt{-g_{\mu\nu}u^\mu u^\nu}$. This is the same result as in Minkowski space, where $u^\mu = (\gamma c, \gamma\vec{v})$ and

$$-g_{\mu\nu}u^\mu u^\nu = -\gamma^2(-c^2+v^2) = c^2$$

(b) We know that if a vector $\vec{v}$ is parallel propagated along a curve $c^\mu(\tau)$, then $\nabla_{\vec{u}}V^\alpha = 0$ where $\vec{u}$ is the tangent vector $u^\beta = dx^\beta/d\tau$. We also note that $\nabla_{\vec{u}}V^\alpha = u^\beta\nabla_\beta V^\alpha$.

Now consider

$$\nabla_{\vec{u}}(g_{\alpha\beta}V^\alpha V^\beta) = g_{\alpha\beta}(V^\alpha\nabla_{\vec{u}}V^\beta + V^\beta\nabla_{\vec{u}}V^\alpha)$$

where we have used the Leibniz rule for differentiation of a product and made use of the fact that $\nabla_{\vec{u}} g_{\alpha\beta} = 0$.

Now, by assumption, $\nabla_{\vec{u}} V^\alpha = 0$. Therefore, $\nabla_{\vec{u}}(g_{\alpha\beta} V^\alpha V^\beta) = 0$ which can be written as $u^\gamma \nabla_\gamma (g_{\alpha\beta} V^\alpha V^\beta) = 0$.

Now, since $g_{\alpha\beta} V^\alpha V^\beta$ is a scalar, we have

$$\nabla_\gamma (g_{\alpha\beta} V^\alpha V^\beta) = \frac{\partial}{\partial x^\gamma}(g_{\alpha\beta} V^\alpha V^\beta)$$

Thus, using the chain rule we have

$$0 = u^\gamma \nabla_\gamma (g_{\alpha\beta} V^\alpha V^\beta) = \frac{dx^\gamma}{d\tau}\frac{\partial}{\partial x^\gamma}(g_{\alpha\beta} V^\alpha V^\beta) = \frac{d}{d\tau}(g_{\alpha\beta} V^\alpha V^\beta)$$

Therefore, we have proved that

$$\frac{d}{d\tau}(g_{\alpha\beta} V^\alpha V^\beta) = 0$$

which implies that for a timelike vector (for which $g_{\alpha\beta} V^\alpha V^\beta < 0$),

$$\frac{d}{d\tau}(-g_{\alpha\beta} V^\alpha V^\beta)^{1/2} = 0$$

This means that the length of V^α is constant along the curve $x^\mu(\tau)$. NOTE: We did not need to assume that $x^\mu(\tau)$ is a geodesic curve. Any curve along which $\vec{v}$ is parallel propagated has the property that the length of $\vec{v}$ is constant along the curve.

(c) We know that the condition for a geodesic curve is $\nabla_{\vec{u}} u^\alpha = 0$ where $u^\alpha = dx^\alpha/d\tau$. From part (b), this means that the tangent vector u^α is parallel propagated along the geodesic curve $x^\alpha(\tau)$. Since u^α is tangent to the geodesic curve at all points along the geodesic, we conclude that a vector tangent to the geodesic at a given point will always remain tangent to the geodesic curve when parallel transported along the geodesic.

In case it is not clear that $\nabla_{\vec{u}} u^\alpha = 0$ is equivalent to the geodesic equation, here is the proof.

$$\nabla_{\vec{u}} u^\alpha = u^\beta \nabla_\beta u^\alpha = u^\beta \left[\frac{\partial u^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma} u^\gamma\right]$$

Now, $u^\beta = dx^\beta/d\tau$, so by the chain rule we have

$$u^\beta \frac{\partial u^\alpha}{\partial x^\beta} = \frac{dx^\beta}{d\tau}\frac{\partial u^\alpha}{\partial x^\beta} = \frac{du^\alpha}{d\tau} = \frac{d}{d\tau}\left(\frac{dx^\alpha}{d\tau}\right) = \frac{d^2 x^\alpha}{d\tau^2}$$

Thus,

$$\begin{aligned}\nabla_{\vec{u}} u^\alpha &= \frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma \\ &= \frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}\end{aligned}$$

Setting $\nabla_{\vec{u}} u^\alpha = 0$ then is equivalent to the geodesic equation

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0$$

EP #37

(a) The coordinate velocity in the radial direction is dr/dt. Light travels along a path characterized by $ds^2 = 0$. Setting $d\theta d\phi = 0$ for the radial path we have

$$0 = \left(1 - \frac{2GM}{c^2r}\right)c^2dt^2 - \left(1 - \frac{2GM}{c^2r}\right)^{-1}dr^2$$

Thus,

$$\left(\frac{dr}{dt}\right)^2 = c^2\left(1 - \frac{2GM}{c^2r}\right)^2$$

or

$$\frac{dr}{dt} = c\left(1 - \frac{2GM}{c^2r}\right)$$

(b) In the transverse direction, set $dr = d\phi = 0$. Then

$$0 = \left(1 - \frac{2GM}{c^2r}\right)c^2dt^2 - r^2d\theta^2$$

Thus,

$$r\frac{d\theta}{dt} = c\left(1 - \frac{2GM}{c^2r}\right)^{1/2}$$

(c) The physical consequences of these results is the slowing of light as it passes a massive body. This leads to measurable time delays for light passing the sun, etc. It has been confirmed experimentally. In addition, in both cases we see that as $r \to \infty$, the coordinate velocity approaches c. Furthermore, we can see that as the radius approaches the Schwarzschild radius, the coordinate velocity goes to zero. This just restates the familiar idea that light cannot escape from inside the event horizon of a black hole. We note several peculiar, but related, effects. First, inside the Schwarzschild radius the radial velocity becomes negative. Second, in the same regime, the tangential velocity becomes complex!

EP #38

(a) Start from the Lagrangian

$$L = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = c^2\left(1 - \frac{2GM}{c^2r}\right)\dot{t}^2 - \left(1 - \frac{2GM}{c^2r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2$$

where we have already set $\theta = \pi/2 =$ constant (so that $\dot{\theta} = 0$. The parameter λ is an affine parameter. Two of Lagrange's equations are:

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \rightarrow r^2\dot{\phi} = \bar{J}$$
$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{t}}\right) = \frac{\partial L}{\partial t} \rightarrow \frac{d}{d\lambda}\left[\left(1 - \frac{2GM}{c^2 r}\right)\dot{t}\right] = 0 \rightarrow \left(1 - \frac{2GM}{c^2 r}\right)\dot{t} = \bar{E}$$

where $\bar{J}$ and $\bar{E}$ are constants of the motion. Finally for photon orbits we use $ds^2 = 0 = -g^{\mu\nu} dx^\mu dx^\nu$ which gives the equation

$$c^2\left(1 - \frac{2GM}{c^2 r}\right)\dot{t}^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = 0$$

Inserting $\dot{t} = \bar{E}\left(1 - \frac{2GM}{c^2 r}\right)^{-1}$ and $\dot{\phi} = \bar{J}/r^2$ gives

$$\bar{E}^2 c^2\left(1 - \frac{2GM}{c^2 r}\right)^{-1} - \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\dot{r}^2 - \frac{\bar{J}^2}{r^2} = 0$$

Solving for $\bar{E}$ gives

$$\bar{E}^2 = \frac{1}{c^2}\dot{r}^2 + \frac{\bar{J}^2}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right)$$

(b) We define an effective potential by

$$\bar{E}^2 = \frac{1}{c^2}\dot{r}^2 + V_{eff} \rightarrow V_{eff}(r) = \frac{\bar{J}^2}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right)$$

First, let us compute the extremum of the potential $V_{eff}(r)$:

$$\frac{dV_{eff}(r)}{dr} = 0 \rightarrow -\frac{2\bar{J}^2}{c^2 r^3}\left(1 - \frac{2GM}{c^2 r}\right) + \frac{\bar{J}^2}{c^2 r^2}\frac{2GM}{c^2 r^2} = 0$$

This simplifies to

$$r\left(1 - \frac{2GM}{c^2 r}\right) = \frac{GM}{c^2} \rightarrow r = \frac{3GM}{c^2} = \frac{3}{2}r_s$$

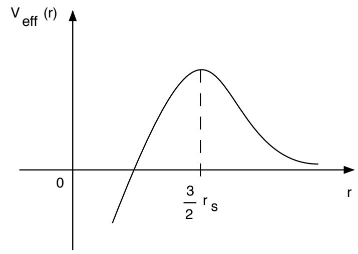
Compute the second derivative:

$$\frac{d^2V_{eff}(r)}{dr^2} = \frac{6\bar{J}^2}{c^2 r^4}\left(1 - \frac{2GM}{c^2 r}\right) - \frac{2\bar{J}^2}{c^2 r^3}\frac{2GM}{c^2 r^2} - \frac{8\bar{J}^2 GM}{c^4 r^5} = \frac{6\bar{J}^2}{c^2 r^4}\left(1 - \frac{3r_s}{r}\right)$$

At $r = 3r_s/2$,

$$\frac{d^2V_{eff}(r)}{dr^2} = -\frac{6\bar{J}^2}{c^2 r^4} < 0$$

Thus $r = 3r_s/2$ is a maximum of $V_{eff}(r)$ (see figure below)



That is, the point $r = 3r_s/2$ corresponds to an unstable circular orbit. Since

$$V_{eff}(r) = \frac{\bar{J}^2}{3c^2 r_s^2}$$

at $r = 3r_s/2$, we see that if

$$\bar{E}^2 = \frac{4\bar{J}^2}{27c^2 r_s^2}$$

then at $r = 3r_s/2$ we have $\dot{r} = 0$ corresponding to circular motion.

(c) Start from the result

$$c^2\left(1 - \frac{2GM}{c^2 r}\right)\dot{t}^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = 0$$

For circular motion $dotr = 0$. Thus,

$$\dot{\phi}^2 = \frac{c^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right)\dot{t}^2$$

Now,

$$\frac{d\phi}{dt} = \frac{d\phi/d\lambda}{dt/d\lambda} = \frac{\dot{\phi}}{\dot{t}} = \frac{2}{3\sqrt{3}}\frac{c}{r_s}$$

Therefore,

$$t = \frac{3\sqrt{3}}{2c}\int d\phi$$

Integrating over one revolution yields a period of

$$\Delta t = \frac{3\sqrt{3}\pi r_s}{c}$$

Now in the Schwarzschild geometry, an observer at a fixed point (r, θ, ϕ) measures a proper time equal to

$$\Delta T = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2}\Delta t = \left(1 - \frac{r_s}{r}\right)^{1/2}\Delta t$$

At $r = 3r_s/2$, the observer measures the time to complete one revolution as

$$\Delta T = \frac{1}{\sqrt{3}} \Delta t = \frac{3\pi r_s}{c}$$

(d) The very distant observer measures Schwarzschild time, so the orbital period measured is

$$\Delta t = \frac{3\sqrt{3}\pi r_s}{c}$$

(e) The orbit equation for the photon is

$$\frac{d^2 u}{d\phi^2} + u = \frac{3GM}{c^2} u^2$$

where $u = 1/r$. In terms of r,

$$\frac{du}{d\phi} = \frac{d}{dr}\left(\frac{1}{r}\right) \frac{dr}{d\phi} = -\frac{1}{r^2} \frac{dr}{d\phi}$$
$$\frac{d^2 u}{d\phi^2} = -\frac{1}{r^2} \frac{d^2 r}{d\phi^2} + \frac{2}{r^2} \left(\frac{dr}{d\phi}\right)^2$$

Therefore,

$$\frac{d^2 r}{d\phi^2} - \frac{2}{r} \left(\frac{dr}{d\phi}\right)^2 = r - \frac{3GM}{c^2} = r - \frac{3r_s}{2}$$

Note that $r = 3r_s/2$ is a solution to this equation.

If we substitute $r = 3r_s/2 + \eta$, we obtain an equation for η, which represents the deviations from a circular orbit

$$\frac{d^2 \eta}{d\phi^2} - \frac{2}{\frac{3r_s}{2} + \eta} \left(\frac{d\eta}{d\phi}\right)^2 = \eta$$

If $|\eta/r_s| \ll 1$, the we can neglect the term proportional to $(d\eta/d\phi)^2$ which is quadratic in η. We are then left with

$$\frac{d^2 \eta}{d\phi^2} = \eta$$

The solution of this equation is $\eta = Ae^{|}phi + Be^{-\phi}$, which exhibits exponetial growth in ϕ. Thgus, the size of the perturbation grows without bound (rather than oscillating as in the case of a stable orbit). Hence, the circular orbit at $r = 3r_s/2$ is unstable.

EP #39

(a) The metric is given by

$$ds^2 = -xdw^2 + 2dwdx$$

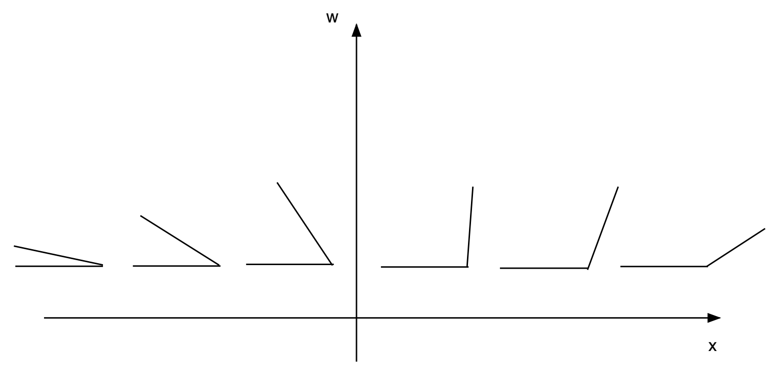
The light cone is given by

$$ds^2 = -xdw^2 + 2dwdx = 0$$

which gives

$$dw = 0 \quad \text{and} \quad \frac{dw}{dx} = \frac{2}{x}$$

These equations represent the two sides of the light cone as shown below.



Thus, one can cross from $x > 0$ to $x < 0$ but not vice versa.

(b) We can write

$$ds^2 = -xdw^2 + 2dwdx - \frac{1}{x}dx^2 + \frac{1}{x}dx^2 = -x\left(dw - \frac{1}{x}dx\right)^2 + \frac{1}{x}dx^2$$

If we then choose

$$dv = dw - \frac{1}{x}dx \to v = w - \ln x$$

the metric equation becomes diagonal, i.e.,

$$ds^2 = -xdv^2 + +\frac{1}{x}dx^2$$

EP #40

Consider a particle in circular orbit around a point mass M (we set $G = c = 1$). The Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

If a particle is following a geodesic, then $ds^2 = -d\tau^2$. Thus, dividing through by dt^2, we get

$$\frac{d\tau^2}{dt^2} = \left(1 - \frac{2M}{r}\right) - \left(1 - \frac{2M}{r}\right)^{-1} \frac{dr^2}{dt^2} - r^2 \frac{d\theta^2}{dt^2} - r^2 \sin^2\theta \frac{d\phi^2}{dt^2}$$

Since we are in circular orbit (in a plane) we have $\dot{r} = 0$, $\dot{\theta} = 0$ and if we choose $\theta = \pi/2$, we have

$$\frac{d\tau^2}{dt^2} = \left(1 - \frac{2M}{r}\right) - r^2 \frac{d\phi^2}{dt^2}$$

or

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{2M}{r} - r^2 \frac{d\phi^2}{dt^2}}$$

We now need to determine $d\phi/dt$. Consider the radial component of the geodesic equation

$$\frac{d^2 r}{dt^2} + \Gamma^r_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = 0$$

Now for the the circular orbit we have

$$\frac{d^2 r}{dt^2} = 0$$

Also since $d\theta = dr = 0$, only $\Gamma^r_{\phi\phi}$, $\Gamma^r_{\phi t}$ and Γ^r_{tt} will contribute to the sum. For a diagonal metric the affine connection is given by

$$\Gamma^r_{bc} = \frac{1}{2} g^{rr} (\partial_b g_{rc} + \partial_c g_{rb} - \partial_r g_{bc})$$

Clearly, because the metric is diagonal, only $\Gamma^r_{\phi\phi}$ and Γ^r_{tt} will be nonzero. Finally, we note that the contravariant component of the metric will be

$$g^{rr} = 1 - \frac{2M}{r}$$

Thus, calculating these quantities, we have

$$\Gamma^r_{tt} = -\frac{1}{2} g^{rr} \partial_r g_{tt} = \frac{M}{r^2} \left(1 - \frac{2M}{r}\right)$$
$$\Gamma^r_{\phi\phi} = -\frac{1}{2} g^{rr} \partial_r g_{\phi\phi} = -r \left(1 - \frac{2M}{r}\right)$$

Thus, rearranging the geodesic equation, we have

$$\Gamma^r_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2 + \Gamma^r_{tt}\left(\frac{dt}{d\tau}\right)^2 = 0 \tag{5}$$

or

$$\left(\frac{d\phi/d\tau}{dt/d\tau}\right)^2 = -\frac{\Gamma^r_{tt}}{\Gamma^r_{\phi\phi}} \to \left(\frac{d\phi}{dt}\right)^2 = \frac{M}{r^3}$$

We finally have

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{2M}{r} - r^2\frac{d\phi^2}{dt^2}} = \sqrt{1 - \frac{3M}{r}}$$

Since this is constant with respect to coordinate time, we can easily integrate to obtain

$$\frac{\tau}{t} = \sqrt{1 - \frac{3M}{r}}$$

as required.

EP #41

(a) Done in earlier problems and in text.

(b) Done in Problem #40.

$$\Omega = \frac{d\phi/d\tau}{dt/d\tau} = \frac{d\phi}{dt} = \sqrt{-\frac{\Gamma^r_{tt}}{\Gamma^r_{\phi\phi}}} = \sqrt{\frac{M}{r^3}} = \frac{2\pi}{\text{period}}$$

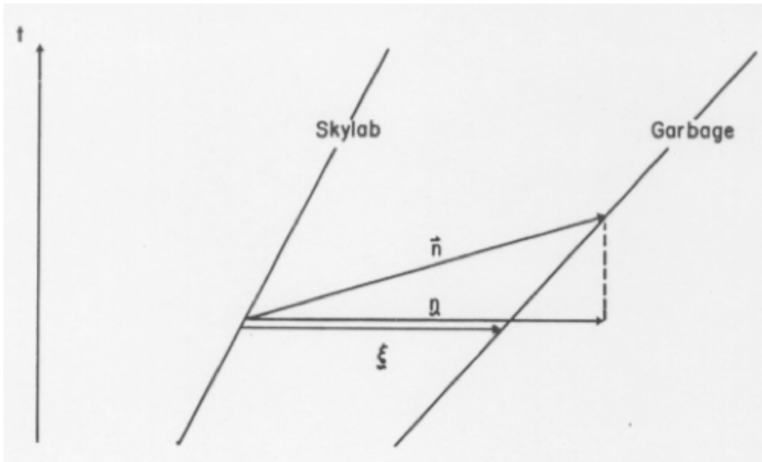
which is the same as the Newtonian result.

(c) Now if we have two nearby geodesics, then they have close values of affine parameter λ. If we let $u^\alpha = dx^\alpha/d\lambda$ be the tangent to one of the geodesics and we let $\vec{n}$ be the differential vector connecting points of equal affine parameter on the two geodesics, then it is proved in the text that the equation of *geodesic deviation* is

$$\frac{D^2 n^\alpha}{d\lambda^2} + R^\alpha_{\beta\gamma\delta}u^\beta u^\gamma u^\delta = 0$$

where $D = \partial + \Gamma$.

We are required in the problem to the vector $\vec{\xi}$ defined in the problem, which is the separation of points on the two geodesics at a given coordinate time. From the diagram below



it is clear that the fractional difference vetween $\vec{n}$ and $\vec{\xi}$ is proportional to the relative velocity of the grabage and the space shuttle or SkyLab, and hence can be ignored to lowest order. Now, since all the Christoffel symbols in the t, x, y, z system are of order M/r^2, and $d/d\tau$ is of order $\omega \sim (M/r)^{1/2} r^{-1}$, we can approximate

$$\frac{D}{d\tau} = \frac{d}{d\tau} + \Gamma u \approx \frac{d}{d\tau}$$

Furthermore

$$\frac{d}{d\tau} = u^0 \frac{d}{dt} = \left[1 + O\left(\frac{M}{r}\right)\right] \frac{d}{dt} \tag{6}$$

so we can approximate $D^2/d\tau^2 \approx d^2/dt^2$. Using these approximations in the equation of geodesic deviation for the vector $\vec{\xi}$ we have

$$\frac{d^2\xi^i}{dt^2} + R^i_{0j0}(u^0)^2 \xi^j = 0 \qquad i = x, y, z$$

To lowest order the Riemann components are

$$R^i_{0j0} \approx \Gamma^i_{00,j} - \Gamma^i_{0j,0} = -\frac{1}{2} g_{00,ij} = \frac{M}{r^3}\left(\delta_{ij} - \frac{3x^i x^j}{r^2}\right)$$

If we choose to describe the Skylab or space shuttle orbit as

$$x = r\cos\Omega t \quad , \quad y = r\sin\Omega t$$

then the equations of motion for $\vec{\xi}$ become

$$\begin{aligned}
\ddot{\xi}^x + \Omega^2 \xi^x &= 3\Omega^2 \cos\Omega t(\cos\Omega t \xi^x + \sin\Omega t \xi^y) \\
\ddot{\xi}^y + \Omega^2 \xi^y &= 3\Omega^2 \sin\Omega t(\cos\Omega t \xi^x + \sin\Omega t \xi^y) \\
\ddot{\xi}^z + \Omega^2 \xi^z &= 0
\end{aligned}$$

The relative motion in the $z-$direction is clearly

$$\xi^z = L \cos \Omega t$$

if the garbage was jettisoned at $t = 0$ and $\xi^z(0) = L$. To find the relative motion in the $x - y$ plane introduce new variables η^1, η^2 defined by

$$\xi^x = \eta^1 \cos \Omega t + \eta^2 \sin \Omega t \quad , \quad \xi^y = \eta^1 \sin \Omega t - \eta^2 \cos \Omega t$$

and the equations of motion become

$$\begin{aligned} &\ddot{\eta}^2 - 2\Omega \dot{\eta}^1 = 0 \\ &\ddot{\eta}^1 + 2\Omega \dot{\eta}^2 - 3\Omega^2 \eta^1 = 0 \end{aligned}$$

These can be solved easily to find the four independent solutions

$$[\eta^1, \eta^2] = \left[1, \frac{3}{2}\Omega t\right], [0, 1], [\cos \Omega t, 2 \sin \Omega t], [\sin \Omega t, -2 \cos \Omega t]$$

The linear combination of solution which correspond to the garbage jettisoned at $t = 0$ with $\xi^x(0) = \xi^y(0) = 0$ is

$$\begin{aligned} \xi^x &= A[\cos 2\Omega t - 3 + 2 \cos \Omega t + 3\Omega t \sin \Omega t] + B[4 \sin \Omega t - \sin 2\Omega t] \\ \xi^y &= A[\sin 2\Omega t + 2 \sin \Omega t - 3\Omega t \cos \Omega t] + B[\cos 2\Omega t + 3 - 4 \cos \Omega t] \end{aligned}$$

(d) The constants A and B depend on the x and y components of the velocity with which the garbage was jettisoned. The nonperiodic terms in the solution correspond to the fact that the two orbits are of slightly different period, so the relative distance will exhibit a secular growth in time . They get rid of the garbage!

EP #42

(a) The Schwarzschild line element is

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^\theta d\phi^2)$$

The geodesics are computed from the Lagrangian $L = -g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$.

$$L = \left(1 - \frac{2GM}{c^2 r}\right) c^2 \dot{t}^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)$$

The geodesic equation for variable r follows from

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$$

This gives

$$\frac{d}{d\tau}\left(-2\dot{r}\left(1-\frac{2GM}{c^2r}\right)^{-1}\right)=\frac{2GM}{r^2}\dot{t}^2+\left(1-\frac{2GM}{c^2r}\right)^{-2}\left(\frac{2GM}{c^2r^2}\right)\dot{r}^2$$
$$-2r(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)$$

Thus, doing the derivative and rearranging we have

$$\left(1-\frac{2GM}{c^2r}\right)^{-1}\ddot{r}+\frac{GM}{r^2}\dot{t}^2-\left(1-\frac{2GM}{c^2r}\right)^{-2}\dot{r}^2-r(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)=0$$

For a circular orbit in the plane $\theta=\pi/2$, we have $\dot{r}=\ddot{r}=0$ and $\dot{\theta}=0$. Thus,

$$\frac{GM}{r^2}\dot{t}^2=r\dot{\phi}^2\rightarrow\frac{\dot{\phi}}{\dot{t}}=\left(\frac{GM}{r^3}\right)^{1/2}$$

Now the constants of motion are

$$e=\left(1-\frac{r_s}{r}\right)\dot{t}\quad,\quad\ell=r^2\dot{\phi}$$

Thus,

$$\frac{\ell}{e}=r^2\left(1-\frac{r_s}{r}\right)^{-1}\frac{\dot{\phi}}{\dot{t}}=(GMr)^{1/2}\left(1-\frac{r_s}{r}\right)^{-1}$$

Finally, we note that $GM=c^2r_s/2$. thus,

$$\frac{\ell}{e}=c\left(\frac{1}{2}r_sr\right)^{1/2}\left(1-\frac{r_s}{r}\right)^{-1}$$

(b) The relationship $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=-c^2$ gives

$$-\left(1-\frac{2GM}{c^2r}\right)c^2\dot{t}^2+\left(1-\frac{2GM}{c^2r}\right)^{-1}\dot{r}^2+r^2(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)=-c^2$$

Again, we insert $\dot{r}=\dot{\theta}=0$ and $\theta=\pi/s$ for the circular orbit. Thus,

$$-\left(1-\frac{2GM}{c^2r}\right)c^2\dot{t}^2+r^2\dot{\phi}^2=-c^2$$

Inserting

$$\dot{t}=\left(1-\frac{r_s}{r}\right)^{-1}e\quad\text{and}\quad\dot{\phi}=\frac{\ell}{r^2}\quad\text{with}\quad r_s=\frac{2GM}{c^2}$$

we get

$$-\left(1-\frac{r_s}{r}\right)^{-1}c^2e^2+\frac{\ell^2}{r^2}=-c^2$$

From part (a)

$$ce = \ell \left(1 - \frac{r_s}{r}\right) \left(\frac{1}{2} r_s r\right)^{-1/2}$$

Hence,

$$-\ell^2 \left[\left(1 - \frac{r_s}{r}\right) \frac{2}{r_s r} - \frac{1}{r^2}\right] = -c^2$$

Simplifying the last expression yields

$$\frac{2\ell^2}{r_s r} \left(1 - \frac{3}{2} \frac{r_s}{r}\right) = c^2$$

or

$$\ell = c \left(\frac{1}{2} r_s r\right)^{1/2} \left(1 - \frac{3}{2} \frac{r_s}{r}\right)^{-1/2}$$

Now,

$$\frac{d\tau}{d\phi} = \frac{1}{d\phi/d\tau} = \frac{r62}{\ell}$$

Hence,

$$\frac{d\tau}{d\phi} = \frac{r^2}{c} \left(\frac{1}{2} r_s r\right)^{-1/2} \left(1 - \frac{3}{2} \frac{r_s}{r}\right)^{1/2}$$

(c) The observer at rest inside the orbiting spacecraft measures proper time. Thus, the period of an orbit as measured by the orbiting astronaut, T_0, is given by

$$T_0 = \int_{\text{one period}} d\tau = \int_0^{2\pi} \frac{d\tau}{d\phi} d\phi = 2\pi \frac{d\tau}{d\phi}$$

or

$$T_0 = \frac{2\pi r^2}{c} \left(\frac{1}{2} r_s r\right)^{-1/2} \left(1 - \frac{3}{2} \frac{r_s}{r}\right)^{1/2}$$

(d) Consider an astronaut outside the spacecraft at fixed r, $\theta = \pi/2$ and $\phi = 0$. This astronaut also measures proper time, but in this case

$$c^2 d\tau^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2) = \left(1 - \frac{r_s}{r}\right) c^2 dt^2$$

Since $dr = d\theta = d\phi = 0$ for the outside astronaut. This stationary astronaut outside the spacecraft measures a period T_s given by

$$T_s = \int_{\text{one period}} d\tau = \left(1 - \frac{r_s}{r}\right)^{1/2} \int_{\text{one period}} dt$$

Now t is Schwarzschild time, so any observer can agree on the value of $\int_{\text{one period}} dt$. Since the orbiting spacecfraft has angular velocity $d\phi/dt$ in Schwarzschild coordinates,

$$\int_{\text{one period}} = \int_0^{2\pi} \frac{dt}{d\phi} d\phi$$

By the chain rule,

$$\frac{dt}{d\phi} = \frac{\dot{t}}{\dot{\phi}} = \left(\frac{GM}{r^3}\right)^{-1/2} = \frac{r^2}{c}\left(\frac{1}{2}r_s r\right)^{-1/2}$$

where we have used the result for $\dot{\phi}/\dot{t}$ from part (a). Hence

$$\int_{\text{one period}} dt = \frac{2\pi r^2}{c}\left(\frac{1}{2}r_s r\right)^{-1/2}$$

and

$$T_s = \frac{2\pi r^2}{c}\left(\frac{1}{2}r_s r\right)^{-1/2}\left(1 - \frac{r_s}{r}\right)^{1/2}$$

Comparing the measurements of the period

$$\frac{T_s}{T_0} = \left(\frac{r - r_s}{r - \frac{3}{2}r_s}\right)^{1/2} > 1$$

This indicates that the stationary astronaut ages more than the the orbiting astronaut, i.e., the stationary astronaut outside the spacecraft ages faster than the orbiting astronaut.

EP #43

(a) The text derives the weak field metric or you can derive it yourself. We get $g_{ij} = (1 - 2\phi)\delta_{ij}$ so that $\alpha = 1$.

(b) For a boost with speed v in the $x-$direction, we have $x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu}$ or $t' = \gamma(t + vx)$, $x' = \gamma(x + vt)$, $y' = y$, $z' = z$. The transformed metric is

$$[g'] = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -(1+2\phi) & 0 & 0 & 0 \\ 0 & 1-2\alpha\phi & 0 & 0 \\ 0 & 0 & 1-2\alpha\phi & 0 \\ 0 & 0 & 0 & 1-2\alpha\phi \end{pmatrix} \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

or

$$[g'] = \begin{pmatrix} -1 - 2\gamma^2\phi(1 + \alpha v^2) & -2\gamma^2 v\phi(1+\alpha) & 0 & 0 & \\ -2\gamma^2 v\phi(1+\alpha) & 1 - 2\gamma^2\phi(\alpha + v^2) & 0 & 0 & \\ & 0 & 0 & 1-2\alpha\phi & 0 \\ & 0 & 0 & 0 & 1-2\alpha\phi \end{pmatrix}$$

(c) There are (at least) two ways to do this. First, use the geodesic equation in the geometry of part (b). Let the undeflected trajectory be $t = \lambda$, $x = -b$, $y = \lambda$, with tangent $p^{\mu} = (1, 0, 1, 0)$. The geodesic equation gives

$$\begin{aligned} \frac{dp_{\mu}}{d\lambda} &= \Gamma_{\alpha\mu\beta}p^{\alpha}p^{\beta} = \frac{1}{2}h_{\alpha\beta,\mu}p^{\alpha}p^{\beta} \\ &= \frac{1}{2}(h_{00} + 2h02 + h_{22})_{,\mu} = -\gamma^2(1+\alpha)\phi_{,\mu} \end{aligned}$$

The derivatives are

$$\phi_{,t} = \gamma^2 \frac{(vt-b)M}{r^3} \quad , \quad \phi_{,x} = \gamma^2 \frac{(b-vt)M}{r^3} \quad , \quad \phi_{,y} = \frac{yM}{r^3}$$

On integrating over the full trajectory, the net changes are

$$\Delta p_t = -\gamma v \Delta\theta_0 \quad , \quad \Delta p_x = \gamma \Delta\theta_0 \quad , \quad \Delta p_y = \gamma v \Delta\theta_0 \tag{7}$$

Or, second, boost to the rest frame, where the mass M sits at the origin and where we know from the text that $\Delta\theta_0 = 2(1+\alpha)M/b$. In the rest frame

$$\begin{aligned}
&t' = \gamma(t - vx) = \gamma(\lambda + bv) \quad , \quad p'^t = \gamma(p^t - vp^x) = \gamma \\
&x' = \gamma(x - vt) = -\gamma(b + v\lambda) \quad , \quad p'^x = \gamma(p^x - vp^t) = -\gamma v \\
&y' = y = \lambda \quad , \quad p'^y = p^y = 1
\end{aligned}$$

The distance of closest approach is still b, achieved now at $t' = 0$ in the rest frame or at $t = -vb$ in the moving frame. The deflection is a rotation by $\Delta\theta_0$ about the $z-$axis,

$$\begin{aligned}
p''^t &= p'^t = \gamma \\
p''^x &= p'^x + \Delta\theta_0 p'^y = -\gamma v + \Delta\theta_0 \\
p''^y &= p'^y - \Delta\theta_0 p'^x = 1 + \gamma v \Delta\theta_0
\end{aligned}$$

Finally, boost this back to the original frame,

$$\begin{aligned}
p'''^t &= \gamma(p''^t + vp''^x) = 1 + \gamma v \Delta\theta_0 \\
p'''^x &= \gamma(p''^x + vp''^t) = -\gamma\Delta\theta_0 \\
p'''^y &= p''^y = 1 + \gamma v \Delta\theta
\end{aligned}$$

The deflection angle is $\delta\theta = \Delta p_x / p_y$,

$$\Delta\theta = \gamma\Delta\theta_0 = \frac{\gamma(1+\alpha)M}{b}$$

(d) The same two calculations give

$$\omega''' = (1 + \gamma v \Delta\theta_0)\omega$$

EP #44

(a) the factor

$$1 - \frac{2M(r)}{r} = \begin{cases} 1 - \frac{2M}{r} & r > R \\ 1 - \frac{2M}{R}\left(\frac{r}{R}\right)^2 & r < R \end{cases}$$

Since $R > 2M$, this factor is never 0, i.e.,

$$\frac{2M}{R}\left(\frac{r}{R}\right)^2 < 1$$

for $r < R$ and $2M/R < 1$.

(b) Given $\xi^\alpha = (1,0,0,0) \quad , \quad \eta^\alpha = (0,0,0,1)$, we have

$$e \equiv -\vec{\xi}\cdot\vec{u} = \left(1 - \frac{2M(r)}{r}\right)\frac{dt}{d\lambda}$$

$$\ell \equiv \vec{\eta}\cdot\vec{u} = r^2\frac{d\phi}{d\lambda} \quad \theta = \pi/2$$

(c) Now $\vec{u}\cdot\vec{u} = 0$ or

$$\left(1 - \frac{2M(r)}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 - \left(1 - \frac{2M(r)}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 = 0$$

or

$$e^2 - \left(\frac{dr}{d\lambda}\right)^2 + \frac{\ell^2}{r^2}\left(1 - \frac{2M(r)}{r}\right) = 0$$

or

$$\frac{e^2}{\ell^2} \equiv \frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}$$

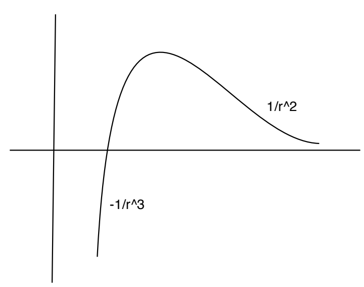
where

$$W_{eff} = \frac{1}{r^2} - \frac{2M(r)}{r^3} = \begin{cases} \frac{1}{r^2} - \frac{2M}{r^3} & r > R \\ \frac{1}{r^2} - \frac{2M}{R^3} & r < R \end{cases}$$

(d) In the standard point mass Schwarzschild case we would have

$$W_{eff} = \frac{1}{r^2} - \frac{2M}{r^3}$$

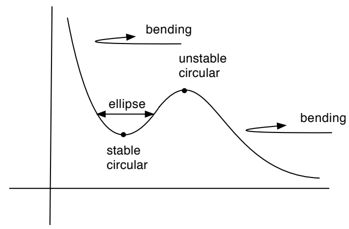
which looks like



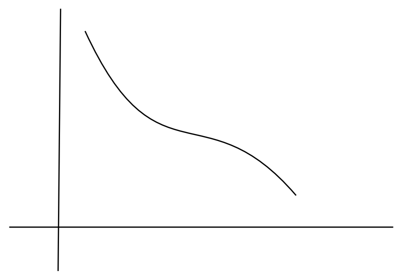
Recall that in the standard Schwarzschild case we have

$$\frac{dW_{eff}}{dr} = -\frac{1}{r^3} + \frac{6M}{r^4} = 0 \rightarrow \text{ maximum when } r = 3M$$

In this problem, however, we have for the case $2M < R < 3M$



and for the case $R > 3M$



Thus, there is no horizon where photons fall in and cannot escape!

(e) We have

$$\frac{dt}{dr} = \frac{1}{1 - \frac{2M(r)}{r}}$$

and since $r < R$, we have

$$\frac{2M(r)}{r} = \frac{2M}{R}\frac{r^2}{R^2}$$

which gives

$$\begin{aligned} t &= \int_0^R \frac{dr}{1 - ar^2} \quad , \quad a = \frac{2M}{R^3} \\ &= \left(\frac{R^3}{2M}\right)^{1/2} \tanh^{-1}\left(\left(\frac{2M}{R^3}\right)^{1/2} R\right) = R\left(\frac{R}{2M}\right)^{1/2} \tanh^{-1}\left(\left(\frac{2M}{R}\right)^{1/2}\right) \end{aligned}$$

(f) If $R \gg M$, then we have

$$\begin{aligned} t &= R\left(\frac{R}{2M}\right)^{1/2}\left(\left(\frac{2M}{R}\right)^{1/2}\right) + \frac{1}{3}\left(\frac{2M}{R}\right)^{1/2} \\ &= R\left(1 + \frac{1}{3}\frac{2M}{R}\right) \end{aligned}$$

Thus,

$$\frac{\delta t}{t} = \frac{1}{3}\frac{2M}{R} = \frac{2 \times 1.5\,km}{3 \times 7 \times 10^5\,km} = 1.4 \times 10^{-6}$$

EP #45

(a) For fixed r, θ and ϕ we have

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 = -c^2 dT^2 \to dT = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} dt$$

(b) For fixed t, θ and ϕ we have

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 = dR^2 \to dR = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} dr$$

(c) The geodesic equation is obtained from $L = -g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$, i.e.,

$$L = \left(1 - \frac{2GM}{c^2 r}\right) c^2 \dot{t}^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \dot{r}^2 - r^2\dot{\theta}^2 - r^2 \sin^2\theta \dot{\phi}^2$$

Then using

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{t}}\right) - \frac{\partial L}{\partial t} = 0 \tag{8}$$

with

$$\frac{\partial L}{\partial t} = 0 \tag{9}$$

we obtain

$$\frac{d}{d\tau}\left[\left(1 - \frac{2GM}{c^2 r}\right)\dot{t}\right] = 0 \tag{10}$$

where $\dot{t} = dt/d\tau$. This means that

$$k = \left(1 - \frac{2GM}{c^2 r}\right)\frac{dt}{d\tau}$$

is a constant of the motion independent of τ. Writing out the component of the geodesic equation in more detail,

$$\left(1 - \frac{2GM}{c^2 r}\right)\ddot{t} + \frac{2GM}{c^2 r^2}\dot{t}\dot{r} = 0$$

which we can write as

$$\ddot{t} + \frac{2GM}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1}\dot{t}\dot{r} = 0$$

Comparing this with the formal expression for the geodesic

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = 0$$

we conclude that

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{GM}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1}$$

and

$$\Gamma^t_{\alpha\beta} = 0$$

for any other choice of α, β.

(d) Assume that the particle follows a radial geodesic in which the particle initially starts at rest when $r \to \infty$. Since

$$k = \left(1 - \frac{2GM}{c^2 r}\right) \frac{dt}{d\tau}$$

is independent of τ, we can evaluate k at any point along its trajectory. In particular, as $r \to \infty$, we assert that $dt/d\tau = 1$ since the particle is initially at rest (so $\gamma \to$ as $r \to \infty$). From $dt = \gamma d\tau$, our assertion is verified. Hence $k = 1$ for $r \to \infty$ and since k is constant, we can conclude that $k = 1$ everywhere along the radial path. Thus,

$$\frac{dt}{d\tau} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1}$$

(e) To compute $v = dr/dt$, we start from

$$g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -c^2$$

Since $d\theta/d\tau = d\phi/d\tau = 0$ for a radial path, we have

$$g_{00} c^2 \left(\frac{dt}{d\tau}\right)^2 + g_{rr} \left(\frac{dr}{d\tau}\right)^2 = -c^2$$

But for the Schwarzschild metric

$$g_{00} = -\left(1 - \frac{2GM}{c^2 r}\right) \quad , \quad g_{rr} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1}$$

Therefore,

$$-\left(1 - \frac{2GM}{c^2 r}\right) \left(\frac{dt}{d\tau}\right)^2 + \frac{1}{c^2} \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \left(\frac{dr}{d\tau}\right)^2 = 1$$

Using $dr/d\tau = (dr/dt)(dt/d\tau)$ we get

$$\left(\frac{dt}{d\tau}\right)^2 \left[1 - \frac{2GM}{c^2 r} - \frac{1}{c^2} \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \left(\frac{dr}{dt}\right)^2\right] = 1$$

Inserting

$$\frac{dt}{d\tau} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1}$$

from part(d) we obtain

$$1 - \frac{2GM}{c^2 r} - \frac{1}{c^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1}\left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2GM}{c^2 r}\right)^2$$

Divide both sides by $1 - \frac{2GM}{c^2 r}$ to get

$$1 - \frac{1}{c^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-2}\left(\frac{dr}{dt}\right)^2 = 1 - \frac{2GM}{c^2 r}$$

Therefore,

$$\left(\frac{dr}{dt}\right)^2 = \frac{2GM}{r}\left(1 - \frac{2GM}{c^2 r}\right)^2$$

Taking the square root

$$\frac{dr}{dt} = -\left(1 - \frac{2GM}{c^2 r}\right)\left(\frac{2GM}{r}\right)^{1/2}$$

We have taken the negative root since $dr/dt < 0$ for radial motion toward the origin. Inverting this equation yields

$$\frac{dt}{dr} = -\left(1 - \frac{2GM}{c^2 r}\right)^{-1}\left(\frac{r}{2GM}\right)^{1/2}$$

or

$$t = -\left(\frac{1}{2GM}\right)^{1/2}\int_{r_0}^{r_s} \frac{r^{1/2}\,dr}{1 - \frac{2GM}{c^2 r}}$$

where t is the elapsed coordinate time it takes the particle to move from r_0 to r_s. If $r_s = 2GM/c^2$ is the Schwarzschild radius, then

$$t = -\left(\frac{1}{2GM}\right)^{1/2}\int_{r_0}^{r_s} \frac{r^{3/2}\,dr}{r - r_s}$$

Note that $r_0 > r_s$, so t is positive. But, due to the singularity in the integrand at $r = r_s$, the integral diverges logarithmically, i.e., $t = \infty$, meaning that it takes an infinite coordinate time for the particle to reach the Schwarzschild radius, $r = r_s$.

(f) Using the results of parts (a) and (b),

$$\frac{dR}{dT} = \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\frac{dr}{dt}$$

Thus, from the result of part (e),

$$\frac{dR}{dT} = -\left(\frac{r}{2GM}\right)^{1/2} = -c\left(\frac{r_s}{r}\right)^{1/2}$$

Indeed, $|dR/dT| \to c$ as $r \to r_s$.

EP #46

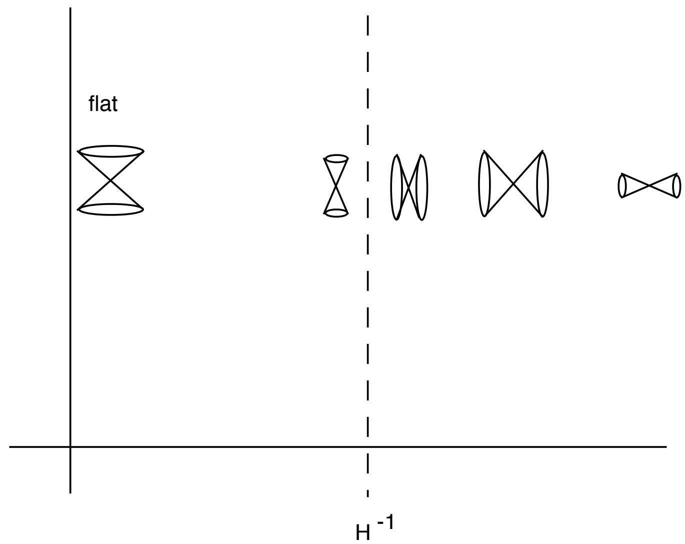
(a) The light cone structure in the (r,t) plane is determined by

$$\frac{dt}{dr} = \pm \left(1 - H^2 r^2\right)^{-1}$$

which has these properties

$$\frac{dt}{dr} \to \pm\infty \text{ for } r \to H^{-1} \text{ and } \frac{dt}{dr} \to 0 \text{ for } r \to \infty$$

This looks like



(b) (1) We have spherical symmetry, so pick $\theta = \pi/2$ plane and $d\theta = 0$. No time-dependence, so let $t = constant$, $dt = 0$.

(2) We pick

$$ds^2_{3D} = d\rho^2 + \rho^2 d\varphi^2 + dz^2$$

Choose $\varphi = \phi$, $\rho = r$, $z = z(r)$ so that

$$ds^2_{3D} = dr^2 + r^2 d\phi^2 + (z')^2 dr^2 = (1 + (z')^2) dr^2 + r^2 d\phi^2$$

We thus require

$$1 + (z')^2 = \frac{1}{1 - H^2 r^2}$$

This gives

$$(z')^2 = \frac{1}{1 - H^2 r^2} - 1 \to z' = \frac{rH}{(1 - H^2 r^2)^{1/2}}$$

Integrating we get

$$z = \int \frac{rHdr}{(1 - H^2r^2)^{1/2}} = \frac{1}{H}\left(1 - H^2r^2\right)^{1/2}$$

Thus $z = 1/H$ at $r = 0$. This is a sphere!

EP #47

Let

$$x^0 = t\ ,\ x^1 = r\ ,\ x^2 = \theta\ ,\ x^3 = \phi$$

Then

$$g_{00} = -c^2\left(1 - \frac{k}{r}\right)\ ,\ g_{11} = -\left(1 - \frac{k}{r}\right)^{-1}\ ,\ g_{22} = r^2$$
$$g_{33} = r^2\sin^2\theta\ ,\ g_{\mu\nu}0 \text{ for } \mu \neq \nu$$

where $k = 2GM/c^2$. For convenience we choose coordinates such that the Earth(E), the Sun(S) and Jupiter(J) are in the $\theta = \pi/2$ plane. The equations of the geodesic along which the radio pulse propagates are

$$\frac{d}{ds}\left(g_{\mu\nu}\frac{dx^\nu}{ds}\right) = \frac{1}{2}g_{\alpha\beta,\mu}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}$$

where s is some orbital parameters. These equations give two first integrals

$$g_{3\nu}\frac{dx^\nu}{ds} = g_{33}\frac{dx^3}{ds} = r^2\sin^2\theta\frac{d\phi}{ds} = r^2\frac{d\phi}{ds} = \text{constant}$$
$$g_{0\nu}\frac{dx^\nu}{ds} = g_{00}\frac{dx^0}{ds} = -c^2\left(1 - \frac{k}{r}\right)\frac{dt}{ds} = \text{constant}$$

since

$$g_{\alpha\beta,3} = \frac{\partial g_{\alpha\beta}}{\partial\phi} = 0 \quad , \quad g_{\alpha\beta,0} = \frac{\partial g_{\alpha\beta}}{\partial t} = 0$$

Taking the ratio of the integrals gives

$$\frac{r^2}{1 - \frac{k}{r}}\frac{d\phi}{dt} = F = \text{constant}$$

The time taken for one round trip of the pulse if the sun did not exert any effect on the speed of light is

$$T_0 = \frac{2}{c}\left(\sqrt{r_1^2 - r_0^2} + \sqrt{r_2^2 - r_0^2}\right)$$

while the actual time is

$$T = 2\int_0^{\sqrt{r_1^2 - r_0^2}}\frac{d\xi}{c'} + 2\int_0^{\sqrt{r_2^2 - r_0^2}}\frac{d\xi}{c'}$$

where$\xi = \sqrt{r^2 - r_0^2}$. As $d\xi = rdr/\xi$, we get

$$
\begin{aligned}
T &\approx \frac{2}{c}\left[\int_{r_0}^{r_1}\left(1+\frac{k}{r}\right)\frac{rdr}{\sqrt{r^2-r_0^2}} + \int_{r_0}^{r_2}\left(1+\frac{k}{r}\right)\frac{rdr}{\sqrt{r^2-r_0^2}}\right] \\
&\approx \frac{2}{c}\left[\int_{r_0}^{r_1}\left(1+\frac{k}{r}\right)\frac{rdr}{\sqrt{r^2-r_0^2}} + \int_{r_0}^{r_2}\left(1+\frac{k}{r}\right)\frac{rdr}{\sqrt{r^2-r_0^2}}\right] \\
&= \frac{2}{c}\left[\sqrt{r_1^2-r_0^2} + \sqrt{r_2^2-r_0^2} + k\ln\left(\frac{r_1+\sqrt{r_1^2-r_0^2}}{r_0}\right) + k\ln\left(\frac{r_2+\sqrt{r_2^2-r_0^2}}{r_0}\right)\right] \\
&= \frac{2}{c}\left[\sqrt{r_1^2-r_0^2} + \sqrt{r_2^2-r_0^2} + k\ln\left(\frac{4r_1r_2}{r_0^2}\right)\right]
\end{aligned}
$$

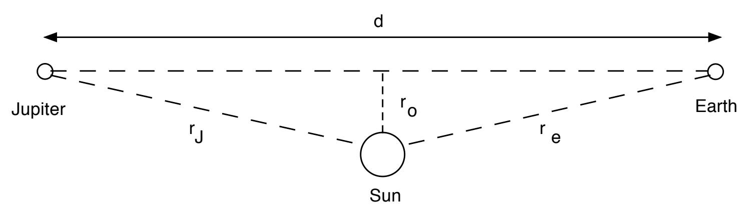
Hence, the delay time is

$$\Delta T = T - T_0 \approx \frac{4GM}{c^3}\ln\left(\frac{4r_1r_2}{r_0^2}\right)$$

Taking $r_0 \approx$ the Sun's radius and using the given data we find

$$\Delta T = 2.7 \times 10^{-4}\, sec$$

Alternative solution: Consider the figure below



If r_0 is small compared to r_e, r_J, then we can say $d \approx r_e + r_J$. We want to find the gravitational correction to the time it takes for the signal to travel (to lowest order in G).

We can parameterize the worldline of the photon via σ - the distance the photon has traveled. Then we have

$$
\begin{aligned}
x(\sigma) &= r_e - \sigma \to dx = -d\sigma \\
y(\sigma) &= r_0 \to dy = 0 \\
z(\sigma) &= 0 \to dz = 0
\end{aligned}
$$

In SR, the metric is (w/o gravitation)

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

For a photon, $ds^2 = 0$, so we have

$$\frac{dt}{d\sigma} = \frac{1}{c}$$

so then

$$t = \int_0^d \frac{1}{c} d\sigma = \frac{d}{c} - \frac{r_e + r_J}{c}$$

is the time it takes for half the journey.

For weak fields we have

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 + \frac{2GM}{c^2 r}\right)(dx^2 + dy^2 + dz^2)$$

and again for photons $ds^2 = 0$ so that

$$\left(\frac{dt}{d\sigma}\right)^2 = \frac{1}{c^2} \frac{1 + \frac{2GM}{c^2 r}}{1 - \frac{2GM}{c^2 r}}$$

or

$$\frac{dt}{d\sigma} = \frac{1}{c}\left(\frac{1 + \frac{2GM}{c^2 r}}{1 - \frac{2GM}{c^2 r}}\right)^{1/2} \approx \frac{1}{c}\left(\left(1 + \frac{2GM}{c^2 r}\right)\left(1 + \frac{2GM}{c^2 r}\right)\right)^{1/2} \approx \frac{1}{c}\left(1 + \frac{2GM}{c^2 r}\right)$$

Then t for half the journey is

$$t = \int_0^d \frac{1}{c}\left(1 + \frac{2GM}{c^2 r}\right) d\sigma = \frac{d}{c} + \int_0^d \frac{2GM}{c^3\sqrt{x^2 + r_0^2}} d\sigma$$

But $dx = -d\sigma$, $x(0) = r_e$, $x(d) = -r_J$, so

$$\begin{aligned} t &= \frac{d}{c} - \int_{r_e}^{-r_J} \frac{2GM}{c^3\sqrt{x^2 + r_0^2}} dx \\ &= \frac{d}{c} + \frac{2GM}{c^3} \int_{-r_J}^{r_e} \frac{dx}{\sqrt{x^2 + r_0^2}} \\ &= \frac{d}{c} + \frac{2GM}{c^3} \sinh^{-1}\left(\frac{x}{r_0}\right)\Big|_{-r_J}^{r_e} \\ &= \frac{d}{c} + \frac{2GM}{c^3}\left[\sinh^{-1}\left(\frac{r_e}{r_0}\right) + \sinh^{-1}\left(\frac{r_J}{r_0}\right)\right] \end{aligned}$$

Thus, the correction for both halves is

$$\Delta t = \frac{4GM}{c^3}\left[\sinh^{-1}\left(\frac{r_e}{r_0}\right) + \sinh^{-1}\left(\frac{r_J}{r_0}\right)\right]$$

Plugging in numbers we get

$$\Delta T = 2.73 \times 10^{-4}\, sec$$

EP #48

(a) The spinning motion of the particle is given by the equation

$$\frac{dS_\mu}{d\tau} = \Gamma^\lambda_{\mu\nu} S_\lambda \frac{dx^\nu}{d\tau}$$

where S_μ is the spin vector. The left-hand side is the time rate of change of the spin vector and the right-hand side gives the effect of the gravitational force, the whole equation describing the precession of the spinning body in free fall. If gravitational forces are absent, $\Gamma^\lambda_{\mu\nu} = 0$, giving $dS_\mu/d\tau = 0$. Hence a particle under not force will have a constant spin.

(b) Let $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$. As the particle is at rest at $r < R$, $H = constant$ and

$$\frac{dx^j}{d\tau} = 0 \text{ for } j = 1, 2, 3$$

the equation of motion reduces to

$$\frac{dS_\mu}{d\tau} = \Gamma^\lambda_{\mu 0} S_\lambda \frac{dx^0}{d\tau}$$

or

$$\begin{aligned} \frac{dS_i}{dt} &= \Gamma^\lambda_{i0} S_\lambda \\ &= \frac{1}{2} g^{\lambda\rho} (\partial_i g_{\rho 0} + \partial_0 g_{\rho i} - \partial_\rho g_{i0}) S_\lambda \\ &= \frac{1}{2} S^\rho (\partial_i g_{\rho 0} - \partial_\rho g_{i0}) \quad (i = 1, 2, 3) \end{aligned}$$

as $g_{\mu\nu}$ does not depend on x^0 explicitly. Note that $S_0 = 0$ as the particle is at rest. This can be written in 3-dimensional form as

$$\frac{d\vec{S}}{dt} = \frac{1}{2} \vec{S} \times (\nabla \times \vec{\xi})$$

where

$$\xi_i = \frac{g_{i0}}{\sqrt{g_{ii}}}$$

(c) In the given metric if we set $\phi' = \phi - \Omega t$, then the space and time parts of the line element separate out, showing that the spherical shell rotates with angular frequency Ω about the $z-$axis. Th equation of motion in 3D form shows that the spin vector $\vec{S}$ precesses with angular velocity

$$\vec{\Omega}' = -\frac{1}{2} \nabla \times \vec{\xi}$$

As

$$g_{30} = -\frac{\Omega r^2 \sin^2\theta}{H} \quad , \quad g_{33} = -\frac{r^2 \sin^2\theta}{H}$$

we have

$$\xi_3 = -\frac{\Omega r \sin\theta}{\sqrt{H}} \approx -\Omega r \sin\theta$$

since

$$H^{-1/2} = \left(1 - \frac{2GM}{Rc^2}\right)^{-1/2} \approx 1 + \frac{GM}{Rc^2} \approx 1$$

Hence we can write

$$\vec{\xi} = \vec{r} \times \vec{\Omega}$$

Then

$$\vec{\Omega}' = -\frac{1}{2}\nabla \times (\vec{r} \times \vec{\Omega}) = \vec{\Omega}$$

Hence, the precession angular velocity is in the $z-$direction and has magnitude

$$|\vec{\Omega}'| = \Omega = \frac{4GM\omega}{3Rc^2}$$

For a rough estimate take the Earth as a spherical shell of $M = 6.0 \times 10^{27}\, g$, $R = 6.4 \times 10^8\, cm$. Then $\Omega' = 6.74 \times 10^{-14}\, rad/sec = 1.39 \times 10{-}8\, second-of-arc/sec$.

Actually, the coordinate transformation

$$t' = t \quad , \quad r' = r \quad , \quad \theta' = \theta \quad , \quad \phi' = \phi - \Omega t$$

would give rise to an Euclidean space-time in which $\vec{S}$ is constant. Then in the original frame $\vec{S}$ will precess with angular velocity Ω about the $z-$axis.

EP #49

(a) It is easiest to work backwards:

$$\begin{aligned} du' &= \frac{1}{4M}\left(\frac{r}{2M} - 1\right)^{1/2} e^{(r+t)/4M}(dr + dt) + \frac{1}{4M}\left(\frac{r}{2M} - 1\right)^{-1/2} e^{(r+t)/4M} dr \\ &= \frac{u'}{4M}(dr + dt) + \frac{u'}{4M}\left(\frac{r}{2M} - 1\right)^{-1} dr = \frac{u'}{4M}\left[\frac{dr}{1 - \frac{2M}{r}} + dt\right] \end{aligned}$$

$$\begin{aligned} dv' &= -\frac{1}{4M}\left(\frac{r}{2M} - 1\right)^{1/2} e^{(r-t)/4M}(dr - dt) - \frac{1}{4M}\left(\frac{r}{2M} - 1\right)^{-1/2} e^{(r-t)/4M} dr \\ &= \frac{v'}{4M}(dr - dt) + \frac{v'}{4M}\left(\frac{r}{2M} - 1\right)^{-1} dr = \frac{v'}{4M}\left[\frac{dr}{1 - \frac{2M}{r}} - dt\right] \end{aligned}$$

so

$$du'dv' = \frac{u'v'}{16M^2}\left[\frac{dr^2}{\left(1 - \frac{2M}{r}\right)^2} - dt^2\right]$$

Now $u'v' = -(r/2M - 1)e^{r/2M}$, so

$$\begin{aligned}
-\frac{32M^3}{r}e^{-r/2M}du'dv' &= \frac{2M}{r}\left(\frac{r}{2M}-1\right)\left[\frac{dr^2}{(1-\frac{2M}{r})^2} - dt^2\right] \\
&= -\left(1-\frac{r}{2M}\right)dt^2 + \left(1-\frac{r}{2M}\right)^{-1}dr^2
\end{aligned}$$

Thus, the two metrics are the same.

(b) There is a singularity at $r = 0$, which corresponds to $u'v' = 1$, where the $du'dv'$ component of the metric blows up. Although it's not properly speaking a place in spacetime, we might be worried about using this metric for $r \to \infty$, as the determinant goes to zero there, so the inverse metric is blowing up.

(c) Since the radial light rays in these coordinates are $u' =$ constant and $v' =$ constant, to be able to send signals to $u' = u_0$, $v' = v_0$, an event must lie at $u' \leq u_0$, $v' \leq v_0$. Note that this is a necessary, but not a sufficient condition; some events that satisfy this condition will still not be able to send signals to a given event if their angular separation is large enough.

(d) By symmetry, it must lie at $u' \geq u_0$, $v' \geq v_0$.

(e) From the coordinate transformation, we see that $u'v'$ is a function of r only. Thus,

$$u'v' = -\left(\frac{r}{2M}-1\right)e^{r/2M} = -2e^3$$

That is, the orbit follows a hyperbola in the $u'v'$ plane. Assuming we are initially in the region $u' > 0$, $v' < 0$ (as opposed to $u' < 0$, $v' > 0$), we will always stay in this region.

(f) The region $v' > 0$ cannot send signals to this observer. The region $u' < 0$ cannot receive signals from this observer.

EP #50

First approach: Using equations of motion and numerical solution of ODEs.

We have the following orbital (motion in the plane $\theta = \pi/2$) equations from the text (7.47) and (7.52):

$$\begin{aligned}
&\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2}\left(1-\frac{r_s}{r}\right) - \frac{GMm}{r} = E \\
&J = mr^2\dot{\phi}
\end{aligned}$$

In addition, from the Schwarzschild Lagrangian we can write the r Lagrange equation as

$$\left(1-\frac{r_s}{r}\right)^{-1}\ddot{r} + \frac{r_s c^2}{2r^2}\dot{t}^2 - \left(1-\frac{r_s}{r}\right)^{-2}\frac{r_s}{2r^2}\dot{r}^2 - r\dot{\phi}^2 = 0$$

The energy equation can be re-written using $r_s = 2GM/c^2$ as

$$\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2}\left(1 - \frac{r_s}{r}\right) - \frac{1}{2}mc^2\frac{r_s}{r} = E$$

or

$$\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2}\left(1 - \frac{2GM}{c^2 r}\right) - \frac{GMm}{r} = E$$

Using the equation for $\ddot{r}$ we can consider circular motion, where $\ddot{r} = \dot{r} = 0$. This gives

$$\frac{r_s c^2}{2r^2}\dot{t}^2 - r\dot{\phi}^2 = 0$$

or

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{GM}{r^3}$$

Thus the change in coordinate time t for one revolution is

$$\Delta t = 2\pi\left(\frac{r^3}{GM}\right)^{1/2}$$

which is the same as the Newtonian expression, i.e., Kepler's third law is unchanged.

We also have

$$\frac{mr_s c^2}{2R^2}\dot{t}^2 = mR\dot{\phi}^2 = \frac{J^2}{mR^3}$$

or

$$\dot{t} = \sqrt{\frac{J^2}{m^2 GMR}} = \frac{dt}{d\tau} \approx 1$$

This implies that

$$d\tau = \sqrt{\frac{m^2 GMR}{J^2}}dt \approx dt$$

We then have the two ODEs

$$\frac{d\phi}{dt} = \sqrt{\frac{GM}{r^3}}$$

and

$$\frac{dr}{dt} = \sqrt{\frac{2E}{m} + \frac{2GM}{r} - \frac{J^2}{m^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right)}$$

To determine the initial conditions (initially start in circular orbit) we use the effective potential energy approach. We have

$$\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2}\left(1 - \frac{2GM}{c^2 r}\right) - \frac{GMm}{r} = E$$

which we can write as

$$\frac{1}{2}m\dot{r}^2 + V_{eff}(r) = E$$

where

$$V_{eff}(r) = \frac{J^2}{2mr^2}\left(1 - \frac{2GM}{c^2 r}\right) - \frac{GMm}{r}$$

The condition for circular motion is an extremum of the effective potential energy. Therefore we calculate

$$\frac{dV_{eff}(r)}{dr} = 0 = -\frac{J^2}{mr^3}\left(1 - \frac{2GM}{c^2 r}\right) + \frac{J^2}{2mr^2}\left(1 - \frac{2GM}{c^2 r}\right) - \frac{GMm}{r}$$

$$J = m\sqrt{\frac{r_s R c^2}{2\left(1 - \frac{3r_s}{R}\right)}}$$

and

$$E = -\frac{GMm}{R} + \frac{GMm}{2R}\frac{\left(1 - \frac{r_s}{R}\right)}{\left(1 - \frac{3r_s}{R}\right)}$$

Re-writing the ODEs and re-inserting r_s we have

$$\frac{d\phi}{dt} = \sqrt{\frac{r_s c^2}{2r^3}}$$

$$\frac{dr}{dt} = \sqrt{\frac{2E}{m} + \frac{r_s c^2}{r} - \frac{J^2}{m^2 r^2}\left(1 - \frac{r_s}{r}\right)}$$

with initial conditions

$$\phi(0) = 0 \quad \text{and} \quad r(0) = R$$

and

$$J = m\sqrt{\frac{r_s R c^2}{2\left(1 - \frac{3r_s}{R}\right)}} \quad \text{and} \quad E = -\frac{mc^2 r_s}{2R}\left(1 - \frac{1}{2}\frac{\left(1 - \frac{r_s}{R}\right)}{\left(1 - \frac{3r_s}{R}\right)}\right)$$

so that

$$\frac{d\phi}{dt} = c\sqrt{\frac{r_s}{2r^3}}$$

$$\frac{dr}{dt} = c\sqrt{-\frac{r_s}{2R}\left(1 - \frac{1}{2}\frac{\left(1 - \frac{r_s}{R}\right)}{\left(1 - \frac{3r_s}{R}\right)}\right) + \frac{r_s}{r} - \frac{r_s R}{2r^2}\left(1 - \frac{r_s}{r}\right)\left(1 - \frac{3r_s}{r}\right)^{-1}}$$

with initial conditions

$$\phi(0) = 0 \quad \text{and} \quad r(0) = R$$

MATLAB Code:

```
phi=0;
n=4;
GM=1.327*10^20;
c=3*10^8;
rs=2*GM/c^2;
R=n*rs;
r=R;
m=50000;
dt=0.000001;
x=zeros(1,m+1);y=zeros(1,m+1);
x(1)=r*cos(phi);y(1)=r*sin(phi);
for k=1:m
phi=phi+c*sqrt(rs/(2*r^3))*dt;
r=r+c*sqrt(-(rs/(2*R))*(1-0.5*(1-rs/R)/(1-3*rs/R)) ...
                              +rs/r-((rs*R)/(2*r^2))*(1-rs/R)/(1-3*rs/R))*dt;
x(k+1)=real(r*cos(phi));y(k+1)=real(r*sin(phi));
end
plot(x,y)
axis('square');
```

Second approach: Using effective potential energy. We have

$$V_{eff}(r) = \frac{J^2}{2mr^2}\left(1 - \frac{2GM}{c^2 r}\right) - \frac{GMm}{r}$$

The condition for circular motion is an extremum of the effective potential energy. Therefore we calculate

$$\frac{dV_{eff}(r)}{dr} = 0 = -\frac{J^2}{mr^3}\left(1 - \frac{2GM}{c^2 r}\right) + \frac{J^2}{2mr^2}\left(1 - \frac{2GM}{c^2 r}\right) - \frac{GMm}{r}$$

$$J = m\sqrt{\frac{r_s R c^2}{2\left(1 - \frac{3r_s}{R}\right)}}$$

and

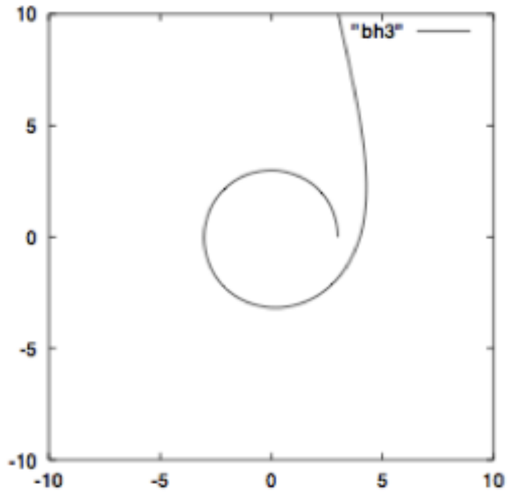
$$E = -\frac{GMm}{R} + \frac{GMm}{2R}\frac{\left(1 - \frac{r_s}{R}\right)}{\left(1 - \frac{3r_s}{R}\right)}$$

The solution I have goes as follows: We use the values for a circular orbit:

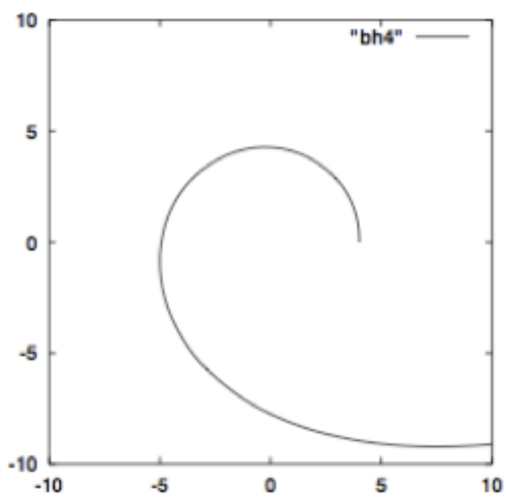
$$\tilde{L}^2 = \frac{Mr^2}{r - 3M} \quad \text{and} \quad \tilde{E}^2 = \frac{1}{r}\frac{(r-2M)^2}{r-3M} + \epsilon$$

The term ϵ is a small perturbation.

For $r = 3M$, the particle must have zero mass. It escapes with impact parameter $b = L/E = 3\sqrt{3}M$ as shown below

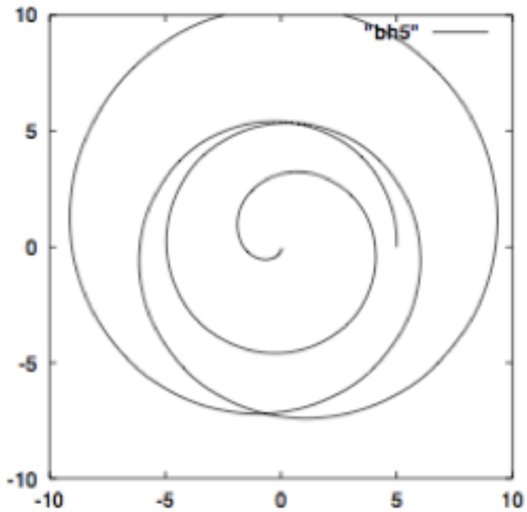


For $r = 4M$, the particle has reduced energy $\tilde{E}$ slightly greater than 1 so it barely escapes to infinity. It has $\tilde{L} = 4$ so its impact parameter will be $b = 4M/u$ for some nonrelativistic velocity $u \ll 1$ as shown below.

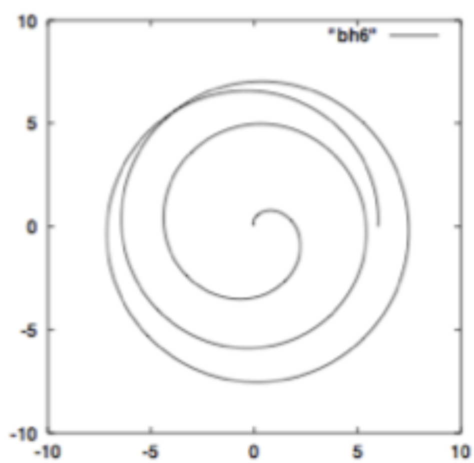


For $r = 5M$, the particle has reduced energy $\tilde{E} = \sqrt{9/10} < 1$ so it cannot escape: it will reach a maximum radius determined by $\tilde{E}^2 = \tilde{V}^2$, that is, $r = 10M$ (it is a cubic equation, but we can divide through y our known solution $r = 5M$). Upon reflection by the effective potential barrier, it will return to

$r = 5M$ and pass over it (having slightly more energy than the unsatble orbit there), falling into the black hole as shown below.



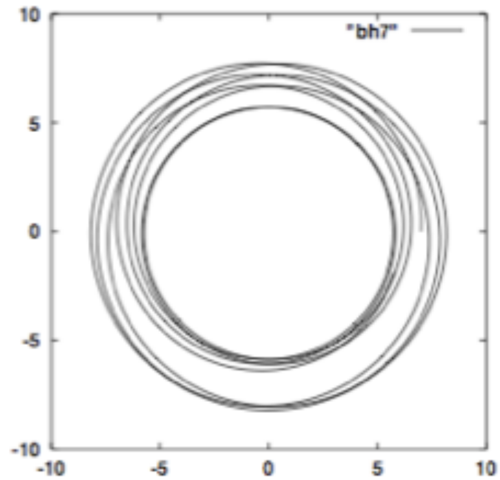
For $r = 6M$, the orbit is marginally stable, i.e., the effective potential has a point of inflection. Any perturbation will move outwards slightly before falling into the black hole as shown below.



For $r = 7M$, the orbit is stable, so the particle will remain in a stable orbit. The ratio of the oscillation periods in the ϕ and r directions is

$$\frac{T_r}{T_\phi} = \frac{1}{\sqrt{1 - \frac{6M}{r}}} = \sqrt{7}$$

so the particle will go around about two and a half times between each closest approach as shown below.



EP #51

Choosing $\theta = \pi/2$, the conserved energy and angular momentum in the metric are $E = \dot{t}$ and $L = r^2\dot{\phi}$. For null geodesics,

$$0 = -E^2 + (1 - \lambda r^2)^{-1}\dot{r}^2 + \frac{L^2}{r^2}$$

that is,

$$\dot{r}^2 = \frac{E^2 - L^2}{r^2}(1 - \lambda r^2)$$

Now,

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{\dot{r}^2}{\dot{\phi}^2} = \frac{E^2 - L^2}{r^2}(1 - \lambda r^2)\frac{r^4}{L^2} = r^2(1 - \lambda r^2)(\mu r^2 - 1)$$

where $\mu = E^2/L^2$. Thus,

$$\phi = -\int \frac{dr}{r(1 - \lambda r^2)^{1/2}(\mu r^2 - 1)^{1/2}} = \frac{1}{2}\int \frac{dv}{(v - \lambda)^{1/2}(\mu - v)^{1/2}} = \frac{1}{2}\int \frac{dv'}{(\alpha^2 - v'^2)^{1/2}}$$

where we have set $r^2 = 1/v$ and then $v = v' + (\lambda + \mu)/2$ and $\alpha = (\lambda - \mu)/2$. Performing the integral gives

$$\phi = \frac{1}{2}\sin^{-1}\left(\frac{v'}{\alpha}\right) + \phi_0$$

which can be written as

$$\frac{1}{r^2} = \frac{\lambda + \mu}{2} + \frac{\lambda - \mu}{2} \sin(2\phi - \phi_0)$$

Let us take $\phi_0 = 0$, and move to Cartesian coordinates $x = r\cos\phi$, $y = r\sin\phi$. Then

$$1 = \frac{\lambda + \mu}{2}(x^2 + y^2) + (\lambda - \mu)xy$$

which is the equation for an ellipse.

EP #52

(a) Zeros of $1 - 2M/r + Q^2/r^2$ occur at

$$r_\pm = M \pm \sqrt{M^2 - Q^2}$$

If $Q > M$, there are no real roots, so the metric components are bounded until we reach $r = 0$.

(b) If $Q < M$, there are zeros as given above. At these values or r, g_{rr} diverges, so there is a singularity in the metric in these coordinates.

(c) As witht he Schwarzschild case, we look for a coordinate adapted to the causal structure. We define the so-called tortoise coordinate r_* by

$$dr_* = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr$$

which can be solved to obtain

$$r_* = r + \frac{r_+^2}{r_+ - r_-} \ln\left(\frac{r}{r_+} - 1\right) - \frac{r_-^2}{r_+ - r_-} \ln\left(\frac{r}{r_-} - 1\right)$$

If we then define $u = t + r_*$, ingoing light rays will be given by $u = \text{constant}$, so this coordinate is adapted to the causal structure in the desired way, and

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) du^2 + 2du dr + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

so the metric is regular at $r = r_+$ in these coordinates (in fact, it is also regular at $r = r_-$).

EP #53

(a) For any particle we must have $v^a v_a = 1 > 0$. Thus,

$$1 = \left(1 - \frac{r_s}{r}\right)\dot{t}^2 - \left(1 - \frac{r_s}{r}\right)^{-1} \dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$$

so if $r < r_s$, the coefficient of $\dot{t}^2$ is negative and the coefficient of $\dot{r}^2$ is positive. This means that $\dot{r}$ must be nonzero so that the RHS is positive and equal to 1.

Thus, $\dot{r}$ can never go from negative to positive because it would have to pass through 0 first, which is not allowed by the above argument.

Moreover, $\dot{r}^2$ can never decrease because it would throw off the delicate balance. Further, as $r \to 0$, $\dot{r}^2$ must increase because the coefficient of $\dot{t}^2$ grows more negative while the coefficient of $\dot{r}^2$ grows less positive. $\dot{r}^2$ must increase to fill this gap.

Finally, if $\dot{\theta}^2$, $\dot{\phi}^2$ are positive, then $\dot{r}$ must simply be bigger to compensate for the more negative contributions.

Thus, we see that: accelerating in the $+\hat{r}$ direction is not allow; accelerating in the $\hat{\theta}$, $\hat{\phi}$ directions only makes $\dot{r}$ increase faster.

Therefore, we want no acceleration, which implies that the particle follows a geodesic.

Now, the energy is conserved and

$$E = \left(1 - \frac{r_s}{r}\right)\dot{t}$$

so we can rewrite

$$1 = \frac{E^2}{1 - \frac{r_s}{r}} - \frac{\dot{r}^2}{1 - \frac{r_s}{r}}$$

or

$$-\dot{r} = \sqrt{\frac{r_s}{r} - 1 + E^2}$$

This gives

$$\frac{d\tau}{dr} = \frac{-1}{\sqrt{\frac{r_s}{r} - 1 + E^2}}$$

and

$$\Delta\tau = \int_{r_s}^{0} \frac{-dr}{\sqrt{\frac{r_s}{r} - 1 + E^2}} = \int_{0}^{r_s} \frac{dr}{\sqrt{\frac{r_s}{r}r - 1 + E^2}}$$

which decreases for larger E. So we want to have $E = 0$ and then

$$\Delta\tau_{max} = \int_0^{r_s} \frac{dr}{\sqrt{\frac{r_s}{r} - 1}} = \frac{\pi}{2} r_s$$

(b) If we start from rest at $r = r_s$ and we move in a radial motion with $dt = d\theta = d\phi = 0$, then

$$d\tau = \frac{-dr}{\sqrt{\frac{r_s}{r}r - 1}}$$

where negative sign is to keep $d\tau$ positive. We then get as before

$$\Delta\tau = \int_0^{r_s} \frac{dr}{\sqrt{\frac{r_s}{r} - 1}} = \frac{\pi}{2} r_s$$

Thus apparently this situation attains the maximum. We should start at r_s at rest because

$$\frac{dr}{d\tau} = \sqrt{-\left(1 - \frac{r_s}{r}\right)} = 0 \text{ at } r = r_s$$

(c) Since starting from rest at $r = r_s$ attains the maximum $\Delta\tau$, all other initial conditions must yield equal or smaller values. Thus, the astronaut should slow down to rest before she reaches $r = r_s$.

EP #54

The key to this problem is to recognize that in the ejection event, energy-momentum is conserved, but rest mass in not necessarily conserved. Let m and $\vec{u}$ be the rest mass and 4-velocity of the rocket hovering at radius R. Thus,

$$u^\alpha = \left[\left(1 - \frac{2M}{R}\right)^{-1}, 0, 0, 0\right]$$

Let m_{esc}, $\vec{u}_{esc}$ and m_{ej}, $\vec{u}_{ej}$ be the corresponding quantities for the escaping and ejected fragment. The minimum 4-velocity for escape corresponds to an orbit with $E = 1$ and $L = 0$ and is

$$u^\alpha_{esc} = \left[\left(1 - \frac{2M}{R}\right)^{-1}, +\left(\frac{2M}{R}\right)^{1/2}, 0, 0\right]$$

Derivation: in order for the observer to escape with the maximum rest mass, they will want to escape radially. In this case, we must have ($d\theta = d\phi = 0$)

$$\begin{aligned}
ds^2 &= \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2 \\
1 &= \left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 \\
1 &= \frac{E^2}{\left(1 - \frac{2M}{r}\right)} - \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 \\
\left(\frac{dr}{d\tau}\right)^2 &= E^2 - \left(1 - \frac{2M}{r}\right)
\end{aligned}$$

For a particle to escape radially, we must have $E = 1$, $L = 0$. Therefore

$$\frac{dr}{d\tau} = \sqrt{1 - \left(1 - \frac{2M}{r}\right)} = \sqrt{\frac{2M}{r}} \quad , \quad \frac{dt}{d\tau} = \frac{E}{\left(1 - \frac{2M}{r}\right)} = \frac{1}{\left(1 - \frac{2M}{r}\right)}$$

Continuing, we have from conservation of three-momentum

$$0 = m_{esc}\sqrt{\frac{2M}{r}} + m_{ej}u^r_{ej}$$

or

$$u^r_{ej} = -\left(\frac{m_{esc}}{m_{ej}}\right)\sqrt{\frac{2M}{r}}$$

where we have assumed that the fragment was ejected in the radial direction. Then, imposing the condition $\vec{u}\cdot\vec{u} = -1$, the 4-velocity of the ejected fragment has the time component

$$u^t_{ej} = \left(1 - \frac{2M}{r}\right)^{-1}\left[1 - \frac{2M}{R}\left(1 - \left(\frac{m_{esc}}{m_{ej}}\right)^2\right)\right]^{1/2}$$

Conservation of energy gives

$$mu^t = m_{esc}u^t_{esc} + m_{ej}u^t_{ej}$$

which implies that

$$\left(1 - \frac{2M}{r}\right)^{1/2} = \left(\frac{m_{esc}}{m}\right) + \left(\frac{m_{ej}}{m}\right)\left[1 - \frac{2M}{R}\left(1 - \left(\frac{m_{esc}}{m_{ej}}\right)^2\right)\right]^{1/2}$$

The largest fragment that can escape is the largest value of m_{esc}/m that can satisfy this relation as m_{ej}/m varies from 0 to 1. Plotting the function for a few cases shows that m_{esc}/m is maximized when $m_{ej} = 0$, i.e., all the rest mass is turned into energy. Then

$$\frac{m_{esc}}{m} = \sqrt{\frac{R - 2M}{R + 2M}}$$

This vanishes for $R = 2M$.

EP #55

(a) The total mass-energy of an isolated system is conserved, so it is not possible to radiate monopole gravitational radiation. Also, since the total momentum of an isolated system is conserved, which means that the second time derivative of the mass dipole moment $\sum m\vec{r}$ is zero, it is not possible to radiate dipole gravitational radiation. The lowest multipole gravitational radiation is quadrupole.

(b) For a system of two stars in a circular orbit around their common center of mass, the rate of energy loss by radiation of gravitational waves is

$$-\frac{dE}{dt} = \frac{32G^4}{5c^5r^5}(m_1m_2)^2(m_1 + m_2)$$

where r is their mutual distance, which is constant for motion in a circular orbit. With $m_1 = m_2 = m$ and the data given we have

$$-\frac{dE}{dt} = \frac{64G^4}{5c^5r^5}m^5 = 1.57 \times 10^{31}\, erg/sec$$

With

$$E = -\frac{Gm^2}{2r}$$

$$\frac{dr}{dt} = \frac{2r^2}{Gm^2}\frac{dE}{dt} = -\frac{128Gm^3}{5c^5r^3}$$

Since dr/dt is the rate at which the two stars approach each other, the time taken for a complete collapse of the orbit is

$$\tau = \int_r^0 \frac{dr}{\left(\frac{dr}{dt}\right)} = -\frac{5c^5}{128G^3m^3}\int_r^0 r^3 dr$$

$$= \frac{5c^5}{512G^3m^3}r^4 = 2.4 \times 10^{15}\, sec.$$

EP #56

Take the center of mass to be the origin of a Cartesian reference frame whose $z-$axis is oriented along the spring. If the mass at $z = +L/2$ is displaced by an amount δz, and the mass at $z = -L/2$ by $-\delta z$, then the center of mass is unchanged and the magnitude of the restoring force on either mass is

$$|F_z| = 2k\delta z$$

The masses oscillate with a frequency $\omega = (2k/M)^{1/2}$ according to

$$\delta z(t) = A\cos\omega t$$

The only non-vanishing component of the second mass moment I_{ij} is I_{zz} because the masses never stray from the $z-$axis. This is

$$I_{zz} = 2M\left(\frac{L}{2} + \delta z(t)\right)^2$$

We then find to lowest non-vanishing order in A that

$$\dddot{I}_{zz} = 2MAL\omega^2\cos\omega t \tag{11}$$

We can write the gravitational wave amplitude far from the source as

$$\tilde{h}^{ij}(t,\vec{x}) \lim_{r\to\infty} \frac{2}{r}\ddot{I}^{ij}(t - r/c)$$

This expression, however, is not in the transverse traceless gauge. To put the result into the transverse traceless gauge for a direction making an angle θ with the $z-$axis, in the $y-z$ plane, we make a rotation about the $x-$axis to a new set of axes in which the $z'-$axis makes an angle θ with respect to the $z-$axis. This rotation is given by the analog of the equations below:

$$\begin{aligned} x' &= x \\ y' &= -z\sin\theta + y\cos\theta \\ z' &= z\cos\theta + y\sin\theta \end{aligned}$$

and the fact that I_{ij} transforms according to

$$I'_{ij}(x') = \frac{\partial x^k}{\partial x'^i}\frac{\partial x^l}{\partial x'^j} I_{kl}(x)$$

The result is

$$I'_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sin^2\theta & -\cos\theta\sin\theta \\ 0 & -\cos\theta\sin\theta & \cos^2\theta \end{pmatrix} I_{zz}$$

The simple algorithm for transforming to the transverse-traceless gauge is as follows:

1. Set all non-transverse parts of the metric to zero
2. Subtract out the trace from the remaining diagonal elements to make it traceless

We get

$$\tilde{h}'_{ij} \to \frac{2}{r}\begin{pmatrix} -\frac{1}{2}\sin^2\theta & 0 & 0 \\ 0 & \frac{1}{2}\sin^2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \ddot{I}_{zz}$$

Evidently the wave is linearly polarized. The time-averaged luminosity radiated into a solid angle $d\Omega_{sa}$ is then

$$\frac{dL}{d\Omega_{sa}} = \frac{\omega^2}{32\pi}\left(\ddot{I}_{zz}\sin^2\theta\right)^2 = \frac{1}{8\pi}M^2A^2L^2\omega^6\sin^4\theta$$

EP #57

Consider a particle moving on the $z-$axis with position $(0,0,z(t))$ in non-relativistic approximation. Then using

$$I_{jk} = m\left[x_j x_k - \frac{1}{3}\delta_{jk}(x_1^2 + x_2^2 + x_3^2)\right]$$

We then have

$$I_{jk} = 0 \text{ for } j \neq k \;\;,\;\; I_{11} = I_{22} = -\frac{1}{3}z(t)^2 \;\;,\;\; I_{33} = \frac{2}{3}z(t)^2$$

so that we have for the reduced inertia tensor

$$\tilde{I} = \begin{pmatrix} -\frac{2}{3}\dot{z}^2 - \frac{2}{3}\ddot{z}z & 0 & 0 \\ 0 & -\frac{2}{3}\dot{z}^2 - \frac{2}{3}\ddot{z}z & 0 \\ 0 & 0 & \frac{4}{3}\dot{z}^2 + \frac{4}{3}\ddot{z}z \end{pmatrix}$$

Then in the transverse traceless gauge, when observing at $(0, 0, L)$, we have

$$\ddot{\tilde{I}}^{TT} = \hat{P}\ddot{\tilde{I}}\hat{P} - \frac{1}{2}\hat{P}(\ddot{I}_{jk}P_{jk}$$

where $P_{ab} = \delta_{ab} - n_a n_b$ and $n_a = x_a/r$ for the point of observation. We then have $n_a = \delta_{az}$ and

$$P_{ab} = \delta_{ab} - \delta_{az}\delta_{az} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We thus have

$$\begin{aligned} \ddot{\tilde{I}}^{TT} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{2}{3}\dot{z}^2 - \frac{2}{3}\ddot{z}z & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -\frac{2}{3}\dot{z}^2 - \frac{2}{3}\ddot{z}z & 0 & 0 \\ 0 & -\frac{2}{3}\dot{z}^2 - \frac{2}{3}\ddot{z}z & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3}\dot{z}^2 + \frac{1}{3}\ddot{z}z & 0 & 0 \\ 0 & -\frac{1}{3}\dot{z}^2 - \frac{1}{3}\ddot{z}z & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

The radiation is then approximately given by

$$\begin{aligned} h^{TT}_{ab} &\approx \frac{1}{L}\ddot{\tilde{I}}^{TT}(t - r/c) \\ &= \begin{pmatrix} \frac{1}{3L}\dot{z}^2(t-r/c) + \frac{1}{3L}\ddot{z}(t-r/c)z(t-r/c) & 0 & 0 \\ 0 & -\frac{1}{3L}\dot{z}^2(t-r/c) - \frac{1}{3L}\ddot{z}(t-r/c)z(t-r/c) & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Now we have

$$z(t) = \begin{cases} \text{constant} & t > T\,,\, t < -T \\ \frac{1}{2}gt^2 & -T < t < T \end{cases}$$

Thus,

$$\dot{z}(t) = \begin{cases} 0 & t > T\,,\, t < -T \\ gt & -T < t < T \end{cases}$$

and

$$\ddot{z}(t) = \begin{cases} 0 & t > T\,,\, t < -T \\ g & -T < t < T \end{cases}$$

Therefore,

$$h^{TT}_{xx} = -h^{TT}_{yy} = \begin{cases} \frac{g^2}{2L}(t-L)^2 & L - T < t < L + T \\ 0 & \text{elsewhere} \end{cases}$$

EP #58

We consider two battleships on the axis that collide. We have $m = 70000\, tons = 6.34 \times 10^7\, kg$, $v_0 = 40\, km/h = 11.1\, m/s$, $v_f = 0\, m/s$, and $\Delta t = 2\, sec$. Then

$$\begin{aligned} I_{jk} &= m_1 \left(x_{1j}x_{1k} - \frac{\delta_{jk}}{3}(x_{11}^2 + x_{12}^2 + x_{13}^2) \right) + m_2\, (........) \\ &= 0 \quad \text{when } j \neq k \end{aligned}$$

Thus,

$$\begin{aligned} I_{xx} &= \frac{2m}{3}(x_1^2(t) + x_2^2(t)) \\ I_{yy} = I_{zz} &= -\frac{m}{3}(x_1^2(t) + x_2^2(t)) \end{aligned}$$

The power radiated is given by

$$P = \frac{dE}{dt} = \frac{1}{5}\langle \dddot{I}_{jk} \dddot{I}_{jk} \rangle$$

Note that $\dddot{I}_{jk}$ has units of J/sec.

Now $a_1 = \Delta v/\Delta t = 5.55\, m/sec^2 = \text{ constant } = \ddot{x}_1$ and $a_2 = -5.55\, m/sec^2 = \ddot{x}_2$. Therefore, $\dddot{x} = 0$. We then integrate to get

$$\begin{aligned} \dot{x}_i &= a_i t + v_0 \\ x_i &= \frac{1}{2}a_i t^2 + v_{i0}t + x_{i0} \end{aligned}$$

We also have

$$\Delta x = \frac{v_f^2 - v_0^2}{2a} = x_f - x_0$$

Since $v_f = 0$ and $x_f = 0$ we have $x_0 = v_0^2/2a$ and then

$$x_i(t) = \frac{v_{i0}^2}{2a_i} + v_{i0}t + \frac{1}{2}a_i t^2$$

Then

$$\dddot{I}_{jk} = \text{constant}\left[\dddot{x}_1^2 + \dddot{x}_2^2\right]$$

Now

$$\begin{aligned} \left[\dddot{x}_1^2 + \dddot{x}_2^2\right] &= 6\ddot{x}_1\dot{x}_1 + 6\ddot{x}_2\dot{x}_2 \\ &= 6[a_1(a_1 t + v_{10}) + a_2(a_2 t + v_{20})] = 6[a_1^2 + a_1 v_0 + (-a_1)^2 t +) - a_1)(-v_0)] \\ &= 12[a_1^2 t + a_1 v_0] \end{aligned}$$

Then

$$\begin{aligned} \dddot{I}_{xx} &= \frac{2m}{3}12[a_1^2 t + a_1 v_0] = 8m[a_1^2 t + a_1 v_0] \\ \dddot{I}_{yy} = \dddot{I}_{zz} &= -\frac{m}{3}12[a_1^2 t + a_1 v_0] = -4m[a_1^2 t + a_1 v_0] \end{aligned}$$

We then get (putting constants back)

$$
\begin{aligned}
P &= \frac{G}{5c^5}[16m^2[a_1^2 t + a_1 v_0] + 16m^2[a_1^2 t + a_1 v_0] + 64m^2[a_1^2 t + a_1 v_0]] \\
&= \frac{96}{5} m^2 (a_1^4 t^2 + a_1^2 v_0^2 + 2a_1^3 v_0 t)
\end{aligned}
$$

We then have the total energy radiated as

$$
\begin{aligned}
E &= \int_0^{2.0} P dt = \frac{96}{5} m^2 \left(\frac{a_1^4 (2)^3}{3} + a_1^2 v_0^2 (2) + a_1^3 v_0 (2)^2 \right) \\
&= 5.37 \times 10^{-33} \, J
\end{aligned}
$$

Using $E = \hbar\omega = h\nu = hc/\lambda$ we get

$$
\lambda = \frac{hc}{E} = 1.06 \times 10^9 \, m
$$

This is approximately three times the earth-moon distance.

EP #59

No solution yet.

EP #60

We have the following relations:

$$
w_{ab} = h_{ab} - \frac{1}{2} h \eta_{ab}
$$

which is called the trace-reversal of h_{ab} (see below)

The gauge is fixed up by

$$
h_{ab} \to h_{ab} + \partial_a z_b + \partial_b z_a
$$

where $\Box z_a = 0$.

Now

$$
w_{ab} = A_{ab} \cos{(n_c x^c)} + B_{ab} \sin{(n_c x^c)}
$$

are the harmonic plane wave solutions to $\Box w_{ab} = 0$

Aside: We have

$$
\begin{aligned}
W &= W_{00} - W_{xx} - W_{yy} - W_{zz} \\
&= (h_{00} - h/2) - (h_{xx} + h/2) - (h_{yy} + h/2) - (h_{zz} + h/2) \\
&= (h_{00} - h_{xx} - h_{yy} - h_{zz}) - 2h = h - 2h = -h
\end{aligned}
$$

which is why w_{ab} is called the trace-reversal of h_{ab}. Note also that $h_{ab} = w_{ab} - \eta_{ab} w/2$. Note that is $h = 0$, then $w_{ab} = h_{ab}$, which is good. So let us see if we can comer up with a gauge transformation to make h_{ab} traceless.

For clarity, we do a simple example. Assume the wave is propagating in the z direction. Then $n^c = (1, 0, 0, 1)$, $n_c = (1, 0, 0, -1)$ and $n_c x^c = t - z$. Let us assume that $B_{ab} = 0$ so that we have

$$w_{ab} = A_{ab} \cos(t - z)$$

Now consider a gauge transformation of the form

$$z_a = c_a \sin(t - z)$$

We then have

$$\begin{aligned}
\partial_0 z_a &= c_a \cos(t - z) \\
\partial_x z_a &= \partial_y z_a = 0 \\
\partial_z z_a &= -c_a \cos(t - z)
\end{aligned}$$

and thus

$$\begin{aligned}
h'_{00} &= h_{00} + 2\partial_0 z_a = h_{00} + 2c_0 \cos(t - z) \\
h'_{xx} &= h_{xx} \ , \ h'_{yy} = h_{yy} \\
h'_{zz} &= h_{zz} - 2c_z \cos(t - z)
\end{aligned}$$

and hence

$$\begin{aligned}
h' &= h'_{00} - h'_{xx} - h'_{yy} - h'_{zz} \\
&= (h_{00} - h_{xx} - h_{yy} - h_{zz}) + 2(c_0 + c_z) \cos(t - z) \\
&= -W + 2(c_0 + c_z) \cos(t - z) \\
&= -(A_{00} - A_{xx} - A_{yy} - A_{zz}) \cos(t - z) + 2(c_0 + c_z) \cos(t - z)
\end{aligned}$$

So it suffices to choose $2(c_0 + c_z) = A_{00} - A_{xx} - A_{yy} - A_{zz}$ to make $h' = 0$ and hence w'_{ab} traceless.

Let us now take a look at the independent components of A_{ab}. First, note that since w_{ab} is symmetric, then A_{ab} is also symmetric. This leave 10 independent components. Next we use the Lorentz gauge condition $\partial_a w^{ab} = 0$, which says that

$$A_{ab} n^b = A_{a0} + A_{az} = 0$$

This gives us

$$\begin{aligned}
A_{00} &= -A_{0z} = -A_{z0} = A_{zz} \\
A_{x0} &= -A_{xz} \\
A_{y0} &= -A_{yz}
\end{aligned}$$

These eliminate 4 components and we are left with:

$$[A_{ab}] = \begin{pmatrix} A_{00} & A_{0x} & A_{0y} & -A_{00} \\ A_{0x} & A_{xx} & A_{xy} & -A_{0x} \\ A_{0y} & A_{xy} & A_{yy} & -A_{0y} \\ -A_{00} & -A_{0x} & -A_{0y} & A_{00} \end{pmatrix}$$

i.e., we have 6 independent components.

Now, imposing the transverse gauge condition, i.e., $t^a h_{ab} = 0$ where t^a is a unit vector along the time axis, gives $t^a w_{ab} = 0 \to t^a A_{ab} = 0 \to A_{0b} = 0$ for all b. Finally imposing the traceless gauge condition above yields $A_{xx} = -A_{yy}$. We are thus left with just two degrees of freedom:

$$[A_{ab}] = A_{xx} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + A_{xy} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = a\epsilon_{\oplus} + b\epsilon_{\otimes}$$

There are nice animations of these polarizations in a Wolfram Mathematica Player app. (see URL below)

http://demonstrations.wolfram.com/GravitationalWavePolarizationAndTestParticles/

$\epsilon_{\oplus}$ makes particles particles in a circle in the $x - y$ plane oscillate like this: $\epsilon_{\oplus}$ does the same rotated by 45°. Linear combinations can yield circular polarization, etc.

EP #61

The line element can be written in the form

$$ds^2 = A(dx^0)^2 + B(dx^1)^2 + C(dx^2)^2 + D(dx^3)^2$$

where

$$x^0 = t \ , \ x^1 = r \ , \ x^2 = \theta \ , \ x^3 = \phi$$

with

$$A = 1 \ \ , \ \ B = -\frac{t^2}{1+r^2} \ \ , \ \ C = -t^2 r^2 \ \ , \ \ D = -t^2 r^2 \sin^2\theta$$

where we have used $a(t) = t$ for convenience. To prove that the space is flat we have to show that the curvature or the Riemann-Christoffel tensor vanishes:

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\frac{\partial^2 g_{\mu\sigma}}{\partial x^\nu \partial x^\rho} + \frac{\partial^2 g_{\nu\rho}}{\partial x^\mu \partial x^\sigma} - \frac{\partial^2 g_{\mu\rho}}{\partial x^\nu \partial x^\sigma} - \frac{\partial^2 g_{\nu\sigma}}{\partial x^\mu \partial x^\rho}\right) + g_{\alpha\beta}\left(\Gamma^\alpha_{\mu\sigma}\Gamma^\beta_{\nu\rho} - \Gamma^\alpha_{\mu\rho}\Gamma^\beta_{\nu\sigma}\right) = 0$$

where $\Gamma^\alpha_{\mu\sigma} = \frac{1}{2} g^{\alpha\lambda}\left(\frac{\partial g_{\lambda\mu}}{\partial x^\sigma} + \frac{\partial g_{\lambda\sigma}}{\partial x^\mu} - \frac{\partial g_{\mu\sigma}}{\partial x^\lambda}\right)$, etc.

The curvature tensor has the symmetry properties

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} \quad , \quad R_{\mu\nu\rho\sigma} = R_{\mu\nu\sigma\rho} \quad , \quad R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$$

and satisfies the identities

$$R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0$$

Then it follows that all components $R_{\mu\nu\rho\sigma}$ in which $\mu = \nu$ or $\rho = \sigma$ are zero and that for 4-dimensional space the curvature tensor has only 20 independent components

0101 0112 0202 0213 0312 1212 1313
0102 0113 0203 0223 0313 1213 1323
0103 0123 0212 0303 0323 1223 2323

with $R_{0123} + R_{0312} + R_{0231} = 0$.

The RobertsonWalker spacetime has

$$g_{00} = A \quad , \quad g_{11} = B \quad , \quad g_{22} = C \quad , \quad g_{33} = D \quad , \quad g_{\mu\nu} = 0 \text{ for } \mu \neq \nu$$

We denote

$$\alpha = \frac{1}{2A} \quad , \quad \beta = \frac{1}{2B} \quad , \quad \gamma = \frac{1}{2C} \quad , \tag{12}$$

$$delta = \frac{1}{2D} \quad ,$$

$$A_\mu = \frac{\partial A}{\partial x^\mu} \quad , \quad B_\mu = \frac{\partial B}{\partial x^\mu} \quad , \quad A_{\mu\nu} = \frac{\partial^2 A}{\partial x^\mu \partial x^\nu} \quad , \quad \text{etc}$$

Then by direct computation we find

$$\begin{aligned} R_{0123} &= 0 \\ R_{0102} &= \frac{1}{2}(-A_{12} + \alpha A_1 A_2 + \beta A_1 A_2 + \gamma A_2 C_1) = 0 \end{aligned}$$

$$\begin{aligned} R_{0101} &= \frac{1}{2}(-A_{11} - B_{00} + \alpha(A_0 B_0 + A_1^2) + \beta(A_1 B_1 + B_0^2) - \gamma A_2 B_2 - \delta A_3 B_3) \\ &= \frac{1}{2}(-B_{00} + \beta B_0^2) = \frac{1}{2}\left(\frac{2}{1+r^2} - \frac{2}{1+r^2}\right) = 0 \end{aligned}$$

Since the assignment of the indices is arbitrary, we can obtain the other components by interchanging indices. For example, R_{0202} can be obtained from R_{1010} by the interchange $(1, B, \beta) \leftrightarrow (2, C, \gamma)$, which gives

$$\begin{aligned} R_{2020} &= \frac{1}{2}(-A_{22} - C_{00} + \alpha(A_0 C_0 + A_2^2) + \gamma(A_2 C_2 + C_0^2) - \beta A_1 C_1 - \delta A_3 C_3) \\ &= \frac{1}{2}(-C_{00} + \gamma C_0^2) = \frac{1}{2}\left(2r^2 - 2r^2\right) = 0 \end{aligned}$$

In this manner, we rfind all $R_{\mu\nu\rho\sigma} = 0$. showing that the space is indeed flat.

Let $R = rt$ and the coefficient of $d\theta^2 + \sin^2\theta d\phi^2$ becomes $-R^2$. The remaining part of the line element can be rewritten

$$\begin{aligned} dt^2 - \frac{(tdr)^2}{1+r^2} &= dt^2 - \frac{(tdR - Rdt)^2}{t^2+R^2} \\ &= \frac{(tdR + Rdt)^2}{t^2+R^2} - dR^2 \\ &= d\tau^2 - DR^2 \end{aligned}$$

if we set

$$\tau = \sqrt{t^2+R^2}$$

Hence the transformation

$$R = rt \quad , \quad \tau = \sqrt{t^2+R^2}$$

can reduce the Robertson-Walker line element to the Minkowski form.

EP #62

(a) Einstein's equations are

$$\rho = \frac{3(k+\dot{a}^2)}{8\pi a^2 G} \quad \text{and} \quad p = -\frac{2a\ddot{a} + \dot{a}^2 + k}{8\pi a^2}$$

Since $k = 0$ in a flat universe, the first one gives us

$$\rho = \frac{3\dot{a}^2}{8\pi a^2 G} \rightarrow \dot{a}^2 = \frac{8\pi G}{3}\rho a^2$$

Combining this with the second equation, we get

$$\frac{\ddot{a}}{a} = -\frac{4\pi G(3p+\rho)}{3}$$

In a matter-dominated universe, the ρ term is much larger than the p term, and we have

$$\ddot{a} = -\frac{4\pi G}{3}\rho a$$

(b) For light moving radially, we have

$$ds^2 = -dt^2 + a^2(t)dr^2 = 0$$

Therefore,

$$\int_{t_e}^{t_o} \frac{dt}{a(t)} = \int_{r_e}^{r_o} dr = d = \text{ spatial separation}$$

Now, consider light emitted one period later. Then emitted at $t'_e = t_e + \lambda_e/c$ and received at $t'_o = t_o + \lambda_o/c$. The distance between the emitter and observer is unchanged, so the RHS is still d. We thus have

$$\int_{t_e}^{t_o} \frac{dt}{a(t)} = d = \int_{t_e+\lambda_e/c}^{t_o+\lambda_o/c} \frac{dt}{a(t)}$$

If we assume that $a(t)$ is constant over small time periods, we get

$$\frac{t_o}{a_o} - \frac{t_e}{a_e} = \frac{t_o + \lambda_o/c}{a_o} - \frac{t_e + \lambda_e/c}{a_e}$$

or

$$\frac{\lambda_o}{\lambda_e} = \frac{a_0}{a_e}$$

Therefore,

$$z = \frac{\lambda_o}{\lambda_e} - 1 = \frac{a_0}{a_e} - 1$$

(c) The proper diameter D of a galaxy at a distance r away at a constant angle ϕ is measured to be

$$ds^2 = a^2 r^2 d\theta^2 \to D = ar\Delta\theta \to \Delta\theta = \frac{D}{ar} = \delta$$

Now consider a light ray traveling to us from the galaxy. We then have

$$-\int_t^{t_0} \frac{dt}{a} = \int_0^r dr = r$$

as before. Now, since D is constant, for δ to be minimum, $a(t)r$ must be a maximum. Thus, at minimum δ,

$$\frac{d}{dt}(ar) = \dot{a}r + a\dot{r} = \dot{a}r - \frac{a}{a} = 0 \to r = \frac{1}{\dot{a}}$$

Since

$$\ddot{a} = \frac{d}{dt}(\dot{a}) = \frac{da}{dt}\frac{d\dot{a}}{da} = \frac{1}{2}\frac{d\dot{a}^2}{da}$$

Einstein's two equations combine to give

$$-\frac{d\rho}{\rho} = 3\frac{da}{a}$$

or

$$\rho a^3 = b = \text{ a constant}$$

The first equation then gives

$$a^{1/2} da = \sqrt{\frac{8\pi G b}{3}} dt$$

which integrates to

$$a(t) = Ct^{2/3}$$

where $C = \left(\frac{6\pi Gb}{3}\right)^{1/3}$ is a constant. Hence,

$$r = \frac{1}{C}\int_t^{t_0} t^{-2/3}dt = \frac{3}{C}(t_0^{1/3} - t^{1/3})$$

Therefore, minimum δ occurs when

$$r = \frac{1}{\dot{a}} = \frac{3}{2C}t^{1/3} = \frac{3}{C}(t_0^{1/3} - t^{1/3})$$

i.e., at

$$t_1 = \frac{8}{27}t_0$$

when

$$z_1 = z_{crit} = \frac{a_0}{a_1} - 1\left(\frac{t_0}{t_1}\right)^{2/3} - 1 = \frac{5}{4}$$

EP #63

(a) In polar coordinates, the metric is

$$ds^2 = dt^2 - R^2(t)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2)$$

Letting $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, we have

$$g_{00} = 1 \;\;,\;\; g_{11} = -R^2 \;\;,\;\; g_{22} = -R^2 r^2 \;\;,\;\; g_{33} = -R^2 r^2 \sin^2\theta$$
$$g_{\mu\nu} = 0 \text{ for } \mu \neq \nu$$

Since $g_{\mu\nu}$ is diagonal, for which $g^{\mu\mu} = g_{\mu\mu}$, we have

$$\begin{aligned}
\Gamma^\mu_{\alpha\beta} &= 0 \text{ for } \mu \neq \alpha \neq \beta \\
\Gamma^\mu_{\mu\beta} &= \Gamma^\mu_{\beta\mu} = \frac{1}{2}g^{\mu\mu}\frac{\partial g_{\mu\mu}}{\partial x^\beta} = \frac{1}{2g_{\mu\mu}}\frac{\partial g_{\mu\mu}}{\partial x^\beta} \\
&= \frac{\partial}{\partial x^\beta}(\log|g_{\mu\mu}|^{1/2})
\end{aligned}$$

where there is no summation for repeated indices above. Therefore, the Christoffel symbols are

$$\begin{aligned}
&\Gamma^\mu_{0\mu} = \Gamma^\mu_{\mu 0} = \frac{1}{R}\frac{\partial R}{\partial t} \quad (\mu = 1,2,3) \\
&\Gamma^\mu_{1\mu} = \Gamma^\mu_{\mu 1} = \frac{1}{r} \quad (\mu = 2,3) \\
&\Gamma^\mu_{2\mu} = \Gamma^\mu_{\mu 2} = \Gamma^\mu_{3\mu} = \Gamma^\mu_{\mu 3} = \Gamma^\mu_{\mu\mu} = 0 \quad (\mu = 0,1,2,3)
\end{aligned}$$

The motion of a particle is described by the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

where $d\tau = ds$, τ being the local proper time. For $\mu = 1$, the above becomes

$$\frac{d^2r}{d\tau^2} + \frac{21}{R}\frac{dR}{dt}\frac{dt}{d\tau}\frac{dr}{d\tau} = 0$$

Multiplying both sides by R^2 gives

$$R^2\frac{d}{d\tau}\left(\frac{dr}{d\tau}\right) + \frac{dR^2}{d\tau}\frac{dr}{d\tau} = 0 \to \frac{d}{d\tau}\left(R^2\frac{dr}{d\tau}\right) = 0$$

Hence

$$R^2\frac{dr}{d\tau} = \text{ constant}$$

The momentum 4-vector of the particle is by definition

$$p^\alpha = mu^\alpha = m\frac{dx^\alpha}{d\tau}$$
$$p_\alpha = g_{\alpha\beta}p^\beta = g_{\alpha\beta}mu^\beta$$

Then,

$$p^\alpha p_\alpha = g_{\alpha\beta}p^\alpha p^\beta = m^2 g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$$

or

$$m^2\left(\frac{dt}{d\tau}\right)^2 - R^2m^2\left(\frac{dr}{d\tau}\right)^2 = m^2$$

where we have used the metric equation. Writing the LHS as $E^2 - p^2$, we have the momentum

$$p = mR\frac{dr}{d\tau}$$

and the energy

$$E = \sqrt{m^2 + p^2}$$

Initially, $p = p_0$, $E = E_0$, so that

$$R^2\frac{dr}{d\tau} = \frac{Rp}{m} = \frac{R_0p_0}{m}$$

or

$$p(t) = \frac{R_0}{R}p_0$$

and thus

$$\begin{aligned}E(t) &= \sqrt{m^2 + \left(\frac{R_0}{R}\right)^2 p_0^2} \\ &= \sqrt{E_0^2 - p_0^2 + \left(\frac{R_0}{R}\right)^2 p_0^2} \\ &= \sqrt{E_0^2 - p_0^2 \left[1 - \left(\frac{R_0}{R}\right)^2\right]}\end{aligned}$$

where $R_0 = R(t_0)$.

(b) If the gas of photons is in thermal equilibrium at time t_0, then according to Planck's theory of black body radiation, the number in volume $V(t_0)$ of photons with frequencies between ν and $\nu + d\nu$ are

$$dN(t_0) = \frac{8\pi\nu^2 V(t_0) d\nu}{c^3 \left[e^{\left(\frac{h\nu}{kT_0} - 1\right)}\right]}$$

where h and k are Planck's and Boltzmann's constants and $T_0 = T(t_0)$ is the temperature. At a later time t', photons with original frequency ν have frequency ν' given by (EP-62)

$$\frac{\nu'}{\nu} = \frac{R(t_0)}{R(t')}$$

Also, the volume of the gas has changed as the scaling factor $R(t)$ changes:

$$\frac{V(t')}{V(t_0)} = \frac{R^3(t')}{R^3(t_0)}$$

Hence

$$dN(t') = dN(t_0) = \frac{8\pi \left(\frac{\nu' R(t')}{R(t_0)}\right)^2 \frac{R^3(t')}{R^3(t_0)} V(t') \frac{R(t')}{R(t_0)} d\nu'}{c^3 \left[e^{\left(\frac{h\nu' R(t')}{kT_0 R(t_0)} - 1\right)}\right]}$$

If we set

$$T(t') = \frac{R(t_0)}{R(t')} T(t_0)$$

the distribution will still retain the black body form:

$$dN(t') = \frac{8\pi\nu'^2 V(t') d\nu'}{c^3 \left[e^{\left(\frac{h\nu'}{kT(t')} - 1\right)}\right]}$$

(c) Consider non-interacting massive particles as an ideal gas. In thermal equilibrium the number of particles with momenta between p and $p + dp$ is

$$dN_p = \frac{gVp^2dp}{2\pi^2 h^3}\left[e^{\left(\frac{E-\mu}{kT}\right)} \pm 1\right]^{-1}$$

where $E = \sqrt{m^2 + p^2}$ and μ the chemical potential, can be set to zero since the particles are non-interacting. Since the expansion of the gas is adiabatic, $TV^{\gamma-1} =$ constant, and hence $T \propto R^{-3(\gamma-1)}$ as $V \propto R^2$. Since $p \propto 1/R$ and $E = \sqrt{m^2 + p^2}$, the initial thermal equilibrium distribution cannot be maintained as the universe expands.

(d) Considering the photon gas, we have from (b)

$$\frac{R(t)}{R(t_0)} = \frac{T(t_0)}{T(t)} = \frac{mc^2}{3 \times 10^{-4}}$$

For neutrinos, we then have

$$\frac{p(t)}{p(t_0)} = \frac{R(t_0)}{R(t)} = \frac{3 \times 10^{-4}}{mc^2}$$

At the initial time t_0, the neutrinos have kinetic energy $\approx mc^2$, i.e.,

$$\sqrt{p^2c^2 + m^2c^4} - mc^2 \approx mc^2$$

Hence, $p(t_0) \approx \sqrt{3}mc$. It follows that at the present time t,

$$p(t) \approx 3\sqrt{3} \times 10^{-4}\, eV/c$$

$$v(t) = \frac{p(t)}{m} \approx \frac{3\sqrt{3} \times 10^{-4}}{mc^2} c$$

where mc^2 is in eV. As $v \ll c$, the kinetic energy is approximately

$$\frac{m}{2}v^2 = \frac{mc^2}{2}\left(\frac{v}{c}\right)^2 = \frac{1.35 \times 10{-7}}{mc^2}\, eV$$

EP #64

(a) The model is evidently flat since $k = 0$.

(b) This is not a matter dominated universe where we would have $a(t) \propto t^{2/3}$. Here we have $a(t) \propto t^{1/2}$.

(c) Using the $k = 0$ solution of the FRW model we get

$$\rho = \frac{3}{8\pi}\left(\frac{\dot{a}}{a}\right)^2$$

But $a(t) \propto t^{1/2}$ from the given metric, which gives

$$\rho(t) = \frac{3}{32\pi t^2}$$

Are the dimensions right?

EP #65

We already have shown that

$$1+z=\frac{R(t_0)}{R(t)}$$

For a matter-dominated FRW cosmology we have $R(t) \propto t^{2/3}$. Hence

$$1+z=\left(\frac{t_0}{t}\right)^{2/3}$$

If $z=1$, then

$$\left(\frac{t_0}{t}\right)^{2/3}=2 \rightarrow \frac{t_0}{t}=2^{-3/2}=0.354$$

Therefore, the light left its galaxy when the universe was about 35% its current size.

EP #66

(a) A flat dust universe with zero cosmological constant has $k=0$ and ds^2 is of the form

$$ds^2=dt^2-a^2(t)(dr^2+r62d\theta^2+r^2\sin^2\theta d\phi^2)$$

or

$$[g]=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -a(t) & 0 & 0\\ 0 & 0 & -a(t) & 0\\ 0 & 0 & 0 & -a(t)\end{pmatrix}$$

Calculation of the Ricci tensor yields the equations:

$$G_{tt}=3\left(\frac{\dot{a}}{a}\right)^2=8\pi T_{tt}=8\pi\rho(t)$$

in the rest frame of the fluid. We now need an expression for $\rho(t)$ so we can solve for $a(t)$.

From relativistic hydrodynamics, we have

$$\vec{u}\cdot(\nabla\cdot\hat{T})=0$$

which actually yields the *first law of thermo*

$$\frac{d}{d\tau}(\rho V)=-p\frac{dV}{d\tau}$$

where V is the volume of any fluid element.

If the matter density is small enough, then $p_m \approx 0$. For a given radiation density, $p_r = \rho_r/3$. Thus, we set $\rho = \rho_m + \rho_r$ and $p = p_m + p_r \approx p_r$ and obtain

$$\frac{d}{d\tau}(\rho_m V) + \frac{d}{d\tau}(\rho_r V) + \frac{1}{3}\rho_r \frac{dV}{d\tau} = 0$$

We can separate these equations to obtain

$$\frac{d}{d\tau}(\rho_m V) = 0 \quad \text{and} \quad \frac{d}{d\tau}(\rho_r V) + \frac{1}{3}\rho_r \frac{dV}{d\tau} = 0$$

These give the two solutions

$$\rho_m V = \text{ constant} \quad \text{and} \quad \rho_r V^{4/3} = \text{ constant}$$

Now we know that $V \propto a^3(t)$, which shows that

$$\rho_m(t)a^3(t) = \text{ constant} = \rho_m(0)a^3(0) \to \rho_m(t) = \frac{a^3(0)}{a^3(t)}\rho_m(0)$$

$$\rho_r(t)a^4(t) = \text{ constant}\rho_r(0)a^4(0) \to \rho_r(t) = \frac{a^4(0)}{a^4(t)}\rho_r(0)$$

Thus, the equation for $a(t)$ becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho(t) = \frac{8\pi}{3}\frac{a^3(0)}{a^3(t)}\rho_m(0) + \frac{8\pi}{3}\frac{a^4(0)}{a^4(t)}\rho_r(0)$$

This is a nonlinear, nonseparable ODE and is not solvable in general. However, if $\rho_r(0) = 0$, then we can solve it to obtain

$$\dot{(a)}a^{1/2} = \sqrt{\frac{8\pi\rho_m(0)a^3(0)}{3}} = \gamma^{1/2} \to a^{3/2} = \gamma^{1/2}(t - t_0)$$

Therefore, we have arrived at

$$a(t) = \gamma^{1/3}(t - t_0)^{2/3}$$

for a $k = 0$ dust universe with $\Omega = 0$.

(b) Consider a photon emitted at t_e and received at t_r. We know that

$$\frac{\dot{\lambda}}{\lambda} = \frac{\dot{a}}{a}$$

so that

$$\int_{t_e}^{t_r} \frac{\dot{\lambda}}{\lambda}\, dt = \int_{t_e}^{t_r} \frac{\dot{a}}{a}\, dt$$

which implies that

$$\ln\frac{\lambda_r}{\lambda_e} = \int_{t_e}^{t_r} \frac{2}{3}\frac{\gamma^{1/3}(t - t_0)^{-1/3}}{\gamma^{1/3}(t - t_0)^{2/3}}\, dt = \frac{2}{3}\ln\left(\frac{t_r - t_0}{t_e - t_0}\right)$$

Therefore,

$$z = \frac{\lambda_r}{\lambda_e} - 1 = \left(\frac{t_r - t_0}{t_e - t_0}\right)^{2/3} - 1$$

Now in a small amount of time dt, a photon travels a distance dictated by $ds^2 = 0$. Since

$$ds^2 = 0 = dt^2 - a^2(t)dr^2 \rightarrow dr = \frac{dt}{a(t)}$$

this piece of space expands with time after the photon travels it. The new length is

$$dr_{now} = \frac{a(t_{now})}{a(t)}dt$$

This means the distance to the emitter, s, is

$$s = \int dr_r = \int_{t_e}^{t_r} \frac{a(t_r)}{a(t)}dt = \int_{t_e}^{t_r} \frac{\gamma^{1/3}(t_r - t_0)^{2/3}}{\gamma^{1/3}(t - t_0)^{2/3}}dt = 3(t_r - t_0)\left[1 - \frac{1}{\sqrt{1+z}}\right]$$

This becomes the answer in the problem for the choice $t_r = t_0$, $t_0 = 0$. I do not know why!

(c) If $\rho_r(0) \neq 0$, then the equations are difficult to solve. If, however, $\dot{a} > 0$ for all t, we can say that since

$$\rho_m(t) \propto a^{-3} \quad \text{and} \quad \rho_r(t) \propto a^{-4}$$

then

$$\frac{\rho_r(t)}{\rho_m(t)} \propto a^{-1} \rightarrow 0 \text{ as } t \rightarrow \infty$$

Thus, the universe will eventually be dominated by the dust!

EP #67

If a photon is emitted it recedes for two reasons. It is traveling through space and space itself is expanding. Assume a photon was emitted at $t = 0$, how far has it gone in a time $t = t_0$/ Since $ds^2 = 0$, we have

$$\frac{dr}{dt} = \frac{1}{a(t)} \rightarrow D = \in_0^{t_0} \frac{1}{a(t)}dt = 3\gamma^{1/3}(t_0)^{1/3}$$

This is the present distance to the photon in the fixed 3-space metric. The physical distance at fixed times is given by

$$h_p = a(0)D = 3t_0$$

EP #68

(a) If the metric inside is flat, then there are coordinates (T, R, Θ, Φ) in which the metric can be written

$$ds^2 = -dT^2 + dR^2 + R^2(d\Theta^2 + \sin^2\Theta d\Phi^2) \tag{13}$$

The world line of the shell will be the curve $R(\tau)$. The metric in the three-surface of the collpasing shell is then,

$$ds^2 = -dT^2 + R^2(\tau)(d\Theta^2 + \sin^2\Theta d\Phi^2) \tag{14}$$

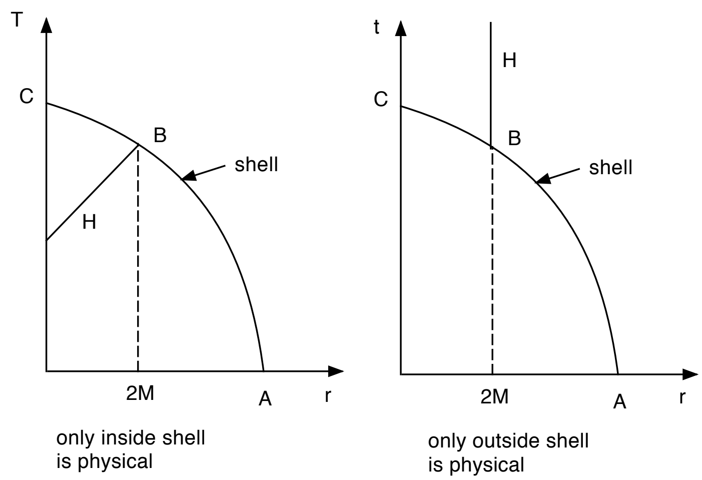
Outside the story is similar using Schwarzschild (t, r, θ, ϕ) coordinates. The metric on the surface of the collapsing shell is

$$ds^2 = -dt^2 + r^2(\tau)(d\theta^2 + \sin^2\theta d\phi^2) \tag{15}$$

The inside and outside geometries will match if $r = R$, $\theta = \Theta$, $\phi = \Phi$. In particular, $R(\tau) = r(\tau)$. The function $T(\tau)$ can then be computed from the normalization condition

$$-\left(\frac{dT}{d\tau}\right)^2 + \left(\frac{dr}{d\tau}\right)^2 = -1$$

and the given $r(\tau)$. We can then plot a (T, r) diagram .



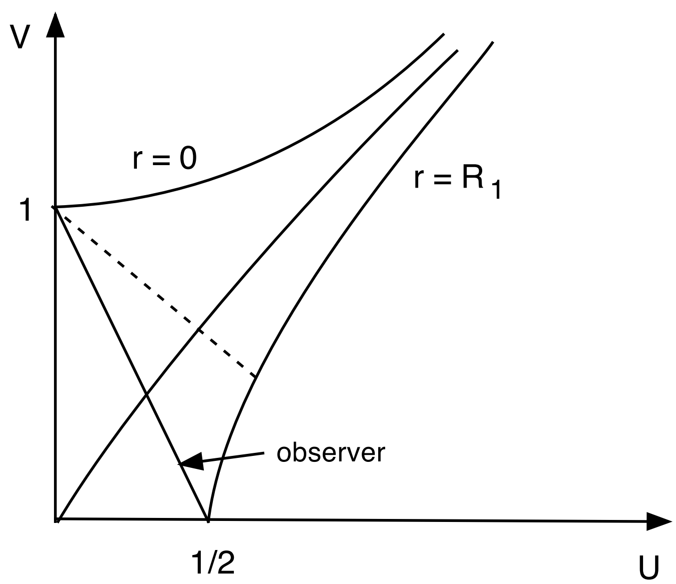
The figure on the left refers to the geometry inside the shell. The figure on the right is an Eddington-Finkelstein diagram referring to the outside.

The world line of the collpasing shell is shown schematically in both diagrams. It's not the same curve because, although $R(\tau) = r(\tau)$, $T(\tau)$ and $t(\tau)$ will be different. That's why two separate plot are required. They match up across the surface of the shell. For example, points A, B, and C match.

(b) The horizon H is the surface generated by light rays that neither escape to infinity nor collapse to the singularity. Inside the shell, those are radial light rays that just make it to the surface at $r = 2M$ as shown. After passing through the shell, they remain stationary at $r = 2M$. The area increases from zero inside the shell and remains stationary outside.

EP #69

(a) The figure below shows the solution:



(b) The straight line has a slope of 2 which means the observer is within the 45° lines which are the light cones.

(c) The latest time is the value of t at which the 45° dotted line from $U = 0$, $V = 1$ intersects the curve $r = R$. The equation of the 45° line is $V = 1 - U$, so

$$\left(\frac{R}{2M} - 1\right)^{1/2} e^{\frac{R}{2M}} \sinh\left(\frac{t}{4M}\right) = 1 - \left(\frac{R}{2M} - 1\right)^{1/2} e^{\frac{R}{2M}} \cosh\left(\frac{t}{4M}\right)$$

or

$$\frac{1}{2}\sinh\left(\frac{t}{4M}\right) = 1 - \frac{1}{2}\cosh\left(\frac{t}{4M}\right)$$

so that

$$t = 4M \log(2)$$

EP #70

(a) Suppose in the coordinate frame of the galaxy and observer successive crests of a light wave are emitted by the galaxy at times t_0, $t_0 + \Delta t_0$ and received by the observer at times t_1, $t_1 + \Delta t_1$. The world line of each crest is a radial null geodesic along which θ and ϕ remain constant so that

$$0 = -dt^2 + R^2 dx^2$$

or

$$dx = \frac{dt}{R(t)}$$

along each world line. Integrating for each crest gives

$$x_0 = \int_0^{x_0} dx = \int_{t_)}^{t_1} \frac{dt}{R(t)} = \int_{t_0+\Delta t_0}^{t_1+\Delta t_1} \frac{dt}{R(t)}$$

Since Δt_0, Δt_1 are small, the above equation implies that

$$\frac{\Delta t_0}{R(t_0)} = \frac{\Delta t_1}{R(t_1)}$$

If λ_0, λ_1 are the emitted and received wavelengths then

$$\frac{\lambda_1}{\lambda_0} = \frac{\Delta t_1}{\Delta t_0} = \frac{R(t_1)}{R(t_0)}$$

and the red shift is

$$z = \frac{\lambda_1 - \lambda_0}{\lambda_0} = \frac{\lambda_1}{\lambda_0} - 1 = \left(\frac{t_1}{t_0}\right)^{2/3} - 1$$

since $R(t) = R_0 t^{2/3}$.

(b) The angular diameter of the galaxy is by definition

$$\delta = \frac{D}{R(t_0)x_0} = \frac{D}{R_0 x_0} t_0^{-2/3}$$

Now

$$x_0 = \int_{t_)}^{t_1} \frac{dt}{R(t)} = \frac{1}{R_0} \int_{t_)}^{t_1} t^{-2/3} dt = \frac{3}{R_0} \left[\left(\frac{t_1}{t_0}\right)^{1/3} - 1 \right]$$

and

$$\left(\frac{t_1}{t_0}\right)^{1/3} = (z+1)^1/2$$

we have

$$\delta = \frac{D}{3t_0[(z+1)^1/2 - 1]} = \frac{D(z+1)^3/2}{3t_1[(z+1)^1/2 - 1]}$$

(c) Differentiating the above gives

$$\frac{d\delta}{dz} = A(z)\left[(z+1)^1/2 - \frac{3}{2}\right]$$

where $A(z) > 0$. Hence

$$\frac{d\delta}{dz} = 0 \quad \text{when } z = \frac{5}{4}$$

For $z < 5/4$, $d\delta/dz < 0$ and the angular diameter decreases as the red shift increases. For $z > 5/4$, $d\delta/dz > 0$ and the angular diameter increases as the red shift increase. Thus, the angular diameter reaches a minimum at $z = 5/4$.

EP #71

(a) For the metric in this problem

$$g_{00} = -1 \;\; , \;\; g_{11} = \frac{a^2(t)}{1-kr^2} \;\; , \;\; g_{22} = a^2(t)r^2 \;\; , \;\; g_{33} = a^2(t)r^2\sin^2\theta$$

One can then calculate $G_{\mu\nu}$ to get

$$\begin{aligned} &G_{00} = \frac{3(k+\dot{a}^2)}{a^2} \quad , \quad G_{rr} = \frac{1}{r^2} \\ &G_{\theta\theta} = \frac{1}{r^2\sin^2\theta} \quad , \quad G_{\phi\phi} = -\frac{k+\dot{a}^2+2a\ddot{a}}{1-kr^2} \\ &G_{ij} = 0 \text{ for } i \neq j \end{aligned}$$

We have a matter-dominated universe, i.e., $T_{00} = \rho$, $T_{ij} = 0$ otherwise. Therefore the field equations $G_{ab} = 8\pi G T_{ab}$ give only two equations

I. $\frac{3(k+\dot{a}^2)}{a^2} = 8\pi G\rho$

II. $k + \dot{a}^2 + 2a\ddot{a} = 0$

Now using a trick we have

$$\frac{d(\dot{a}^2)}{da} = \frac{dt}{da}\frac{d(\dot{a}^2)}{dt} = \frac{1}{\dot{a}}(2\dot{a}\ddot{a}) = 2\ddot{a}$$

Thus, (II) becomes

$$k + \dot{a}^2 + a\frac{d(\dot{a}^2)}{da} = 0$$

Integrating with respect to a, we have

$$ak + \int \dot{a}^2 da + \int a\frac{d(\dot{a}^2)}{da}da = ak + \int \dot{a}^2 da + a\dot{a}^2 - \int a\frac{d(\dot{a}^2)}{da}da = C$$

or

$$k + \dot{a}^2 = \frac{C}{a} \tag{16}$$

for some constant C. We have used integration by parts in the above derivation.

Substituting this into (I) yields

$$\frac{3G}{a^3} = 8\pi G\rho \to \rho a^3 = \frac{3C}{8\pi G} = \text{ constant}$$

So we can describe the field equations by

$$\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2 \quad , \quad \rho a^3 = \text{ constant}$$

as desired.

(b) Let L_r be the distance(radial) from the origin to a particle at position r at a fixed time t. Then, from the metric, we have

$$ds = a(t)\frac{dr}{\sqrt{1-kr^2}} = \sqrt{g_{rr}}$$

Thus,

$$L_r = a(t)\int_0^r \frac{dx}{\sqrt{1-kx^2}} = \begin{cases} a(t)r & \text{if } k = 0 \\ a(t)\sin^{-1} r & \text{if } k = +1 \\ a(t)\sinh^{-1} r & \text{if } k = -1 \end{cases}$$

(c) We now attempt a classical (Newtonian) derivation. Consider a sphere of uniform density ρ and radisu L. A particle a distance L from the origin will experience an acceleration

$$\ddot{L} = -\frac{GM}{L^2} = -\frac{G}{L^2}\rho\frac{4}{3}\pi L^3 = -\frac{4\pi G}{3}\rho L$$

(d) If M is to be conserved, we must have

$$\rho(t)\frac{4}{3}\pi L^3(t) = \rho(t')\frac{4}{3}\pi L^3(t') \to \rho L^3 = \text{ constant}$$

Again using the trick that

$$\ddot{L} = \frac{1}{2}\frac{d(\dot{L}^2)}{dL}$$

and integrating the equation for $\ddot{L}$ with respect to L, gives

$$\frac{1}{2}\dot{L}^2 = \int -\frac{GM}{L^2}dL = \frac{GM}{L} + C \tag{17}$$

where we have used $M \propto \rho L^3$ is constant and C is a constant. This implies that

$$\dot{L}^2 + C = \frac{2GM}{L} = \frac{2G}{L}\rho\frac{4}{3}\pi L^3 = \frac{8\pi G}{3}\rho L^2$$

These equations are equivalent to those found in part (a).

EP #72

If we use the transformations $r = \sin x$ and $r = \sinh x$ for $\kappa = +1$ and $\kappa = -1$, respectively, we have

$$ds^2 = \begin{cases} -dt^2 + R^2(t)[dx^2 + \sin^2 x d\Omega^2] & \kappa = +1 \\ -dt^2 + R^2(t)[dx^2 + \sinh^2 x d\Omega^2] & \kappa = -1 \end{cases}$$

Suppose the spaceship is launched along a radial directions so that $d\theta = d\phi = d\Omega = 0$. If we introduce the proper time by $-d\tau^2 = ds^2$, the metric then becomes

$$d\tau^2 = dt^2 - R^2(t)dx^2$$

Since all $g_{\alpha\beta}$ do not depend on x explicitly, the geodesic equation

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) - \frac{1}{2}g_{\alpha\beta,\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

gives for $\mu = 1$

$$\frac{d}{d\tau}\left(g_{11}\frac{dx}{d\tau}\right) = 0$$

or

$$R^2(t)\frac{dx}{d\tau} = R^2(t)\frac{dx}{dt}\left(\frac{d\tau}{dt}\right)^{-1} = \frac{R^2(t)\frac{dx}{dt}}{\sqrt{1 - R^2(t)\left(\frac{dx}{dt}\right)^2}} = \text{ constant}$$

Since the length element is $dl = R(t_d x$ and the velocity is thus

$$v = \frac{dl}{dt} = R(t)\frac{dx}{dt}$$

the above can be written as

$$\frac{R(t)v}{\sqrt{1-v^2}} = \text{ constant}$$

or

$$\frac{R(t')v'}{\sqrt{1-v'^2}} = \frac{R(t)v}{\sqrt{1-v^2}}$$

Since $R(t') = (1+z)R(t)$, we have

$$\frac{v'}{\sqrt{1-v'^2}} = \frac{v}{\sqrt{1-v^2}}\frac{1}{1+z}$$

or

$$v'^2 = \frac{v^2}{(1-v^2)(1+z)^2+v^2}$$

i.e.,

$$v' = \frac{v}{(1+z)\sqrt{1-v^2+\left(\frac{v}{1+z}\right)^2}} \approx \frac{v}{1+z}$$

for $v \ll 1$.

EP #73

(a) To calculate $\rho(a)$ we use the fluid equation

$$\dot{\rho} + \frac{3\dot{a}}{a}(\rho + p) = 0$$

which in this problem reads

$$\dot{\rho} + \frac{3\dot{a}}{a}\rho = 0$$

or

$$\frac{d}{dt}(a^{3\gamma}\rho) = 0$$

and therefore

$$\rho(t) = \rho(t_0)\left(\frac{a(t_0)}{a(t)}\right)^{3\gamma}$$

To find $a(t)$, we use the Friedmann equation with $k=0$ and substituting for ρ this is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\rho(t_0)}{3}\left(\frac{a(t_0)}{a(t)}\right)^{3\gamma}$$

which gives

$$a^{3\gamma/2-1}\dot{a} = \sqrt{\frac{8\pi}{3}\rho(t_0)a_0^{3\gamma}}$$

where we have chosen the positive root. Hence, integrating gives

$$a^{3\gamma/2} = \frac{3\gamma}{2}H_0 a_0^{3\gamma/2} t$$

where we have chosen the integration constant by requiring $a(0)=0$ and we have used

$$H_0 = \sqrt{\frac{8\pi\rho_0}{3}}$$

which follows from the definition of H and the Friedmann equation. Finally, putting $t = t_0$ gives the age of the universe

$$t_0 = \frac{2}{3\gamma}H_0^{-1}$$

(b) From part (a) we have

$$\dot{a}^2 = \frac{8\pi}{3}\frac{\rho_0 a_0^{3\gamma}}{a^{3\gamma-2}}$$

Therefore, $\dot{a}$ is constant if $3\gamma = 2$. With this value of γ, the Friedmann equation for general k reads

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi}{3}\frac{\rho_0 a_0^2}{a^2}$$

or

$$\dot{a}^2 + k = C$$

where

$$C = \frac{8\pi}{3}\rho_0 a_0^2$$

and therefore

$$a = (C-k)^{1/2} t$$

(c) From the definition of Ω we have

$$\Omega = \frac{8\pi\rho}{3H^2} = \frac{8\pi\rho a^2}{3\dot{a}^2}$$

Thus, by differentiating we get

$$\dot{\Omega} = \frac{8\pi\dot{\rho}a^2}{3\dot{a}^2} + \frac{16\pi\rho a}{3\dot{a}} - \frac{16\pi\rho a^2\ddot{a}}{3\dot{a}^3}$$

Now we replace ρ in terms of Ω, $\dot{a}/a$ with H and use the acceleration equation to replace $\ddot{a}$:

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho+3p) = -\frac{4\pi}{3}\rho(3\gamma-2) = -\frac{1}{2}\Omega H^2(3\gamma-2)$$

and this gives

$$\dot{\Omega} = (2-3\gamma)H\Omega(1-\Omega)$$

Now if we change to $s = \log a$ we have

$$\frac{d\Omega}{ds} = \dot{\Omega}\frac{dt}{ds} = \frac{\dot{\Omega}}{H}$$

and therefore

$$\frac{d\Omega}{ds} = (2-3\gamma)\Omega(1-\Omega)$$

We can see from this that if $\gamma > 2/3$, then $d\Omega/ds$ is positive for $\Omega > 1$ and negative for $\Omega < 1$ and so $|1-\Omega|$ always gets bigger. For $\gamma < 2/3$, the opposite is true.

EP #74

We consider a flat universe involving a period of inflation. The history is split into 4 periods ....

1. $0 < t < t_3$ radiation only

2. $t_3 < t < t_2$ vacuum energy dominates with an effective cosmological constant $\Lambda = 3t_3^2/4$

3. $t_2 < t < t_1$ a period of radiation dominance

4. $t_1 < t < t_0$ matter domination

(a) For a flat universe, the Einstein equations tell us that

$$\dot{(a)}^2 - \frac{8\pi G\rho}{3}a^2 = 0 \quad , \quad \dot{\rho} + 3\left(\frac{\dot{a}}{a}\right)\left(\rho + \frac{p}{c^2}\right)$$

If we set $p = w\rho c^2$ with $w \neq -1$, then

$$\begin{aligned}
&\dot{\rho} + 3\left(\frac{\dot{a}}{a}\right)\left(\rho + \frac{wpc^2}{c^2}\right) = 0 \\
&\dot{\rho} + 3\left(\frac{\dot{a}}{a}\right)(1+w)\,\rho = 0 \\
&\frac{\dot{\rho}}{\rho} = -3\left(\frac{\dot{a}}{a}\right)(1+w) \\
&\int \frac{d\rho}{\rho} = -3(1+w)\int \frac{da}{a} \\
&\ln\frac{\rho}{\rho_0} = -3(1+w)\ln a \\
&\rho = \rho_0 a^{-3(1+w)}
\end{aligned}$$

On the other hand, if $w = -1$, then $\dot{\rho} = 0$ and thus ρ is constant with time, i.e., the energy density associated with a cosmological constant is constant in time, where $\rho_\Lambda = \Lambda/8\pi G$.

Plugging into the Friedmann equation, we get for $w \neq -1$

$$\begin{aligned}
\dot{a}^2 &= \frac{8\pi G\rho}{3}a^2 \\
\dot{a}^2 &= \frac{8\pi G\rho_0}{3}a^{2-3(1+w)} \\
\dot{a}^2 &= \frac{8\pi G\rho_0}{3}a^{-(1+3w)} \\
\dot{a} &= \sqrt{\frac{8\pi G\rho_0}{3}a^{-(1+3w)/2}} \\
a^{(1+3w)/2}da &= \sqrt{\frac{8\pi G\rho_0}{3}}dt \\
\left(\frac{a}{a_0}\right)^{(3+3w)/2} &= \frac{t}{t_0} \\
\frac{a(t)}{a_0} &= \left(\frac{t}{t_0}\right)^{2/(3+3w)}
\end{aligned}$$

and for $w = -1$ we get

$$\begin{aligned}
\dot{a}^2 &= \frac{8\pi G\rho_\Lambda}{3}a^2 \\
\dot{a}^2 &= \frac{\Lambda}{3}a^2 \\
\frac{da}{a} &= \left(\frac{\Lambda}{3}\right)^{1/2} \\
\frac{a(t)}{a_0} &= e^{(\Lambda/3)^{1/2}(t-t_0)}
\end{aligned}$$

where t_0 is the time at which the period of dominance by a given type of matter begins.

We see, therefore, that in time period (3) above (radiation-dominated so $w = 1/3$),

$$a(t) \propto t^{1/2}$$

while in period (4) (matter-dominated so $w = 0$)

$$a(t) \propto t^{2/3}$$

Now from manipulating the equations above we can write

$$\rho(t) = \frac{3}{8\pi G}\left(\frac{\dot{a}}{a}\right)^2$$

for a flat universe. Then using (in period(3)) $a = Ct^{1/2}$ we have

$$\rho(t) = \frac{3}{32\pi G t^2}$$

and similarly(in period (4)) using $a = Ct^{2/3}$, we get

$$\rho(t) = \frac{1}{6\pi G t^2}$$

(b) From the formalism presented in part(a), we see that for the radiation-dominated universes, period(1) and period(3), where $w = 1/3$, we have $a \propto t^{1/2}$. For the matter-dominated universe, period(4), where $w = 0$, we have $a \propto t^{2/3}$. Finally, for $w = -1$, where the universe is dominated by vacuum energy, we have

$$a \propto e^{(\Lambda/3)^{1/2} t} \propto e^{t/(2t_3)}$$

More specifically, we have

$$a(t) = \begin{cases} t^{1/2} & 0 < t < t_3 \\ a(t_3) e^{-1/2} e^{t/(2t_3)} & t_3 < t < t_2 \\ a(t_2)(t/t_2)^{1/2} & t_2 < t < t_1 \\ a(t_1)(t/t_1)^{2/3} & t_1 < t < t_0 \end{cases}$$

or

$$a(t) = \begin{cases} t^{1/2} & 0 < t < t_3 \\ t_3^{1/2} e^{-1/2} e^{t/(2t_3)} & t_3 < t < t_2 \\ t_3^{1/2} e^{-1/2} e^{t_2/(2t_3)} (t/t_2)^{1/2} & t_2 < t < t_1 \\ t_3^{1/2} e^{-1/2} e^{t_2/(2t_3)} (t_1/t_2)^{1/2} (t/t_1)^{2/3} & t_1 < t < t_0 \end{cases}$$

(c) We have shown that for epoch (2)

$$a(t) = a(t_3) e^{-1/2} e^{t/(2t_3)}$$

Therefore

$$\frac{a(t_2)}{a(t_3)} = e^{(t_2 - t_3)/2t_3}$$

(d) We have derived above that

$$\rho = \rho_0 a^{-3(1+w)}$$

Therefore

$$\begin{aligned} \rho_r(t_0) &= \rho_r(t_1) \left(\frac{a(t_0)}{a(t_1)} \right)^{-4} \\ &= \frac{3}{32\pi t_1^2} \left(\frac{t_0}{t_1} \right)^{-8/3} \end{aligned}$$

Then,

$$\frac{\rho_r(t_0)}{\rho_m(t_0)} = \frac{\frac{3}{32\pi t_1^2} \left(\frac{t_0}{t_1} \right)^{-8/3}}{\frac{1}{6\pi G t_1^2}} = \frac{9}{16} \left(\frac{t_0}{t_1} \right)^{2/3}$$

(e) Using the definitions of $a(t)$ above for a universe expanding from a singularity, we get

at $t_3 = 10^{-35}\, sec$

$a(t_3) = 3.16 \times 10{-}18 \quad \log(a) = -17.5$

at $t_2 = 10^{-32}\, sec$

$a(t_2) = 1.77 \times 104e^{499.5} \quad \log(a) = 199.43$

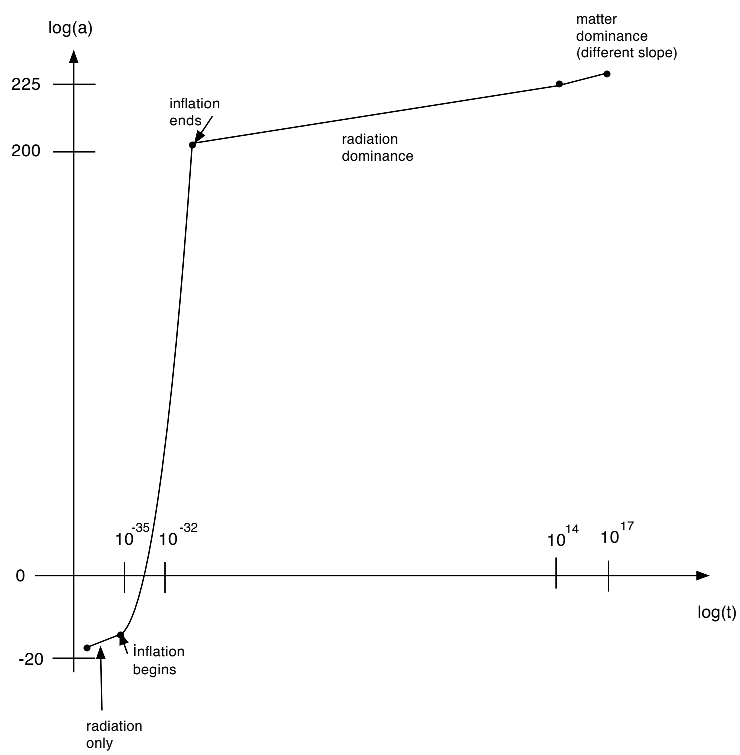
at $t_1 = 10^4\, years\ = 3.14 \times 10^{14}\, sec$

$a(t_1) = 1.77 \times 104e^{499.5} \quad \log(a) = 221.18$

at $t_0 = 10^{10}\, years\ = 3.14 \times 10^{17}\, sec$

$a(t_0) = 1.37 \times 108e^{499.5} \quad \log(a) = 225.17$

which looks like



(f) The particle horizon is defined by the relation

$$\int_{t_0}^{t} \frac{dt'}{a(t')} = \int_0^r \frac{dr'}{\sqrt{1 - kr'^2}}$$

Since $k = 0$ in this universe, we have

$$r = \int_{t_0}^{t} \frac{dt'}{a(t')}$$

is the particle horizon. - the farthest distance at which light emitted at time $t - 0$ can have a causal influence. We have shown above that during inflation, the scale factor is proportional e^{Ht}. Therefore, we see that the change in the particle horizon from t_3 to t_2 is

$$\begin{aligned}
\Delta r &= \int_{t_3}^{t_2} \frac{dt'}{a(t_3)e^{H(t'-t_3)}} \\
&= \frac{1}{a)t_3)} \int_{t_3}^{t_2} e^{-H(t'-t_3)} dt' \\
&= -\frac{1}{a(t_3)H} e^{-H(t_2-t_3)} + \frac{1}{a(t_3)H} \\
&= \frac{1}{a(t_3)H} \left(1 - e^{-H(t_2-t_3)}\right)
\end{aligned}$$

If $H = (\Lambda/3)^{1/3} = 1/(2t_3)$ as above and $t_2 = 10^{-32}\, sec$, $t_3 = 10^{-35}\, sec$, then the second term is negligible and

$$\Delta r = \frac{1}{a(t_3)H} = \frac{2t_3}{a(t_3)}$$

If we put in numbers we find for the increase in the physical distance between one end of the horizon and the other

$$\Delta d = \frac{a(t_2)}{a(t_3)} \frac{1}{H} = 2 \times 10^{-35} e^{499.5}$$

and

$$\log(\Delta d) = 182.23$$

which is a very significant increase in the physical horizon distance!

EP #75

(a) We can find the Christoffel symbols using the relation

$$\Gamma_{\mu\nu\kappa} = \frac{1}{2}(\partial_\nu g_{\kappa\mu} + \partial_\mu g_{\kappa\nu} - \partial_\kappa g_{\mu\nu})$$

One could also use the Euler-Lagrange equations. We have

$$g_{00} = 1 \ , \ g_{11} = -1 \ , \ g_{22} = -(b^2 + r^2) \ , \ g_{33} = -(b^2 + r^2)\sin^2\theta$$

The non-zero partial derivatives are then

$$\begin{aligned}
\partial_1 g_{22} &= -2r \\
\partial_1 g_{33} &= -2r\sin^2\theta \\
\partial_2 g_{33} &= -2(b^2 + r^2)\sin\theta\cos\theta
\end{aligned}$$

All others are zero.

Remember that the Christoffel symbols are symmetric in the first pair of indices, and the metric is diagonal, also only one and two derivatives are ever non-zero. We find (Roman indices are spatial)

$$\begin{aligned}
\Gamma_{000} &= 0 \\
\Gamma_{00i} &= 0 \\
\Gamma_{0i0} &= 0 \\
\Gamma_{ijk} &= \frac{1}{2}(g_{ki,j} + g_{kj,i} - g_{ij,k}) = \frac{1}{2}(\delta_{ki} g_{ki,j} + \delta_{kj} g_{kj,i} - \delta_{ij} g_{ij,k})
\end{aligned}$$

So we always need to keep some pair of indices equal to get a non-zero result. We then find the non-zero Christoffel symbols

$$\begin{aligned}
\Gamma_{122} &= -r \\
\Gamma_{133} &= -r\sin^2\theta \\
\Gamma_{221} &= r \\
\Gamma_{233} &= -(b^2 + r^2)\sin\theta\cos\theta \\
\Gamma_{331} &= r\sin^2\theta \\
\Gamma_{332} &= (b^2 + r^2)\sin\theta\cos\theta
\end{aligned}$$

plus all thus related to these six by symmetry.

(b) For the geodesic equations we need to raise the third index using

$$\Gamma^{\kappa}_{\mu\nu} = g^{\kappa\lambda}\Gamma_{\mu\nu\lambda}$$

Since the metric is diagonal, this is straightforward. We have

$$g^{00} = 1 \ , \ g^{11} = -1 \ , \ g^{22} = -\frac{1}{(b^2 + r^2)} \ , \ g^{33} = -\frac{1}{(b^2 + r^2)\sin^2\theta}$$

which give

$$\begin{aligned}
\Gamma^{1}_{22} &= g^{11}\Gamma_{221} = -r \\
\Gamma^{1}_{33} &= g^{11}\Gamma_{331} = -r\sin^2\theta \\
\Gamma^{2}_{21} &= g^{22}\Gamma_{122} = \frac{r}{(b^2 + r^2)} \\
\Gamma^{2}_{33} &= g^{22}\Gamma_{332} = -\sin\theta\cos\theta \\
\Gamma^{3}_{31} &= g^{33}\Gamma_{133} = \frac{r}{(b^2 + r^2)} \\
\Gamma^{3}_{32} &= g^{33}\Gamma_{233} = \frac{\cos\theta}{\sin\theta}
\end{aligned}$$

Finally, the geodesic equations have the form

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\kappa\lambda}\frac{dx^\kappa}{d\tau}\frac{dx^\lambda}{d\tau}$$

which gives

$$\begin{aligned}
\frac{d^2t}{d\tau^2} &= 0 \\
\frac{d^2r}{d\tau^2} &= -\Gamma^1_{22}\left(\frac{d\theta}{d\tau}\right)^2 - \Gamma^1_{33}\left(\frac{d\phi}{d\tau}\right)^2 = r\left(\frac{d\theta}{d\tau}\right)^2 r\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2 \\
\frac{d^2\theta}{d\tau^2} &= -2\Gamma^2_{12}\left(\frac{dr}{d\tau}\right)\left(\frac{d\theta}{d\tau}\right) - \Gamma^2_{33}\left(\frac{d\phi}{d\tau}\right)^2 \\
&= \frac{-2r}{(b^2+r^2)}\left(\frac{dr}{d\tau}\right)\left(\frac{d\theta}{d\tau}\right) + \sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 \\
\frac{d^2\phi}{d\tau^2} &= -2\Gamma^3_{13}\left(\frac{dr}{d\tau}\right)\left(\frac{d\phi}{d\tau}\right) - 2\Gamma^3_{23}\left(\frac{d\theta}{d\tau}\right)\left(\frac{d\phi}{d\tau}\right) \\
&= \frac{-2r}{(b^2+r^2)}\left(\frac{dr}{d\tau}\right)\left(\frac{d\phi}{d\tau}\right) - \frac{2\cos\theta}{\sin\theta}\left(\frac{d\theta}{d\tau}\right)\left(\frac{d\phi}{d\tau}\right)
\end{aligned}$$

Notice factors of two which come from summing over off-diagonal elements and using the symmetry of the Christoffel symbols.

EP #76 Consider as path through spacetime given by $x_s(t)$. The velocity associated with the curve is given by $v_s(t) = dx_s(t)/dt$ and the distance from any point (x, y, z) from the curve is determined by $r_s^2 = [(x - x_s(t))^2 + y^2 + z^2]$. We now define a smooth, positive function $f(r_s)$ such that $f(0) = 1$ and there exists an r such that $f(r_s) = 0$ for all $r_s > R$. We then consider the metric(warp-drive spacetime) given by

$$ds^2 = dt^2 - [dx - v_s(t)f(r_s)dt]^2 - dy^2 - dz^2$$

or

$$g = \begin{pmatrix} 1 - v_s^2 f^2 & v_s f & 0 & 0 \\ v_s f & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$g^{-1} = \begin{pmatrix} 1 & v_s f & 0 & 0 \\ v_s f & v_s^2 f^2 - 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(a) Null geodesics are given by $ds^2 = 0$. If we look for null geodesics only in the $x-$direction ($dy = dz = 0$), then we have

$$ds^2 = 0 = (1 - v_s^2 f^2)dt^4 + (2v_s f dx)dt + (-dx^2)$$

Then by the quadratic equation we have

$$dt = \frac{-2v_s f \pm \sqrt{4v_s^2 f^2 - 4(1 - v_s^2 f^2)(-1)}}{2(1 - v_s^2 f^2)} dx$$

or

$$\frac{dx}{dt} = \frac{v_s^2 f^2 - 1}{v_s f \pm 1} = \pm 1 + v_s f(r_s)$$

Thus, the forward light cones are tilted along the path $x_s(t)$ so that as we will see shortly this path is timelike. Far away from $x_s(t)$ where $r_s > R$ and thus $f(r_s) = 0$ we have

$$\frac{dx}{dt} = \pm 1$$

and the light cones are vertical just as in flat spacetime. Note also that we move *faster* than light in one direction and *slower* then light in the other!

(b) We consider the path $x^\mu(t, x_s(t), 0, 0)$. First, let us check that this path is inside the forward light cone for this metric. For this path we have

$$\frac{dx}{dt} = v_s(t)$$

and light paths that pass through the points on $x_s(t)$ and moving on the $t - x$ plane have velocities $\pm 1 + v_s f(r_s = 0) = \pm 1 + v_s$. Since

$$-1 + v_s < v_s < 1 + v_s$$

the particle is moving at less than the *local* speed of light and thus the path is inside the forward light cone.

Alternatively, we can check that the path is timelike all along its length. This is also sufficient to say it is inside the forward light cone. If $x = x_s(t)$, $y = z = 0$, $t =$ free, then

$$dx = \frac{dx_s(t)}{dt} dt = v_s(t) dt \quad \text{and} \quad f(r_s = 0) = 0$$

We then have

$$ds^2 = (1 - v_s^2) dt^2 = 2v_s(v_s dt)dt - (v_s dt)^2 = dt62_2 v_s^2 dt^2 - 2v_s^2 dt^2 = dt^2 > 0$$

Thus it is timelike. We also have $d\tau^2 = dt^2$ so that

$$\frac{dt}{d\tau} = 1 \ , \ \frac{d^2 t}{d\tau^2} = 0 \to \tau = t + \text{ constant}$$

The equation for a geodesic then becomes

$$\frac{d^2 x^a}{d\tau^2} + \Gamma^a_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = 0 = \frac{d^2 x^a}{dt^2} + \Gamma^a_{bc} \frac{dx^b}{dt} \frac{dx^c}{dt}$$

Now the velocities are zero for $b, c = 2, 3(y, z)$ so nonzero contributions come only for $b, c = t, x$. Thus,

$$\frac{d^2x^a}{dt^2} + \Gamma^a_{tt} + \Gamma^a_{xx}v^2 + 2\Gamma^a_{xt}v = 0$$

Now, clearly, if $a = y, z$, $\Gamma^a_{xx} = 0$ because $\partial_a g_{xx} = 0$ and $g_{ax} = 0$ for $a \neq x, t$.

It can be shown(see later) that

$$\begin{aligned}
&\Gamma^t_{tt} = \Gamma^t_{xx} = \Gamma^t_{xt} = 0 \quad \text{(along path } x_s(t))\\
&\gamma^{y,z}_{x,t,x} = 0 \quad \text{(along path } x_s(t))\\
&\Gamma^x_{xx} = \Gamma^x_{xt} = 0 \quad \text{(along path } x_s(t))\\
&\Gamma^x_{tt} = -\frac{d^2x_s}{dt^2} \quad \text{(along path } x_s(t))
\end{aligned}$$

so that all geodesic conditions are satisfied.

(c) We saw earlier that $\tau = t +$ constant for a particle moving along the geodesic $x_s(t)$. Thus, if $\Delta = T$, then $\Delta\tau = \Delta t = T$ as well so that there is no time dilation!

(d) A normal vector n^μ to the surface of constant time is a vector orthogonal to the three spatial directions $x^\mu = (0, 1, 0, 0)$, $y^\mu = (0, 0, 1, 0)$, $x^\mu = (0, 0, 0, 1)$. Orthogonal means orthogonal in our metric $n^\mu n_\mu = g_{\mu\nu}n^\mu n^\nu$ and so on.

We seek a 4-vector $n^\mu = n^0, n^1, n^2, n^3)$ orthogonal to the surface or which satisfies

$$\begin{aligned}
n^\mu z_\mu &= -n^3 = 0\\
n^\mu y_\mu &= -n^2 = 0\\
n^\mu x_\mu &= -n^1 + v_s f n^0 = 0
\end{aligned}$$

Therefore, we find

$$n^\mu = (n^0, v_s f n^0, 0, 0)$$

If this is a unit vector, then we must have

$$\begin{aligned}
n^\mu n_\mu = g_{\mu\nu}n^\mu n^\nu = 1 &= (1 - v_s^2 f^2)(n^0)^2 + 2v_s f n^0 n^1 - (n^1)^2\\
&= (n^0)^2((1 - v_s^2 f^2) + 2v_s^2 f^2 - v_s^2 f^2) = (n^0)^2
\end{aligned}$$

so that $n^0 = 1$ and thus $n^\mu = (1, v_s f, 0, 0)$.

(e) We note that the energy density seen by an observer at rest is

$$\rho_{obs} = T^{\mu\nu}n_\mu n_\nu = T_{\mu\nu}n^\mu n^\nu$$

Using Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 8\pi G T_{\mu\nu}$$

we find

$$\rho_{obs} = \frac{1}{8\pi G}\left(R_{\mu\nu}n^\mu n^\nu - \frac{1}{2}g_{\mu\nu}n^\mu n^\nu R\right) = \frac{1}{8\pi G}\left(R_{\mu\nu}n^\mu n^\nu - \frac{1}{2}R\right)$$

We now need to calculate all the curvature quantities involved.

Calculation of Christoffel Symbols and Riemann tensors

We have

$$ds^2 = dt^2 - [dx - v_s(t)f(r_s)dt]^2 - dy^2 - dz^2$$

We make the simplifying change of variables

$$f' = t \ \ , \ \ y' = y \ \ , \ \ z' = z \ \ , \ \ x' = x - x_s(t)$$

We can then use the change of variables formula for the metric

$$g_{\alpha\beta}(x'^\mu) = g_{\kappa\lambda}\frac{dx^\kappa}{dx'^\alpha}\frac{dx^\lambda}{dx'^\beta}$$

to find the new metric

$$g_{\mu\nu} = \begin{pmatrix} 1 - v_s^2(f(r_s') - 1)^2 & v_s(f-1) & 0 & 0 \\ v_s(f-1) & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

where now r_s' is independent of t', $r_s' = \sqrt{x'^{\,2} + y^{\,2} + z'^{\,2}}$. From now on we drop the prime notation and also use $\bar{f} = f - 1$, which is a function that has the same derivatives as the original f. We write $f' = df/dr$. Finally we use the fact that there is a pretty clear cylindrical symmetry and use the coordinates ρ, θ, such that $y = \rho\cos\theta$ and $z = \rho\sin\theta$. This gives the final metric

$$g_{\mu\nu} = \begin{pmatrix} 1 - v_s^2\bar{f}^2 & v_s\bar{f} & 0 & 0 \\ v_s\bar{f} & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\rho^2 \end{pmatrix}$$

A final simplification is that we will imagine that the path $x_s(t)$ has no acceleration, so that $dv_s/dt = 0$. This has the effect of making the metric independent of time. This simplification would be fairly easy to correct in the following but does simplify the calculations without giving up the fact that it is possible to connect any two points in space-time along some trajectory $x_s(t)$.

The inverse of the metric gives

$$g^{\mu\nu} = \begin{pmatrix} 1 & v_s\bar{f} & 0 & 0 \\ v_s\bar{f} & -1 + v_s^2\bar{f}^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1/\rho^2 \end{pmatrix}$$

Now we evaluate the curvature scalar in terms of the elements of the Ricci tensor. (using $R_{10} = R_{01}$)

$$\begin{aligned} R_0^0 &= g^{0\nu} R_{\nu 0} = R_{00} + v_s \bar{f} R_{01} \\ R_1^1 &= g^{1\nu} R_{\nu 1} = v_s \bar{f} R_{01} - (1 - v_s^2 \bar{f}^2) R_{11} \\ R_2^2 &= -R_{22} \ \ , \ \ R_3^3 = -R_{33}/\rho^2 \end{aligned}$$

which gives us

$$R = R_{00} + 2v_s\bar{f}R_{01} - (1 - v_s^2\bar{f}^2)R_{11} - R_{22} - R_{33}/\rho^2$$

Substituting into the formula for the energy density gives

$$\rho_{obs} = \frac{1}{16\pi G}(R_{00} + 2v_s\bar{f}R_{01} + (1 + v_s^2\bar{f}^2)R_{11} + R_{22} + R_{33}/\rho^2)$$

Christoffel Symbols

Notice that

$$\begin{aligned} \partial_1 g_{00} &= -2v_s^2 f' \bar{f} \frac{x}{r_s} \\ \partial_2 g_{00} &= -2v_s^2 f' \bar{f} \frac{\rho}{r_s} \\ \partial_1 g_{01} &= v_s^2 f' \frac{x}{r_s} \\ \partial_2 g_{01} &= v_s^2 f' \frac{\rho}{r_s} \\ \partial_2 g_{33} &= -2\rho \end{aligned}$$

with all other derivatives being zero. We can then use the formula

$$\Gamma_{\mu\nu\kappa} = \frac{1}{2}(\partial_\nu g_{\kappa\mu} + \partial_\mu g_{\kappa\nu} - \partial_\kappa g_{\mu\nu})$$

So, for example, $\Gamma_{000} = \partial_0 g_{00}/2 = 0$, while

$$\begin{aligned} \Gamma_{001} &= \frac{1}{2}(\partial_0 g_{01} + \partial_0 g_{01} - \partial_1 g_{00}) = v_s^2 f' \bar{f} \frac{x}{r_s} \\ \Gamma_{002} &= \frac{1}{2}(\partial_0 g_{02} + \partial_0 g_{02} - \partial_2 g_{00}) = v_s^2 f' \bar{f} \frac{\rho}{r_s} \end{aligned}$$

and $\Gamma_{003} = 0$.

In general, the pairs $(0,0)$, $(0,1)$, $(3,3)$ must occur in the three indices since these are the only entries in the metric with non-zero derivatives, and the remaining index of the triple must be 1 or 2 since these are the only variables with which we can differentiate and get something non-zero. Also since the symbols are symmetric in the first two indices we only need to consider cases where $\nu \geq \mu$. Working in *alphabetical* ordering this gives the following combinations as the remaining possibilities (010), (011), (012), (020), (021), (110), (120), (133), (233), (331), (332). Working through these we get

$$
\begin{aligned}
\Gamma_{010} &= \frac{1}{2}(\partial_1 g_{00} + \partial_0 g_{01} - \partial_0 g_{01}) = -v_s^2 f' \bar{f} \frac{x}{r_s} \\
\Gamma_{011} &= \frac{1}{2}(\partial_1 g_{01} + \partial_0 g_{01} - \partial_1 g_{01}) = 0 \\
\Gamma_{012} &= \frac{1}{2}(\partial_1 g_{02} + \partial_0 g_{02} - \partial_2 g_{01}) = -\frac{1}{2} v_s f' \frac{\rho}{r_s} \\
\Gamma_{020} &= \frac{1}{2}(\partial_2 g_{00} + \partial_0 g_{02} - \partial_0 g_{02}) = -v_s^2 f' \bar{f} \frac{\rho}{r_s} \\
\Gamma_{021} &= \frac{1}{2}(\partial_2 g_{01} + \partial_0 g_{12} - \partial_1 g_{02}) = \frac{1}{2} v_s f' \frac{\rho}{r_s} \\
\Gamma_{110} &= \frac{1}{2}(\partial_1 g_{01} + \partial_1 g_{01} - \partial_0 g_{11}) = v_s f' \frac{x}{r_s} \\
\Gamma_{120} &= \frac{1}{2}(\partial_2 g_{01} + \partial_1 g_{02} - \partial_0 g_{12}) = \frac{1}{2} v_s f' \frac{\rho}{r_s} \\
\Gamma_{133} &= \frac{1}{2}(\partial_3 g_{13} + \partial_1 g_{33} - \partial_3 g_{12}) = 0 \\
\Gamma_{233} &= \frac{1}{2}(\partial_3 g_{23} + \partial_2 g_{33} - \partial_3 g_{23}) = -rho \\
\Gamma_{331} &= \frac{1}{2}(\partial_3 g_{13} + \partial_3 g_{13} - \partial_1 g_{33}) = 0 \\
\Gamma_{332} &= \frac{1}{2}(\partial_3 g_{23} + \partial_3 g_{23} - \partial_2 g_{33}) = \rho
\end{aligned}
$$

with all other values zero or related by symmetry.

Now we need to use the metric to raise one index using

$$\Gamma^{\kappa}_{\mu\nu} = g^{\kappa\lambda} \Gamma_{\mu\nu\kappa}$$

Some care needs to be taken since the off-diagonal makes for some non-zero terms that are not matched with all the indices lowered. So we get for example

$$\Gamma^0_{00} = g^{00} \Gamma_{000} + g^{01} \Gamma_{001} = v_s^2 f' \bar{f}^2 \frac{x}{r_s}$$

Clearly, this only affects the cases $\kappa = 0, 1$. We find the following non-zero

Christoffel symbols

$$\begin{aligned}
\Gamma^0_{00} &= v_s^2 f' \bar{f}^2 \frac{x}{r_s} \\
\Gamma^0_{01} &= -v_s^2 f' \bar{f}^2 \frac{x}{r_s} \\
\Gamma^0_{02} &= -\frac{1}{2} v_s^2 f' \bar{f}^2 \frac{\rho}{r_s}
\end{aligned} \tag{18}$$

Notice that we have $\Gamma^0_{00} = -\Gamma^1_{01}$, $\Gamma^0_{01} = -\Gamma^1_{11}$, $\Gamma^0_{02} = -\Gamma^1_{12}$, and $\Gamma^0_{12} = \Gamma^2_{01}$ which will help simplify the expressions below.

Curvature Tensor Elements

EP #77

The meetings of the siblings must occur at common values of space-time coordinates. Thus, we know that they meet at

$$t = t_1 = 0 \quad , \quad r = R \quad , \quad \phi = 0 \quad , \quad \theta = \pi/2$$

and again at

$$t = t_2 \quad , \quad r = R \quad , \quad \phi = 20\pi \quad , \quad \theta = \pi/2$$

In general, we have

$$d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

For Paul's circular orbit we have $r = R =$ constant, $dr = 0$ and if we choose the plane of the orbit to be plane $\theta = \pi/2$, $d\theta = 0$. Therefore, we have for Paul's orbital motion we have

$$d\tau^2 = \left(1 - \frac{2GM}{c^2 R}\right) dt^2 - R^2 d\phi^2$$

and therefore we get using $R = 4GM/c^2$

$$1 = \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2 - R^2 \left(\frac{d\phi}{d\tau}\right)^2$$

Now in general we also have

$$\left(1 - \frac{2GM}{c^2 r}\right) \frac{dt}{d\tau} = k = \text{ constant}$$

and

$$r^2 \frac{d\phi}{d\tau} = h = \text{ constant}$$

Remember h is related to the angular momentum(L) and k is given in terms of the angular momentum(L) and the energy(E). In Paul's orbit we choose k and h such that

$$\left(1 - \frac{2GM}{c^2 r}\right)\frac{dt}{d\tau} = k = \frac{1}{2}\frac{dt}{d\tau} = 1$$

Therefore, we get

$$\frac{dt}{d\tau} = 2 \rightarrow \frac{d\phi}{d\tau} = \frac{1}{R}$$

This then gives(Paul moves through $\Delta\phi = 20\pi$)

$$\Delta\tau_{Paul} = R\Delta\phi = 80\pi\frac{GM}{c^2}$$

Finally since

$$\Delta\tau = \left(1 - \frac{2GM}{c^2 r}\right) dt \rightarrow \Delta\tau_{Paul} = \frac{1}{2}dt \rightarrow t_2 = 2\Delta\tau_{Paul} = 160\pi\frac{GM}{c^2}$$

Now let us consider Patty's orbit. Patty is going along a radial path (out and back). Therefore we have

$$\theta = \pi/2 \ , \ d\theta = 0 \ , \ \frac{d\phi}{d\tau} = 0$$

We have

$$\left(1 - \frac{2GM}{c^2 r}\right)\frac{dt}{d\tau} = \Gamma = \text{ constant}$$

which is just the energy conservation equation. Therefore, we get

$$\left(1 - \frac{2GM}{c^2 r}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 = 1$$

Thus,

$$\Gamma^2 - \left(\frac{dr}{d\tau}\right)^2 = \left(1 - \frac{2GM}{c^2 r}\right) = B(r)$$

Clearly, $\Gamma^2 = B(r_>$, where $r_>$ is the aphelion, which is the maximum height Patty reaches and where $dr/d\tau = 0$. We also see that

$$\frac{dt}{dr} = \frac{dt}{d\tau}\left(\frac{dr}{d\tau}\right)^{-1} = \frac{\Gamma}{B(r)}\left(\Gamma^2 - B(r)\right)^{-1/2}$$

or

$$t_2 = 2\int_R^{r_>} dr \frac{\Gamma}{B(r)}\left(\Gamma^2 - B(r)\right)^{-1/2} = 160\pi\frac{GM}{c^2}$$

Now for Patty we have

$$B(r)\frac{dt}{d\tau} = \Gamma \rightarrow \Delta_{Patty} = \frac{B(r)}{\Gamma}dt$$

or

$$\Delta\tau_{Patty} = 2\int_R^{r_>} dr\left(\Gamma^2 - B(r)\right)^{-1/2}$$

Define a new variable $u = 2GM/r$, then

$$\Delta\tau_{Patty} = 2\int_{u_>}^{1/2} \frac{du}{u^2(1-u)}\left(\frac{1-u_>}{u-u_>}\right)^{1/2}$$

The factor

$$\frac{Gamma}{B(r)} = \frac{(1-u_>)^{1/2}}{1-u}$$

is never much larger than 1 (since the integrand is dominated by small u near $u_>$), hence

$$t_2 = 160\pi\frac{GM}{c^2} \approx \Delta\tau_{Patty} \rightarrow \Delta\tau_{Patty} \approx 2\Delta\tau_{Paul}$$

EP #78

Consider a hollow ball in a bucket filled with water.

The ball feels the force of gravity as well as the bouyancy force due to the pressure created by the weight of the water above it.

If the ball is released and the bucket is dropped, the system accelerates with $\vec{g}$.

In the frame of the bucket, $\vec{g}\,' = 0$, so the water above the ball has no weight and so exerts no pressure.

Thus, bouyancy and gravity are 0 and the ball stays at rest relative to the bucket.

EP #79

If the elastic string obeys Hooke's law and has an equilibrium length of L, then the forces on the ball are

$$\sum F_y = -mg + k(s-L)$$

where s is the stretched length of the string. Since the ball is not moving, $\sum F_y = 0$, and thus

$$k(s-L) = mg$$

Now, if we drop the apparatus vertically, the entire apparatus accelerates with $\vec{g}$.

In this frame,

$$\sum F_y = -mg' + F_s$$

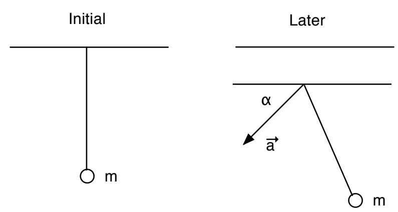
Bit, $g' = 0$, so that

$$\sum F_y = F_s \neq 0$$

and thus the string pulls the ball back into the cup.

EP #80

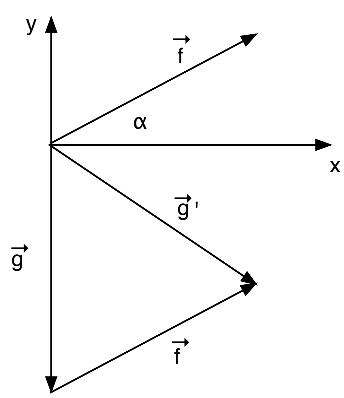
We have the following situation:



The equivalence principle tells us there is no way to distinguish between gravity and constant acceleration with a local measurement. Therefore, we can consider the problem in a frame where the base of the pendulum is at rest and there is a gravitational force in the direction given by the vector sum of $\vec{g}$ and the force felt by the pendulum $\vec{f}$.

$$\vec{g}\,' = \vec{f} + \vec{g} = f\cos\alpha\,\hat{x} + (-f\sin\alpha + g)\hat{y}$$

as shown below (in a frame accelerating with $\vec{a}$).



If $f\cos\alpha = g - f\sin\alpha$, then $g = f(\cos\alpha + \sin\alpha)$ and $\vec{g}\,'$ makes a 45° with the vertical. (The pendulum is initially at rest in the vertical position). Thus, the pendulum oscillates between the vertical and the horizontal.

EP #81

(i) This makes sense.

$$x^a = L^a_b M^b_c \hat{(x)}^c = \sum_{b=0}^{3}\sum_{c=0}^{3} L^a_b M^b_c \hat{(x)}^c$$

(ii) $x^a = L^b_c M^c_d \hat{x}^d$; This makes not sense because the index on the LHS does not match the un-contracted index on the RHS.

(iii) This makes sense.

$$\delta^a_b = \delta^a_c \delta^c_d \delta^d_b = \delta^a_b = \sum_{c=0}^{3}\sum_{d=0}^{3} \delta^a_c \delta^c_d \delta^d_b$$

(iv) $\delta^a_b = \delta^a_c \delta^c_c \delta^c_b$; This makes no sense because the repeated summation index c makes the expression ambiguous.

(v) This makes sense.

$$x^a = L^a_b \hat{x}^b + M^a_b \hat{x}^b = \sum_{b=0}^{3} L^a_b \hat{x}^b + \sum_{b=0}^{3} M^a_b \hat{x}^b$$

(vi) This makes sense.

$$x^a = L^a_b \hat{x}^b + M^a_c \hat{x}^c = \sum_{b=0}^{3} L^a_b \hat{x}^b + \sum_{c=0}^{3} M^a_c \hat{x}^c$$

(vii) $x^a = L^a_c \hat{x}^c + M^b_c \hat{x}^c$; This makes no sense because the non-contracted summation index in the second term on the RHS does not match the free summation index on the LHS or in the other terms.

EP #82

We have

$$Z^a = X^b \partial_b Y^a - Y^b \partial_b X^a$$

Under a change of coordinates we get

$$\begin{aligned}
Z^a &= \left(\frac{\partial x^b}{\partial \tilde{x}^i}\tilde{X}^i\right)\frac{\partial}{\partial x^b}\left(\frac{\partial x^a}{\partial \tilde{x}^j}\tilde{Y}^j\right) - \left(\frac{\partial x^b}{\partial \tilde{x}^i}\tilde{Y}^i\right)\frac{\partial}{\partial x^b}\left(\frac{\partial x^a}{\partial \tilde{x}^j}\tilde{X}^j\right)\\
&= \tilde{X}^i \frac{\partial}{\partial \tilde{x}^i}\left(\frac{\partial x^a}{\partial \tilde{x}^j}\tilde{Y}^j\right) - \tilde{Y}^i \frac{\partial}{\partial \tilde{x}^i}\left(\frac{\partial x^a}{\partial \tilde{x}^j}\tilde{X}^j\right)\\
&= \tilde{X}^i \tilde{Y}^j \frac{\partial^2 x^a}{\partial \tilde{x}^i \partial \tilde{x}^j} + \frac{\partial x^a}{\partial \tilde{x}^j}\tilde{X}^i \frac{\partial}{\partial \tilde{x}^i}\tilde{Y}^j - \tilde{Y}^i \tilde{X}^j \frac{\partial^2 x^a}{\partial \tilde{x}^i \partial \tilde{x}^j} - \frac{\partial x^a}{\partial \tilde{x}^j}\tilde{Y}^i \frac{\partial}{\partial \tilde{x}^i}\tilde{X}^j\\
&= \frac{\partial x^a}{\partial \tilde{x}^j}\left(\tilde{X}^i \frac{\partial}{\partial \tilde{x}^i}\tilde{Y}^j - \tilde{Y}^i \frac{\partial}{\partial \tilde{x}^i}\tilde{X}^j\right) + (\tilde{X}^i \tilde{Y}^j - \tilde{Y}^i \tilde{X}^j)\frac{\partial^2 x^a}{\partial \tilde{x}^i \partial \tilde{x}^j}
\end{aligned}$$

When we sum over i and j, the last term will cancel out (since the order of mixed partials is irrelevant) and we are left with

$$Z^a = \frac{\partial x^a}{\partial \tilde{x}^j} \tilde{Z}^j$$

Thus, it transforms correctly.

EP #83

We have the line element

$$ds^2 = dt^2 - dr^2 - \sin^2 r(d\theta^2 + \sin^2\theta d\phi^2)$$

This gives a Lagrangian

$$L = -g_{\mu\nu}\dot{x}^\mu \dot{x}^\mu = \dot{t}^2 - \dot{r}^2 - \sin^2 r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$$

The geodesic equations are the Lagrange equations

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{\partial L}{\partial x^i}$$

and the geodesic equation can also be written as

$$\frac{d^2x^a}{d\tau^2} + \Gamma^a_{bc}\dot{x}^b\dot{x}^c$$

The Lagrange equations are

$$\begin{aligned}
&\frac{d^2t}{d\tau^2} = 0 \to \Gamma^0_{bc} = 0 \text{ for all } b \text{ and } c \\
&\frac{d^2r}{d\tau^2} = -2\sin r\cos r\,\dot{\theta}^2 - 2\sin r\cos r\sin^2\theta\,\dot{\phi}^2 \\
&\to \Gamma^1_{22} = 2\sin r\cos r \quad , \quad \Gamma^1_{33} = 2\sin r\cos r\sin^2\theta \quad \text{and all others are zero} \\
&\frac{d}{d\tau}(\sin^2 r\dot{\theta}) = \sin^2 r\sin\theta\cos\theta\,\dot{\phi}^2 \\
&\to 2\sin r\cos r\,\dot{r}\dot{\theta} + \sin^2 r\frac{d^2\theta}{d\tau^2} = \sin^2 r\sin\theta\cos\theta\,\dot{\phi}^2 \\
&\to \frac{d^2\theta}{d\tau^2} = \sin\theta\cos\theta\,\dot{\phi}^2 - 2\cot r\,\dot{r}\dot{\theta} \\
&\to \Gamma^2_{33} = -\sin\theta\cos\theta \quad , \quad \Gamma^2_{12} = \Gamma^2_{21} = \cot r \quad \text{and all others are zero} \\
&\frac{d}{d\tau}(\sin^2 r\sin^2\theta\dot{\phi}) = 0 \to \Gamma^3_{bc} = 0 \text{ for all } b \text{ and } c \\
&\to 2\sin r\cos r\sin^2\theta\dot{r}\dot{\phi} + 2\sin\theta\cos\theta\sin^2 r\dot{\theta}\dot{\phi} + \sin^2 r\sin^2\theta\frac{d^2\phi}{d\tau^2} = 0 \\
&\to \frac{d^2\phi}{d\tau^2} = -2\cot r\,\dot{r}\dot{\phi} - 2\cot\theta\,\dot{\theta}\dot{\phi} \\
&\to \Gamma^3_{13} = \Gamma^3_{31} = \cot r \quad , \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta \quad \text{and all others are zero}
\end{aligned}$$

If we impose the constraints r and θ constant or $\dot{r} = \dot{\theta} = 0$ we get the equations

$$\sin r \cos r \sin^2\theta\, \dot{\phi}^2 = 0 \quad , \quad \sin\theta\cos\theta\, \dot{\phi}^2 = 0 \tag{19}$$

or $r = \theta = \pi/2$ Thus, there exists a geodesic where r and θ are constant and equal to $\pi/2$.

EP #84

We have

$$ds^2 = dudv + \log{(x^2+y^2)}du^2 - dx^2 - dy^2$$

$(0 < x^2 + y^2 < 1)$

We find g_{ab} from $ds^2 = g_{ab}dx^a dx^b$

$$g_{ab} = \begin{pmatrix} \log{(x^2+y^2)} & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The geodesic equation is

$$\frac{dx^a}{d\tau^2} + \Gamma^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = 0$$

We also have We find g_{ab} from $ds^2 = g_{ab}dx^a dx^b$

$$g^{ab} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & -4\log{(x^2+y^2)} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc})$$

The only derivatives that are non-zero are

$$\partial_x g_{uu} = \frac{2x}{x^2+y^2} \quad , \quad \partial_y g_{uu} = \frac{2y}{x^2+y^2}$$

This gives

$$
\begin{aligned}
\Gamma^u_{uu} &= 0 \\
\Gamma^u_{uv} &= \Gamma^u_{vu} = 0 \\
\Gamma^u_{ux} &= \Gamma^u_{xu} = \frac{1}{2}g^{ud}(\partial_u g_{xd} + \partial_x g_{ud} - \partial_d g_{xu}) \\
&= \frac{1}{2}g^{uu}(\partial_u g_{xu} + \partial_x g_{uu} - \partial_u g_{xu}) + \frac{1}{2}g^{uv}(\partial_u g_{xv} + \partial_x g_{uv} - \partial_v g_{xu}) = 0 \\
\Gamma^u_{uy} &= \Gamma^u_{yu} = \frac{1}{2}g^{ud}(\partial_u g_{yd} + \partial_y g_{ud} - \partial_d g_{yu}) \\
&= \frac{1}{2}g^{uu}(\partial_u g_{yu} + \partial_y g_{uu} - \partial_u g_{yu}) + \frac{1}{2}g^{uv}(\partial_u g_{yv} + \partial_y g_{uv} - \partial_v g_{yu}) = 0 \\
\Gamma^u_{vx} &= \Gamma^u_{xv} = 0 \\
\Gamma^u_{vy} &= \Gamma^u_{yv} = 0 \\
\Gamma^u_{yx} &= \Gamma^u_{xv} = 0 \\
\Gamma^u_{xx} &= 0 \\
\Gamma^u_{yy} &= 0 \\
\Gamma^u_{vv} &= 0
\end{aligned}
$$

Therefore we have for the u equation

$$\ddot{u} + \Gamma^u_{bc}\dot{x}^b\dot{x}^c = \ddot{u} = 0$$

For the v variable we have

$$\ddot{v} + \Gamma^v_{bc}\dot{x}^b\dot{x}^c = 0$$

Now Γ^v_{bc} vanishes unless $(b, c) = (x, u)$ *or* (y, u). Therefore we have

$$\ddot{v} + \Gamma^v_{xu}\dot{x}\dot{u} + \Gamma^v_{yu}\dot{y}\dot{u} = 0$$

where

$$\Gamma^v_{xu} = \frac{4x}{x^2 + y^2} \quad , \quad \Gamma^v_{yu} = \frac{4y}{x^2 + y^2}$$

and therefore

$$\ddot{v} + \frac{4x}{x^2 + y^2}\dot{x}\dot{u} + \frac{4y}{x^2 + y^2}\dot{y}\dot{u} = 0$$

which gives

$$\frac{d}{d\tau}\left(\frac{1}{2}\dot{v} + \log(x^2 + y^2)\dot{u}\right) = 0$$

Finally we have

$$\ddot{x} + \Gamma^x_{bc}\dot{x}^b\dot{x}^c = 0$$

Only $\Gamma^x_{uu} = x/(x^2 + y^2)$ is nonzero so we have

$$\ddot{x} + \frac{x}{x^2 + y^2}\dot{u}^2 = 0$$

and similarly for y we have

$$\ddot{y} + \frac{y}{x^2 + y^2}\dot{u}^2 = 0$$

Now if we define $K = x\dot{y} - y\dot{x}$, then we have

$$\frac{dK}{d\tau} = x\ddot{y} - y\ddot{x} = 0$$

and thus, K is a constant of the motion.

Now consider the following Newtonian Lagrangian

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 - A^2 \log{(x^2 + y^2)} = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 - A^2 \log{(r^2)}$$

where A^2 is the constant value of $\dot{(u)}$ and r, θ are plane polar coordinates.

From this we see that angular momentum is conserved since

$$\frac{\partial L}{\partial \theta} = 0 \rightarrow r^2\dot{theta} = x\dot{y} - y\dot{x} = K = \text{ constant of the motion}$$

Also L has no explicit time dependence so energy is conserved. L is the Lagrangian for a central force problem with potential $V = A^2 \log{(r^2)}$, so energy conservation yields

$$\frac{1}{2}\left(\dot{r}^2 + \frac{K^2}{r^2}\right) + A^2 \log{(r^2)} = \text{ constant}$$

Thus, for $K \neq 0$, we have

$$A^2 \log{(r^2)} + \frac{K^2}{r^2} \rightarrow \infty \quad \text{as} \quad r \rightarrow 0$$

Hence, for $K \neq 0$, there is no geodesic that can reach $r = 0$.

EP #85

In the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

In the frame of a particle at rest

$$U^a = \left(\frac{dt}{d\tau}, 0, 0, 0\right)$$

Since

$$U^a U_a = g_{00}\left(\frac{dt}{d\tau}\right)^2 = 1$$

we get

$$\frac{dt}{d\tau} = \left(1 - \frac{2m}{r}\right)^{-1/2}$$

If we are along a radial null geodesic, then $ds^2 = d\theta^2 = d\phi^2 = 0$ and we have

$$0 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2$$

or

$$\frac{dt}{dr} = \frac{r}{r - 2m}$$

Thus, if a photon leaves $C_1 = (r_1, \theta, \phi)$ at time t_1 and arrives at $C_2 = (r_2, \theta, \phi)$ at time t_2, integrating the above equation we obtain

$$t_2 - t_1 = \in_{r_1}^{r_2} \frac{r}{r - 2m} dr$$

Since the RHS is independent of time, the coordinate time $t_2 - t_1$ which elapses between the photon leaving C_1 and arriving at C_2 is always the same.

Therefore, if two photons are emitted from C_1 at events A and A', which arrive at C_2 at events B and B'. respectively, we see

$$t(B) - t(A) = t(B') - t(A') \rightarrow t(A') - t(A) = t(B') - t(B)$$

and so the coordinate time interval between A and A' is the same as the coordinate time interval between B and B'.

Then, since $\Delta t_1 = \Delta t_2$ and $dt/d\tau = (1 - 2m/r)^{-1/2}$, we see

$$\begin{aligned} \Delta\tau_1 \left(\frac{\Delta t_1}{\Delta\tau_1}\right) &= \Delta\tau_2 \left(\frac{\Delta t_2}{\Delta\tau_2}\right) \\ \Delta\tau_1 \left(1 - \frac{2m}{r_1}\right)^{-1/2} &= \Delta\tau_2 \left(1 - \frac{2m}{r_2}\right)^{-1/2} \end{aligned}$$

We now consider the difference in proper time between two watches, one on the wrist and one on the ankle of an observer, in the presence of the Earth. In this metric, and in SI units,

$$\Delta\tau_1 \left(1 - \frac{2GM_E}{r_1 c^2}\right)^{-1/2} = \Delta\tau_2 \left(1 - \frac{2GM_E}{r_2 c^2}\right)^{-1/2}$$

or

$$\Delta\tau_2 = \Delta\tau_1 \frac{\left(1 - \frac{2GM_E}{r_1 c^2}\right)^{-1/2}}{\left(1 - \frac{2GM_E}{r_2 c^2}\right)^{-1/2}}$$

Since $2GM_E \ll r_1c^2$ and $2GM_E \ll r_2c^2$, we can write

$$\Delta\tau_2 = \Delta\tau_1 \left(1 + \frac{GM_E}{r_1c^2}\right)\left(1 - \frac{GM_E}{r_2c^2}\right) = \Delta\tau_1 \left(1 + \frac{GM_E}{r_1c^2} - \frac{GM_E}{r_2c^2}\right)$$

to first order. We then have

$$\Delta\tau_2 = \Delta\tau_1 \left(1 + \frac{GM_E}{r_1c^2}\right)\left(1 - \frac{GM_E}{r_2c^2}\right) = \Delta\tau_1 \left(1 + \frac{GM_E}{c^2r_1r_2}(r_2 - r_1)\right)$$

Now since the distance between one's wrist and ankle is $\approx 1\,m$, so that $r_1 \approx r_2 \approx R_E$ and we get

$$\Delta\tau_2 = \Delta\tau_1 \left(1 + \frac{GM_E}{c^2R_E^2}(r_2 - r_1)\right) = \Delta\tau_1 \left(1 + \frac{g}{c^2}(r_2 - r_1)\right)$$

On earth, $g \approx 10\,m/sec^2$, $r_2 - r1 = 1\,m$ and $c = 3 \times 10^8\,m/sec$. If $\Delta\tau_1 = 1\,year = 3.15 \times 10^7\,sec$, we get $\Delta\tau_2 = 3.5 \times 10^{-9}\,sec = 3.5\,ns$.

EP #86

For free particle worldlines in the equatorial plane($d\theta = 0$, $\theta = pi/2$) we have the Lagrangian

$$L = \frac{1}{2}\left(\left(1 - \frac{2m}{r}\right)\dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2\right)$$

The Lagrange equations for t and ϕ are

$$\frac{\partial L}{\partial t} = \frac{d}{d\tau}\frac{\partial L}{\partial \dot{t}} \to 0 = \frac{d}{d\tau}\left(\left(1 - \frac{2m}{r}\right)\dot{t}\right) \to \left(1 - \frac{2m}{r}\right)\dot{t} = E = \text{ constant}$$

$$\frac{\partial L}{\partial \phi} = \frac{d}{d\tau}\frac{\partial L}{\partial \dot{\phi}} \to 0 = \frac{d}{d\tau}\left(r^2\dot{\phi}\right) \to r^2\dot{\phi} = J = \text{ constant}$$

If $E < 1$, then

$$1 > \left(1 - \frac{2m}{r}\right)\dot{t} \to \frac{2m\dot{t}}{\dot{t} - 1} \to r \text{ bounded}$$

Now the 4-velocity squared gives

$$\left(1 - \frac{2m}{r}\right)\dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = 1$$

or

$$\frac{E^2}{1 - \frac{2m}{r}} - \left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = 1$$

which implies that

$$E^2 = \dot{r}^2 + \left(1 - \frac{2m}{r}\right)(1 + r^2\dot{\phi}^2) = \dot{r}^2 + \left(1 - \frac{2m}{r}\right)\left(1 + \frac{J^2}{r^2}\right)$$

Now

$$\frac{d}{dr}(E^2) = 0 = \frac{d\dot{r}^2}{dr} + \left(1 - \frac{2m}{r}\right)\left(-2\frac{J^2}{r^3}\right) + \left(\frac{2m}{r^2}\right)\left(1 + \frac{J^2}{r^2}\right)$$

Now

$$\frac{d\dot{r}^2}{dr} = \frac{d\dot{r}}{dr}\frac{d\dot{r}^2}{d\dot{r}} = 2\dot{r}\frac{d\dot{r}}{\dot{r}} = 2\frac{d}{dt}\dot{r} = 2\ddot{r}$$

so we have

$$\ddot{r} - \frac{J^2}{r^3} + \frac{m}{r^2} + 3\frac{J^2}{r^4} = 0$$

We now consider a circular orbit $r = R$.In this case we have $\dot{r} = \ddot{r} = 0$. From earlier we have

$$\ddot{r} - \frac{J^2}{r^3} + \frac{m}{r^2} + 3\frac{J^2}{r^4} = 0$$

which gives

$$\frac{J^2}{R^3} + \frac{m}{R^2} + 3\frac{J^2}{R^4} = 0 \rightarrow J^2 = \frac{mR^2}{R - 3m}$$

We also have

$$\begin{aligned} E^2 &= \dot{r}^2 + \left(1 - \frac{2m}{r}\right)\left(1 + \frac{J^2}{r^2}\right) \\ &= \left(1 - \frac{2m}{R}\right)\left(1 + \frac{J^2}{R^2}\right) = \frac{R - 2m)^2}{R(R - 3m)} \end{aligned}$$

and

$$E^2 = \left(1 - \frac{2m}{R}\right)\dot{t} \rightarrow \dot{t} = \frac{\sqrt{R}}{\sqrt{R - 3m}}$$

so that

$$\left(\frac{d\phi}{dt}\right)^2 = \left(\frac{\dot{\phi}}{\dot{t}}\right)^2 = \frac{J^2}{R^4}\frac{R - 3m}{R} = \frac{m}{R^3} \rightarrow \frac{d\phi}{dt} = \sqrt{\frac{m}{R^3}}$$

Now suppose that $r(\tau) = R + \epsilon(\tau)$, $\epsilon(\tau)$ small. This implies that

$$\begin{aligned}
0 &= \ddot{r} - \frac{J^2}{r^3} + \frac{m}{r^2} + 3\frac{J^2}{r^4} \\
0 &= \ddot{\epsilon} - \frac{J^2}{(R+\epsilon)^3} + \frac{m}{(R+\epsilon)^2} + 3\frac{J^2}{(R+\epsilon)^4} \\
0 &\approx \ddot{\epsilon} + \frac{m}{R^2}\left(1 - \frac{2\epsilon}{R}\right) - \frac{J^2}{R^3}\left(1 - \frac{3\epsilon}{R}\right) + 3\frac{J^2}{R^4}\left(1 - \frac{4\epsilon}{R}\right) \\
0 &\approx \ddot{\epsilon} + \epsilon\left(-\frac{2m}{R^3} + \frac{3J^2}{R^4} - \frac{12J^2}{R^5}\right) + C \\
0 &\approx \ddot{\epsilon} + \frac{\epsilon}{R^3}\left(\frac{3J^2}{R} - 2m - \frac{12J^2}{R^2}\right) + C \\
0 &\approx \ddot{\epsilon} + \frac{\epsilon}{R^3}\left(-2m + \frac{mR^2}{R-3m}\left(\frac{3}{R} - \frac{12m}{R^2}\right)\right) \\
0 &\approx \ddot{\epsilon} + \frac{m(R-6m)}{R^3(R-3m)}\epsilon + C
\end{aligned}$$

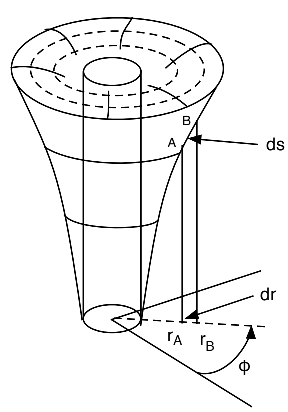
Therefore, circular orbits exhibit small oscillations (not exponential behavior) around the circle and are thus stble for $R > 6m$.

EP #87

We are considering the surface for $\theta = \pi/2 = $ constant, $t = $ constant so that the metric is just

$$ds^2 = \frac{dr^2}{1 - \frac{2m}{r}} + r^2 d\phi^2$$

The embedding of this surface in a flat 3-dimensional space looks like



The $x - y$ plane is the screen of a *record keeper*. The meter stick is positioned as shown (AB).

(a) Now $r_B - r_A = dr = \sqrt{1 - R_S/r}ds$ since $d\phi = 0$ between the ends of the rod. If $ds = 1\,m$, the *record keeper* sees something that is less than $1\,m$ long on her screen.

(b) However, the *record keeper* knows that the physical distance between the ends of the rod is given by ds and not dr. She can use the metric to calculate ds:

$$ds = \left(1 - \frac{R_S}{r}\right)^{-1/2} dr$$

We note that r and ϕ are coordinates that parameterize the surface. Differences between coordinates are not physical differences, however. It is the metric that gives the physical interpretation of differences between coordinates.

EP #88

(a) In the rocket frame, as in all inertial frames, the velocity of light is c, so the time to traverse the proper length L is $t' = L/c$.

(b) There are at least two instructive ways to do this problem:

Direct Calculation in the Earth Frame - Let t be the time as read on Earth clocks that the light signal reaches the nose of the rocket. The signal has traveled a distance equal to the *contracted* length of the rocket plus the distance $(4/5)ct$ it has traveled in the time t. Thus,

$$ct = L\left(1 - (4/5)^2\right)^{1/2} + (4/5)ct$$

Solving for t, one finds $t = 3L/c$

Transforming Back From the Rocket Frame - The event of the signal reaching the nose occurs at $t' = L/c$, $z' = L$ in the rocket frame if c is in the vertical direction. Therefore in the Earth frame

$$t = \frac{t' + (4/5)cz'}{\sqrt{1 - (4/5)^2}} = 3L/c$$

EP #89

The interval between the two events

$$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2$$

must be the same in both frames. In the laboratory frame

$$(\Delta s)^2 = (0)^2 + (3\,m)^2 = 9\,m^2$$

In the moving frame

$$9\,m^2 = -(3 \times 10^8\,m/sec10^{-8}\,sec)^2 + (\Delta x)^2$$

which gives $\Delta x = \sqrt{18\,m^2} = 4.24\,m$

Let (t, x) be coordinates of the inertial frame in which the events are simultaneous, and $(t', x'$ coordinates of a frame moving with respect to this one along the $x-$axis. The Lorentz boost connecting the two frames implies

$$t_2' - t_1' = \gamma\left((t_2 - t_1) - \frac{v}{c^2}(x_2 - x_1)\right)$$

In the unprimed frame, the two events are simultaneous $(\Delta t = t_2 - t_1 = 0)$ and separated by $\Delta x = x_2 - x_1 = 3\,m$. Then

$$\Delta t' = t_2' - t_1' = -\gamma\frac{v}{c^2}\Delta x = 10^{-8}\,sec$$

and solving for γ^2 gives

$$\gamma^2 = 1 + c^2\left(\frac{\Delta t'}{\Delta x}\right)^2 = 2$$

Therefore, the separation of the two events in the moving frame is

$$\Delta x' = \gamma\Delta x = \sqrt{2}(3) = 4.24\,m$$

EP #90

(a)

$$s = \int_0^R (1 - Ar^2)dr = R\left(1 - \frac{AR^2}{3}\right)$$

(b)

$$A = \int_{sphere}(Rd\theta)(R\sin\theta d\phi) = \int_0^\pi d\theta\int_0^{2\pi} R^2\sin\theta d\phi = 4\pi R^2$$

(c)

$$\begin{aligned}
V &= \int_{sphere}((1 - Ar^2)dr)(rd\theta)(r\sin\theta d\phi)\\
&= \int_0^R dr\int_0^\pi d\theta\int_0^{2\pi} d\phi r^2(1 - Ar^2)\sin\theta\\
&= \frac{4\pi R^3}{3}\left(1 - \frac{3}{5}AR^2\right)
\end{aligned}$$

(d)

$$\begin{aligned}
V_4 &= \int (1-Ar^2)dt \int_{sphere} ((1-Ar^2)dr)(rd\theta)(r\sin\theta d\phi) \\
&= \int_0^T (1-Ar^2)dt \int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\phi r^2(1-Ar^2)\sin\theta \\
&= \frac{4\pi R^3 T}{3}\left(1-\frac{6}{5}AR^2+\frac{3}{7}A^2R^4\right)
\end{aligned}$$

EP #91

(a) The geodesic equations can be found using the standard Lagrangian equation approach or by calculating Γs or the conditions for extremal proper time. They are

$$\begin{aligned}
&-\ddot{x} - 2\Omega\dot{y}\dot{t} + \Omega^2 x\dot{t}^2 = 0 \\
&-\ddot{y} + 2\Omega\dot{x}\dot{t} + \Omega^2 y\dot{t}^2 = 0 \\
&\ddot{z} = 0
\end{aligned}$$

(b) For example, in the non-relativistic limit, the x equation becomes

$$\frac{d^2x}{dt^2} = -2\Omega\frac{dy}{dt} + \Omega^2 x$$

The second term on the RHS is the $x-$component of the centrifugal force $\vec{\Omega}\times(\vec{\Omega}\times\vec{x})$ when $\vec{\Omega} = \Omega\hat{z}$. The first term is the Coriolis force $2\vec{\Omega}\times(d\vec{x}/dt)$.

EP #92

In Eddington-Finkelstein coordinates the equation for radial ($d\theta = d\phi = 0$) null geodesics (the light cone equation) is

$$ds^2 = 0 = -\left(1-\frac{2M}{r}\right)dv^2 + 2dvdr$$

An immediate consequence is that some radial light rays move along the curves

$$v = \text{ constant} \quad \text{ingoing radial light rays}$$

These are seen to be ingoing light rays because as t increases, r must decrease to keep v constant. The other possible solution is

$$0 = -\left(1-\frac{2M}{r}\right)dv + 2dr$$

This can be solved for dv/dr and the result integrated to find that these radial light rays move on the curves

$$v - 2\left(r + 2M\log|\frac{r}{2M}-1|\right) = \text{ constant}$$

so that we have radial outgoing light rays for $r > 2M$ and radial ingoing light rays for $r < 2M$.

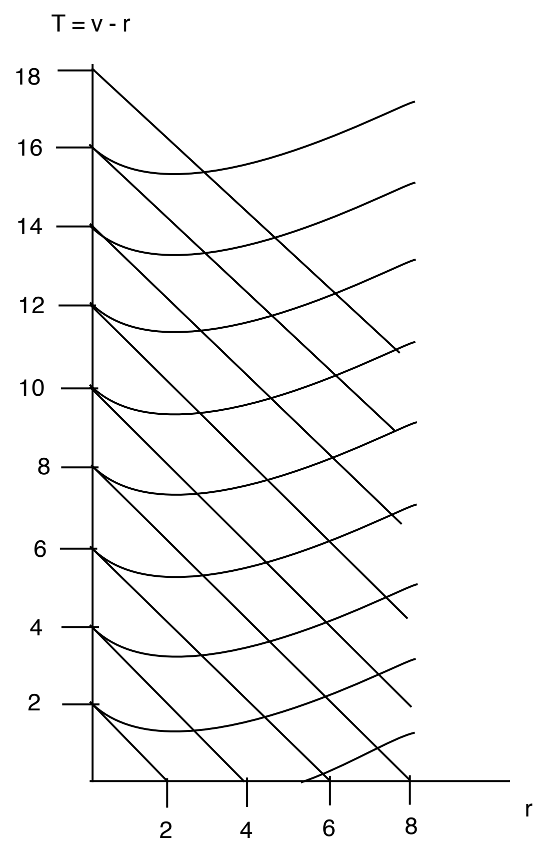
Now when M is negative, we get

$$\frac{dv}{dr} = \frac{1}{2}\left(1 + \frac{2|M|}{r}\right) > 0$$

The light rays are always outgoing! The v = constant light rays are ingoing. We then get the equation

$$v - 2\left(r - 2|M| \log|\frac{r}{2|M|} + 1|\right) = \text{ constant}$$

The resulting Eddington-Finkelstein plot of $T = \tilde{t} = v - r$ versus r is shown below.



From each point there is an outgoing light ray which escapes to infinity. The negative mass Schwarzschild geometry is therefore not a black hole.